MECCANICA RAZIONALE

Ing Civile & Ambientale

Navale

27 aprile 2021

Struttura delle equazioni di Lagrange

$\to$ $\mathcal{L}$, e quindi T.

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} = 0$$

come dipendono dalle $\underline{q}$ $(q_1, \ldots, q_\ell)$

dalle $\underline{\dot{q}}$ $(\dot{q}_1 \ldots \dot{q}_\ell)$ $\left[\frac{d}{dt} q(t)\right]$

e dal tempo t.

Prendiamo $\underline{x}_B = \underline{x}_B(\underline{q}(t); t)$

↑ vincoli mobili

$\underline{x}_B \longrightarrow \underline{v}_B = \frac{d}{dt}\underline{x}_B$

$$\underline{v}_B = \frac{d}{d\tau}\underline{x}_B = \sum_{i=1}^{\ell} \frac{\partial \underline{x}_B}{\partial q_i}\frac{dq_i}{d\tau} + \frac{\partial \underline{x}_B}{\partial \tau}$$

$\underline{x}_B(\underline{q}(\tau),\tau)$ $\qquad$ $\underline{x}_B(q(\tau),\tau)$

$$K = \frac{1}{2}\sum_{B\in S} m_B \|\underline{v}_B\|^2 = \dots$$

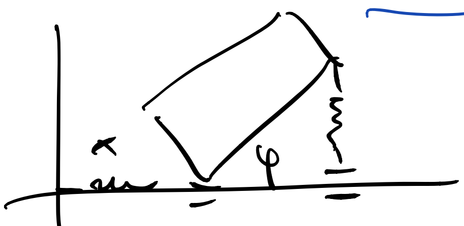
$$= K_0(\underline{q}(\tau);\tau) + \underline{b}(\underline{q}(\tau);t)\cdot\boxed{\dot{\underline{q}}} + \frac{1}{2}\boxed{\dot{\underline{q}}}\cdot A(\underline{q}(\tau);t)\cdot\boxed{\dot{\underline{q}}}$$

A è definita positiva

usualmente $\underline{x}_B = \underline{x}_B(\underline{q}(\tau))$

$K_0 = 0$, $\underline{b} = \underline{0}$ (contengono $\frac{\partial \underline{x}_B}{\partial \tau}$)

$$\hookrightarrow \boxed{K = \frac{1}{2}\dot{\underline{q}}\cdot A(\underline{q}(\tau))\cdot\dot{\underline{q}}}$$



$$K = \frac{1}{2}m\Big[\underbrace{\dot{x}^2}_{\dot{q}_1^2} + \underbrace{\dot{\varphi}^2}_{\dot{q}_2^2}\frac{5}{3}\ell^2 + \dots$$

$$- \sqrt{5}\, \ell\, \dot{x}\, \dot{\varphi} \sin(\gamma + \varphi) \Big]$$

$$\dot{q}_1\ \dot{q}_2$$

$$\frac{1}{2} (\dot{q}_1\ \ \dot{q}_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$$

$$= \frac{1}{2} \Big[\underset{m}{a_{11}} \dot{q}_1^2 + \underset{m \frac{5}{2} \ell^2}{a_{22}} \dot{q}_2^2 + 2 \underset{\ell \sqrt{5} \sin(\gamma + \varphi)}{a_{12}} \dot{q}_1 \dot{q}_2 \Big]$$

Date K, conoscendo Q, V

$$\frac{d}{dt} \frac{\partial K}{\partial \dot{q}_i} - \frac{\partial K}{\partial q_i} = Q_i$$

K dipende da $\dot{q}$

$\frac{d}{dt} \frac{\partial K}{\partial \dot{q}}$ dipende da $\ddot{q}$

$$\Rightarrow \quad \ddot{\underline{q}} = A^{-1}_{(q,t)} F(\dot{q}, q, t)$$

↑ eq. diff. in forma normale

date $\underline{q}(0)$, $\dot{\underline{q}}(0)$ → soluzione esiste ed è unica $\Rightarrow$ deterministico

$$\underline{A\, q} = F \qquad \det A \neq 0$$

A invertibile

Le equazioni di Lagrange hanno la struttura di un sistema dinamico

Esempio

$$\boxed{\underline{F} = m\,\underline{a}} \quad \text{per il punto materiale}$$

$$\underline{F} = m \underbrace{\frac{d^2 \underline{x}}{dt^2}}_{} = m \frac{d}{dt} \underline{v}$$

eq. diff 2nd ordine in

$$\underline{x} \leftarrow \underline{q}$$

sistema dinamico in $(\underline{x}, \underline{p})$

$$\begin{cases} \boxed{\dfrac{d}{dt} \underline{p} = \underline{F}} \end{cases}$$

$$\boxed{\frac{d}{dt}\underline{x} = \frac{\underline{p}}{m}}$$

$\dot{q} = f(q, p, t)$

dalla seconda $\underline{p} = m \frac{d}{dt}\underline{x}$

dalla prima $\underline{F} = \frac{d}{dt}\underline{p} = m \frac{d^2\underline{x}}{dt^2}$

Vediamo come funziona per le equazioni di Lagrange.

Definiamo

$$\boxed{p_i = \frac{\partial K}{\partial \dot{q}_i}}$$

$i = 1, \dots, \ell$

i-esimo momento coniugato (a q_i)

quindi $K = k_0 + \sum_k b_k \dot{q}_k + \sum_{k,j} \frac{1}{2} \dot{q}_k A_{kj} \dot{q}_j$

$$\frac{\partial k}{\partial \dot{q}_i} = 0 + b_i + \sum_{j=1}^{\ell} A_{ij} \dot{q}_j$$

$$p_i = b_i + \sum_{j=1}^{\ell} A_{ij} \dot{q}_j$$

in notazione vettoriale $\underline{p} = (p_1, \dots, p_\ell)$

$$\underline{p} = A \dot{\underline{q}} + \underline{b}$$

Allora $\rightarrow$ $\underline{\dot{q}} = A^{-1}(\underline{p} - \underline{b})$

Quindi possiamo esprimere $\underline{\dot{q}}$ come funzione di $\underline{q}$ e di $\underline{p}$ (ed eventualmente di t).

$$\boxed{\dot{q}_i = \gamma_i(\underline{q}, \underline{p}, t)} \quad i=1,\dots,\ell$$

funzione arbitraria

Abbiamo

$$\frac{d}{dt} p_i = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) =$$

$$= \left.\left(\frac{\partial K}{\partial q_i} + Q_i\right)\right|_{\dot{q}_i = \gamma_i(\underline{q}, \underline{p}, t)}$$

equation di Lagrange $\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} = Q_i$

Vogliamo $\underline{q}, \underline{\dot{q}} \longrightarrow \underline{q}, \underline{p}$

$$\frac{d}{dt} p_i = \left(\frac{\partial k}{\partial q_i} + Q_i \right) \Bigg|_{\dot{q} = \gamma_i(\underline{q}, \underline{p}, t)}$$

$$= f_i(\underline{q}, \underline{p}, t)$$

Quindi possiamo riscrivere le equazioni di Lagrange come un sistema normale di 2ℓ equazioni differenziali del primo ordine nelle incognite $(q_1, \dots, q_\ell, p_1, \dots, p_\ell)$

$$\begin{cases} \dfrac{d}{dt} q_i = \gamma_i(\underline{q}, \underline{p}, t) & i = 1, \dots, \ell \\ \dfrac{d}{dt} p_i = f_i(\underline{q}, \underline{p}, t) \end{cases}$$

Semplifichiamo ancora la notazione

$$\underline{y} = (q_1, \dots, q_\ell, p_1, \dots, p_\ell) \in \mathbb{R}^{2\ell}$$

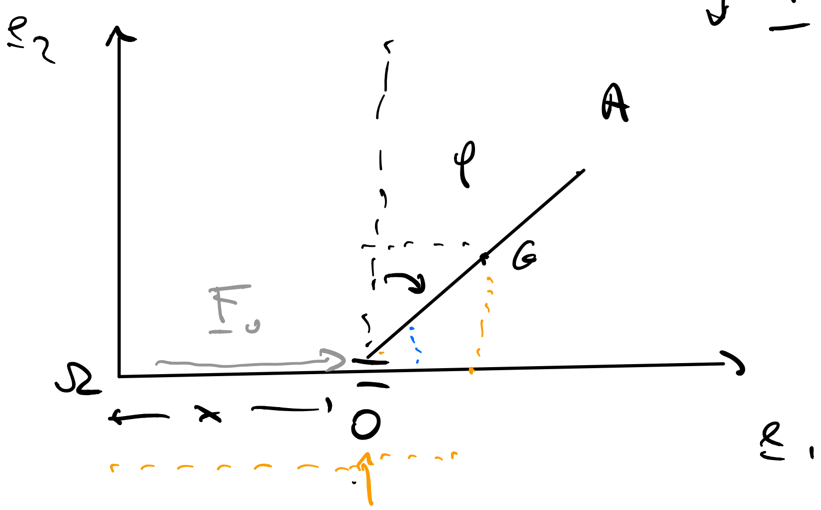
$$\underline{G} = (\gamma_1, \dots, \gamma_\ell, f_1, \dots, f_\ell) \in \mathbb{R}^{2\ell}$$

$$\rightarrow \boxed{\frac{d}{d\tau}\underline{y} = \underline{G}(\underline{y}, t)}$$

Sistema dinamico

Seconda parte

Esercizio



piano verticale

$\overline{OA} = \ell$

massa m

$$\underline{F}_0 = (P\varphi + D\dot{\varphi})\underline{e}_1$$

costanti

Energia cinetica

$$K = \frac{1}{2} m v_G^2 + \frac{1}{2} I_{3G} \dot{\varphi}^2$$

$$\underline{x}_G = \left(x + \frac{\ell}{2}\sin\varphi\right)\underline{e}_1 + \frac{\ell}{2}\cos\varphi\,\underline{e}_2$$

$$\underline{v}_G = \left(\dot{x} + \frac{\ell}{2}\cos\varphi\,\dot{\varphi}\right)\underline{e}_1 - \frac{\ell}{2}\sin\varphi\,\dot{\varphi}\,\underline{e}_2$$

$$v_G^2 = \underline{v}_G \cdot \underline{v}_G = \left(\dot{x} + \frac{\ell}{2}\cos\varphi\,\dot{\varphi}\right)^2 +$$

$$+\left(-\frac{l}{2}\sin\varphi\,\dot\varphi\right)^2$$

$$= \dot x^2 + \frac{l^2}{4}\cos^2\varphi\,\dot\varphi^2 + \dot x\,\dot\varphi\,l\cos\varphi +$$

$$+ \frac{l^2}{4}\sin^2\varphi\,\dot\varphi^2$$

$$= \dot x^2 + \frac{l^2}{4}\dot\varphi^2 + \dot x\,\dot\varphi\,l\cos\varphi$$

Momento di inerzia

$$\boxed{I_{G,3} = m\frac{l^2}{12}} \qquad I_{O,3} = m\frac{l^2}{3}$$

Quindi

$$K = \frac{1}{2}m\left(\dot x^2 + \frac{l^2}{4}\dot\varphi^2 + \dot x\,\dot\varphi\,l\cos\varphi\right) + \frac{1}{2}m\frac{l^2}{12}\dot\varphi^2$$

$$= \frac{1}{2}m\left(\dot x^2 + \frac{l^2}{3}\dot\varphi^2 + \dot x\,\dot\varphi\,l\cos\varphi\right)$$

Forze $\longrightarrow$ la forza peso

$$V = -m\underline{g}\cdot\underline{x}_G = mg\frac{l}{2}\cos\varphi$$

Lavoro virtuale della forza $\underline{F}_0$

$$\underline{F}_0 \cdot \delta \underline{x}_0 = F_0 \, \delta x$$

$$\underline{e}_1 \qquad \uparrow \; \underline{x}_0 = x \, \underline{e}_1$$

$$Q_x = F_0 - \frac{\partial V}{\partial x} = F_0$$

$$Q_\varphi = - \frac{\partial V}{\partial \varphi} = mg \frac{\ell}{2} \sin\varphi$$

$$K = \frac{1}{2} m \left(\dot{x}^2 + \frac{\ell^2}{3} \dot{\varphi}^2 + \dot{x} \dot{\varphi} \, \ell \cos\varphi \right)$$

$$= \frac{1}{2} (\dot{x} \;\; \dot{\varphi}) \, A \begin{pmatrix} \dot{x} \\ \dot{\varphi} \end{pmatrix}$$

$$\boxed{A = m \begin{pmatrix} 1 & \ell/2 \cos\varphi \\ \ell/2 \cos\varphi & \frac{\ell^2}{3} \end{pmatrix}}$$

Infatti:

$$\frac{m}{2} (\dot{x} \;\; \dot{\varphi}) \begin{pmatrix} 1 & \ell/2 \cos\varphi \\ \ell/2 \cos\varphi & \frac{\ell^2}{3} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\varphi} \end{pmatrix} =$$

$$= \frac{m}{2} \begin{pmatrix} \dot{x} & \dot{\varphi} \end{pmatrix} \begin{pmatrix} \dot{x} + \frac{\ell}{2} \cos\varphi \, \dot{\varphi} \\ \dot{x} \frac{\ell}{2} \cos\varphi + \dot{\varphi} \frac{\ell^2}{3} \end{pmatrix}$$

$$= \frac{m}{2} \left(\dot{x}^2 + \frac{\ell}{2} \dot{x}\dot{\varphi} \cos\varphi + \dot{\varphi}\dot{x} \frac{\ell}{2} \cos\varphi + \dot{\varphi}^2 \frac{\ell^2}{3} \right)$$

$$= \frac{m}{2} \left(\dot{x}^2 + \dot{\varphi}^2 \frac{\ell^2}{3} + \underline{\ell \dot{x} \dot{\varphi} \cos\varphi} \right) = K$$

$$A = m \begin{pmatrix} 1 & \frac{\ell}{2} \cos\varphi \\ \frac{\ell}{2} \cos\varphi & \frac{\ell^2}{3} \end{pmatrix}$$

$$\longrightarrow \det A = m^2 \left(\frac{\ell^2}{3} - \frac{\ell^2}{4} \cos^2\varphi \right) > 0$$

Equazioni di Lagrange

$$\begin{cases} \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{x}} \right) - \frac{\partial K}{\partial x} = Q_x \\ \frac{d}{dt} \left(\frac{\partial K}{\partial \dot{\varphi}} \right) - \frac{\partial K}{\partial \varphi} = Q_\varphi \end{cases}$$

esplicitamente

- $$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}}\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\left(\dot{x}^2 + \frac{l^2}{3}\dot{\varphi}^2 + l\dot{x}\dot{\varphi}\cos\varphi\right)\right)$$

$$= \frac{d}{dt}\left[m\dot{x} + \frac{1}{2}ml\dot{\varphi}\cos\varphi\right]$$

$$= m\ddot{x} + \frac{1}{2}ml\ddot{\varphi}\cos\varphi - \frac{1}{2}ml\dot{\varphi}^2\sin\varphi$$

- $$\frac{\partial K}{\partial x} = 0 \leftarrow$$

- $$Q_x = F_0 \leftarrow$$

$$\boxed{m\ddot{x} + \frac{1}{2}ml\ddot{\varphi}\cos\varphi - \frac{1}{2}ml^2\dot{\varphi}^2\sin\varphi = F_0}$$

- $$\frac{d}{dt}\left(\frac{\partial k}{\partial \dot{\varphi}}\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{\varphi}}\left(\frac{1}{2}m\left(\dot{x}^2 + \frac{l^2}{3}\dot{\varphi}^2 + l\dot{x}\dot{\varphi}\cos\varphi\right)\right)$$

$$= \frac{d}{dt}\left[\frac{1}{2}m\left(\frac{2}{3}l^2\dot{\varphi} + l\dot{x}\cos\varphi\right)\right]$$

$$= \frac{1}{2}m\left(\frac{2}{3}l^2\ddot{\varphi} + l\ddot{x}\cos\varphi - l\dot{x}\dot{\varphi}\sin\varphi\right)$$

$$= m\frac{l^2}{3}\ddot{\varphi} + \frac{1}{2}ml\ddot{x}\cos\varphi - \frac{1}{2}ml\dot{x}\dot{\varphi}\sin\varphi$$

$$\frac{\partial K}{\partial \varphi} = \frac{\partial}{\partial \varphi}\left[\frac{1}{2} m\left(\dot{x}^2 + \frac{\ell^2}{3}\dot{\varphi}^2 + \ell \dot{x}\dot{\varphi}\cos\varphi\right)\right]$$

$$= -\frac{1}{2} m \ell \dot{x}\dot{\varphi}\sin\varphi$$

$$Q_\varphi = m g \frac{\ell}{2}\sin\varphi$$

$$m\frac{\ell^2}{3}\ddot{\varphi} + \frac{1}{2} m \ell \ddot{x}\cos\varphi - \cancel{\frac{1}{2} m \ell \dot{x}\dot{\varphi}\sin\varphi}$$

$$+ \cancel{\frac{1}{2} m \ell \dot{x}\dot{\varphi}\sin\varphi} = m g \frac{\ell}{2}\sin\varphi$$

$$\boxed{m\frac{\ell^2}{3}\ddot{\varphi} + \frac{1}{2} m \ell \ddot{x}\cos\varphi = m g \frac{\ell}{2}\sin\varphi}$$

$$\begin{cases} m\ddot{x} + \frac{1}{2} m \ell \ddot{\varphi}\cos\varphi - \frac{1}{2} m \ell \dot{\varphi}^2\sin\varphi = F_0 \\ m\frac{\ell^2}{3}\ddot{\varphi} + \frac{1}{2} m \ell \ddot{x}\cos\varphi = m g \frac{\ell}{2}\sin\varphi \end{cases}$$

$$m\begin{pmatrix} 1 & \frac{\ell}{2}\cos\varphi \\ \frac{\ell}{2}\cos\varphi & \frac{\ell^2}{3} \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\varphi} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} m \ell \dot{\varphi}^2\sin\varphi + F_0 \\ m g \frac{\ell}{2}\sin\varphi \end{pmatrix}$$

$$\underline{A}\,\ddot{\underline{q}} = \underline{F}$$

$$\hookrightarrow \quad \ddot{\underline{q}} = A^{-1}\underline{F}$$

Terzo parte

$$K = \frac{1}{2} m \left(\dot{x}^2 + \frac{l^2}{3}\dot{\varphi}^2 + \dot{x}\dot{\varphi}\, l \cos\varphi \right)$$

$$\frac{1}{2} \begin{pmatrix} \dot{x} & \dot{\varphi} \end{pmatrix} \underbrace{\begin{pmatrix} m & m\frac{l}{2}\cos\varphi \\ m\frac{l}{2}\cos\varphi & m\frac{l^2}{3} \end{pmatrix}}_{A} \begin{pmatrix} \dot{x} \\ \dot{\varphi} \end{pmatrix}$$

$$\hookrightarrow \underbrace{\begin{pmatrix} m & m\frac{l}{2}\cos\varphi \\ m\frac{l}{2}\cos\varphi & \frac{l^2}{3} m \end{pmatrix}}_{A} \begin{pmatrix} \ddot{x} \\ \ddot{\varphi} \end{pmatrix} = \begin{pmatrix} m\frac{l}{2}\dot{\varphi}^2 \sin\varphi + F_0 \\ m\frac{g}{2} l \sin\varphi \end{pmatrix}$$

Adesso riscriviamo queste equazioni nella forma di un sistema dinamico

Definiamo i momenti coniugati

$$x \longrightarrow \quad p_x = \frac{\partial k}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left[\frac{1}{2} m\left(\dot{x}^2 + \dot{\varphi}^2 \frac{\ell^2}{2} + \ell \dot{x}\dot{\varphi}\cos\varphi\right)\right]$$

$$= m\dot{x} + m\frac{\ell}{2}\dot{\varphi}\cos\varphi$$

$$\varphi \longrightarrow \quad p_\varphi = \frac{\partial k}{\partial \dot{\varphi}} = m\frac{\ell}{2}\dot{x}\cos\varphi + m\frac{\ell^2}{3}\dot{\varphi}$$

$$p_i = \frac{\partial k}{\partial \dot{q}_i} = \ldots = \cancel{b_i} + \sum_j A_{ij}\dot{q}_j$$

$$\begin{pmatrix} p_x \\ p_\varphi \end{pmatrix} = A\begin{pmatrix} \dot{x} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} m & m\frac{\ell}{2}\cos\varphi \\ m\frac{\ell}{2}\cos\varphi & m\frac{\ell^2}{3} \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{\varphi} \end{pmatrix}$$

$$= \begin{pmatrix} m\dot{x} + m\frac{\ell}{2}\dot{\varphi}\cos\varphi \\ m\frac{\ell}{2}\cos\varphi\,\dot{x} + m\frac{\ell^2}{3}\dot{\varphi} \end{pmatrix}$$

$$\begin{pmatrix} p_x \\ p_\varphi \end{pmatrix} = A\begin{pmatrix} \dot{x} \\ \dot{\varphi} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \dot{x} \\ \dot{\varphi} \end{pmatrix} = A^{-1}\begin{pmatrix} p_x \\ p_\varphi \end{pmatrix} = B\begin{pmatrix} p_x \\ p_\varphi \end{pmatrix}$$

$$A^{-1} = B$$

$$A = \begin{pmatrix} m & \frac{\ell}{2} m \cos\varphi \\ \frac{\ell}{2} m \cos\varphi & \frac{\ell^2 m}{3} \end{pmatrix}$$

$$B = A^{-1} = \underbrace{\frac{1}{m^2 \ell^2 \left(\frac{1}{3} - \frac{\cos^2\varphi}{4} \right)}}_{\frac{1}{\det A}} \begin{pmatrix} m\frac{\ell^2}{3} & -\frac{\ell}{2} \cos\varphi\, m \\ -\frac{\ell}{2} \cos\varphi\, m & m \end{pmatrix}$$

$$\begin{pmatrix} \dot{x} \\ \dot{\varphi} \end{pmatrix} = B \begin{pmatrix} p_x \\ p_\varphi \end{pmatrix} = \begin{pmatrix} B_{11} p_x + B_{12} p_\varphi \\ B_{12} p_x + B_{22} p_\varphi \end{pmatrix}$$

$$\begin{cases} \dot{x} = B_{11} p_x + B_{12} p_\varphi \\ \dot{\varphi} = B_{12} p_x + B_{22} p_\varphi \end{cases} \qquad \frac{d}{dt} q_i = \gamma_i(q, p, t)$$

$$\frac{dp_i}{dt} = f_i$$

$$\frac{d}{dt} p_i = \frac{d}{dt}\left(\frac{\partial k}{\partial \dot{q}_i} \right) = \left(\frac{\partial k}{\partial q_i} + Q_i \right)\Bigg|_{\dot{q}_i = \gamma_i}$$

eq. di Lagrange

$$\frac{d}{dt} p_x = \frac{d}{dt}\left(\frac{\partial k}{\partial \dot{x}} \right) = \left(\frac{\partial k}{\partial x} + Q_x \right)\Bigg|_{\substack{\dot{x} = B_{11} p_x + B_{12} p_\varphi \\ \dot{\varphi} = \dots}}$$

$$= F_0$$

Primo eq. $\dot{p}_x = F_0$

$$\frac{d}{dt} p_\varphi = \frac{d}{dt}\left(\frac{\partial k}{\partial \dot\varphi}\right) = \left(\frac{\partial k}{\partial \varphi} + Q_\varphi\right)\Bigg|_{\substack{\dot\varphi = B_{12}p_x + B_{22}p_\varphi \\ \dot x = B_{11}p_x + B_{12}p_\varphi}}$$

$$\frac{\partial k}{\partial \varphi} = -\frac{1}{2} m l \dot{x} \dot{\varphi} \sin\varphi$$

$$Q_\varphi = m g \frac{l}{2} \sin\varphi$$

$$\frac{d}{dt} p_\varphi = -\frac{1}{2} m l \sin\varphi \underbrace{(B_{11} p_x + B_{12} p_\varphi)}_{\dot x}\underbrace{(B_{12} p_x + B_{22} p_\varphi)}_{\dot\varphi}$$

$$+ m g \frac{l}{2} \sin\varphi$$

Per riassumere

$$\begin{cases} \dot{x} = B_{11} p_x + B_{12} p_\varphi \\ \dot{\varphi} = B_{12} p_x + B_{22} p_\varphi \\ \dot{p}_x = F_0 \\ \dot{p}_\varphi = -m\frac{l}{2}\sin\varphi\,(B_{11} p_x + B_{12} p_\varphi)(B_{12} p_x + B_{22} p_\varphi) + \end{cases}$$

$+ m g \frac{\ell}{2} \sin \varphi$

$$\boxed{\frac{d}{d\tau} \underline{y} = \underline{G}(\underline{y}, t)}$$

$\underline{y} = (x, \varphi, p_x, p_\varphi)$

Commento extra

eq. Lagrange $\longrightarrow \mathcal{L}$

$$\left(\frac{d}{d\tau} \frac{\partial}{\partial \dot{q}_i} - \frac{\partial}{\partial q_i}\right) \mathcal{L} = 0$$

Hamiltoniana

$$\longrightarrow \mathcal{H} := \left(\sum_{j=1}^{\ell} p_j \dot{q}_j - \mathcal{L}\right)\Bigg|_{\dot{q}_j = \gamma_j(\underline{q}, p, t)}$$

data $\mathcal{H}$ il sistema dinamico

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i} \\ \dot{p}_i = - \dfrac{\partial H}{\partial q_i} \end{cases}$$

Prima $p_i = \frac{\partial K}{\partial \dot q_i} = \frac{\partial \mathcal{L}}{\partial \dot q_i}$ $i=1,\dots,\ell$

Usiamo che $\mathcal{H}$ è funzione di $\underline{q}$ e $\underline{p}$ direttamente e tramite $\dot{\underline{q}}_i = \gamma_i(\underline{q}, \underline{p}, t)$

- $\frac{\partial \mathcal{H}}{\partial \dot q_k} = p_k - \frac{\partial \mathcal{L}}{\partial \dot q_k} = 0$ $\qquad H = \left(\sum p_j \dot q_j - \mathcal{L}\right)_{\dot q_i = \gamma_i}$

- $\frac{\partial \mathcal{H}}{\partial p_i} = \dot q_i + \sum_{k=1}^{\ell} \frac{\partial \mathcal{H}}{\partial \dot q_k} \frac{\partial \dot q_k}{\partial p_i} = \dot q_i$

$\frac{\partial \mathcal{H}}{\partial \dot q_k} = 0$

- $\frac{\partial \mathcal{H}}{\partial q_i} = -\frac{\partial \mathcal{L}}{\partial q_i} + \sum_{k=1}^{\ell} \frac{\partial \mathcal{H}}{\partial \dot q_k} \frac{\partial \dot q_k}{\partial q_i} =$

$$= -\frac{\partial \mathcal{L}}{\partial q_i} = -\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot q_i}\right) = -\frac{d}{dt} p_i$$

eq. di Lagrange

definizione $p_i = \frac{\partial \mathcal{L}}{\partial \dot q_i}$

$$\mathcal{H} = \left(\sum_{j=1}^{\ell} p_j \dot q_j - \mathcal{L}\right)\Big|_{\dot q_j = \gamma_j}$$

$$\begin{cases} \dot{q}_i = \dfrac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial \mathcal{H}}{\partial q_i} \end{cases}$$

Si può verificare

$$\mathcal{L} = K - V = \frac{1}{2}\,\underline{\dot{q}} \cdot A\,\underline{\dot{q}} - V(\underline{q})$$

$$H = \left(\underline{p} \cdot \dot{q} - \frac{1}{2}\,\underline{\dot{q}}\,A\,\underline{\dot{q}} + V(\underline{q}) \right)$$

$\underline{p} = A\underline{\dot{q}}$ $\qquad$ $\underline{\dot{q}} = A^{-1}\underline{p}$

$$= K + V$$

$\mathcal{L} = K - V$, $H = \underbrace{K + V}_{\text{energia meccanica}}$

$\longrightarrow$ meccanica hamiltoniana