$\varphi : X \longrightarrow Y$ $\qquad \varphi^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$

$Y \subseteq \mathbb{A}^n$ closed : $\quad \mathrm{Hom}(X, Y) \underset{\#}{\overset{*}{\longleftrightarrow}} \mathrm{Hom}(\mathcal{O}(Y), \mathcal{O}(X))$

If φ is dominant $\Rightarrow$ $\varphi^* : K(Y) \hookrightarrow K(X)$
X, Y irreducible

$\dim Y \leq \dim X$

If φ is isomorphism : $\mathcal{O}(X) \simeq \mathcal{O}(Y)$

If $Y \subseteq \mathbb{A}^n$ closed $\quad X \simeq Y \Longleftrightarrow \mathcal{O}(X) \simeq \mathcal{O}(Y)$

$X \simeq Y \Longrightarrow K(X) \simeq K(Y)$: $\dim X = \dim Y$.

How to express a regular map if $Y \subseteq \mathbb{P}^n$ (general case).

<u>characterization</u> $\quad \varphi : X \longrightarrow \mathbb{P}^n$

φ is regular $\Longleftrightarrow$ locally φ is given by $n+1$ homog. polynomials $F_0, \ldots, F_n$ of the same degree

$X \subseteq \mathbb{P}^m \qquad \varphi: X \longrightarrow \mathbb{P}^n$

the condition means: $\forall\, P \in X \ \exists\, U_P$ open nbhd and $F_0, \dots, F_n \in K[x_0, \dots, x_m]$ s.t. $\varphi|_{U_P} = [F_0, \dots, F_n]$:

given $x[x_0, \dots, x_m]$, $x \in U_P$ $\quad \varphi(x) = [F_0(x_0, \dots x_m), \dots, F_n(x_0, \dots, x_m)]$

$F_i(\lambda x_0, \dots, \lambda x_m) = \lambda^d F_i(x_0, \dots, x_m) \qquad d = \deg F_i \ \forall i$

The image of x is well defined up to change of coordinates. We need: $\forall\, x \in U_P$ at least one among $F_0, \dots, F_n$ is $\neq 0$ at x.

Pf "$\Rightarrow$" Ass. $\varphi: X \longrightarrow \mathbb{P}^n = U_0 \cup U_1 \cup \dots \cup U_n$ is regular, $P \in X$

$\varphi(P) = Q \in U_0$

$\varphi^{-1}(U_0) \subseteq X$ open, $P \in \varphi^{-1}(U_0)$ $\qquad \varphi|_{\varphi^{-1}(U_0)} : \varphi^{-1}(U_0) \longrightarrow \underline{\underline{U_0 \simeq \mathbb{A}^n}}$ is regular

Now the codomain is $\mathbb{A}^n$: $\forall\, x \in \varphi^{-1}(U_0)\ \exists\, U_x$

$\varphi|_{U_x} = (\varphi_1, \dots, \varphi_n)$, $\quad \varphi_1, \dots, \varphi_n \in \mathcal{O}(U_x)$, $\quad \varphi_1 = \frac{F_1}{G_1}, \ \dots, \ \varphi_n = \frac{F_n}{G_n}$

F_i, G_i have the same degree and homog.

$= \left(\frac{F_1}{G_1}, \dots, \frac{F_n}{G_n}\right) = \left(\frac{F_1'}{G}, \dots, \frac{F_n'}{G}\right) = $ in $K[x_0, \dots, x_m]$

$F_1', \dots, F_n', G$ all homog of the same degree

$= \left[1, \frac{F_1'}{G}, \dots, \frac{F_n'}{G}\right] = [G, F_1', \dots, F_n']$

Note: G is $\neq 0$ $\forall$ point of U_x

"$\Longleftarrow$" Assume $\varphi : X \longrightarrow Y$ map s.t. $\forall\, p \in X\ \exists\, U_p$, $F_0, \dots, F_n \in K[x_0 \dots x_m]$ s.t. $\varphi|_{U_p} = [F_0 \dots F_m]$

$x \in U_p$ $\quad \varphi(x) = [F_0(x), \dots, F_m(x)]$ Assume $F_0(x) \neq 0$

$$= \left[1, \frac{F_1(x)}{F_0(x)}, \dots, \frac{F_m(x)}{F_0(x)}\right] = \left(\frac{F_1}{F_0}(x), \dots, \frac{F_m}{F_0}(x)\right)$$

$\frac{F_1}{F_0}, \dots, \frac{F_m}{F_0}$ define regular fns. where $F_0 \neq 0$:

in $\underbrace{X \setminus V_p(F_0)}_{\text{open} \neq \emptyset} \ni x$ $\quad$ On $U_p \cap (X \setminus V_p(F_0))$

$\Longrightarrow \varphi = \left(\frac{F_1}{F_0}, \dots, \frac{F_m}{F_0}\right)$ regular

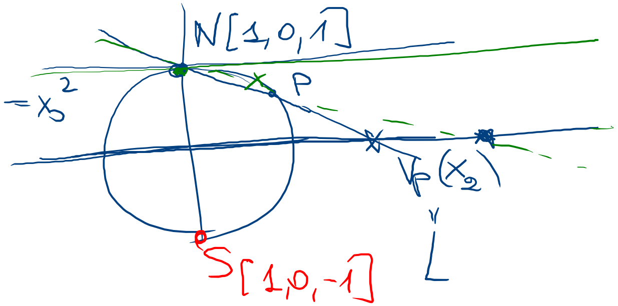
We use the characterization of the aff. case

Examples

1) Stereographic projection.

$X = V_p(x_1^2 + x_2^2 - x_0^2)$

$X : x_1^2 + x_2^2 = x_0^2$



$K \Leftrightarrow V_p(x_2) \setminus [0,1,0]$

$\frac{x_1}{x_0} \longleftrightarrow [x_0, x_1, 0]$, $x_0 \neq 0$

$\lambda \longleftrightarrow [1, \lambda, 0]$

$P \neq N \qquad P \in X$

$f : X \setminus \{N\} \longrightarrow K$

$P \longmapsto \overline{NP} \cap V_p(x_2)$

$f = \frac{x_0 + x_2}{x_1} = \frac{x_1}{x_0 - x_2}$

$(x_0 - x_2)(x_0 + x_2) = x_1^2$ on X

dom $f = X - \{N\}$

Def. $\varphi : X \longrightarrow \mathbb{P}^1 \longleftrightarrow V_p(x_2)$

$[x_0, x_1] \longleftrightarrow [x_0, x_1, 0]$

$[x_0, x_1, x_2] \longrightarrow [x_1, x_0 + x_2]$: this is well def. on $X \setminus V_p(x_1, x_0 + x_2) = X \setminus \{[1,0,-1]\}$

$[x_0 - x_2, x_1]$: well def. on $X \setminus V_p(x_0 - x_2, x_1) = X \setminus N$

On $X \setminus \{N, S\}$: $\operatorname{rk}\begin{pmatrix} x_1 & x_0 + x_2 \\ x_0 - x_2 & x_1 \end{pmatrix} < 2$

$\det\begin{pmatrix} x_1 & x_0 + x_2 \\ x_0 - x_2 & x_1 \end{pmatrix} = x_1^2 - (x_0^2 - x_2^2)$: equation of X

$\varphi : X \longrightarrow \mathbb{P}^1$ is a regular map : stereographic proj.

$[1,0,1] \longrightarrow [0,2] = [0,1]$ point at infinity

$\varphi^{-1} : \mathbb{P}^1 \longrightarrow X$ exists

$A[a_0, a_1] \longmapsto (\overline{AN} \cap X) - N$

$A\,[a_0, a_1, 0]$
$N\,[1, 0, 1]$

$\overline{AN} \begin{cases} x_0 = \lambda a_0 + \mu \\ x_1 = \lambda a_1 \\ x_2 = \mu \end{cases}$ $\lambda = \dfrac{x_1}{a_1}$ $x_0 = \lambda a_0 + x_2$

$a_1 x_0 - a_0 x_1 + a_1 x_2 = 0$

$\begin{cases} x_0^2 - x_1^2 - x_2^2 = 0 \,/ a_1^2 \\ a_1 x_0 - a_0 x_1 + a_1 x_2 = 0 \end{cases}$ An. $a_1 \neq 0$

$a_1 x_0 = a_0 x_1 - a_1 x_2$

$a_1^2 x_0^2 = a_1^2 (x_1^2 + \cancel{x_2^2}) = (a_0 x_1 - a_1 x_2)^2 =$

$= a_0^2 x_1^2 - 2 a_0 a_1 x_1 x_2 + \cancel{a_1^2 x_2^2}$

$x_1 (a_0^2 x_1 - 2 a_0 a_1 x_2 - a_1^2 x_1) = 0$ $x_1 = 0$ N

$(a_0^2 - a_1^2) x_1 - 2 a_0 a_1 x_2 = 0$

$[x_1, x_2] = [2 a_0 a_1, a_0^2 - a_1^2]$

$a_1 x_0 = a_0 (2 a_0 a_1) - a_1 (a_0^2 - a_1^2) =$

$= a_0^2 a_1 + a_1^3$ $\underline{x_0 = a_0^2 + a_1^2}$

$\varphi^{-1} : [a_0, a_1] \longrightarrow [a_0^2 + a_1^2, 2 a_0 a_1, a_0^2 - a_1^2]$

$\mathbb{P}^1 \longrightarrow X$ regular

φ^{-1} is def. by 3 homog. polynomials of deg 2

$\begin{cases} a_0^2 + a_1^2 = 0 \\ 2 a_0 a_1 = 0 \\ a_0^2 - a_1^2 = 0 \end{cases}$ is satisfied only by $(0,0)$

$\varphi : X \longrightarrow \mathbb{P}^1$ is an isomorphism

$\varphi^{-1} : \mathbb{P}^1 \longrightarrow X$ gives a parametrization of X by 2 homog. parameters

$X : \; x_1^2 + x_2^2 = x_0^2$: we have all the solutions of this equation

In particular the solutions in $\mathbb{Z}$, are obtained when $[a_0, a_1] \in \mathbb{P}^1_{\mathbb{Z}}$

2) X $\quad \{\varphi : X \longrightarrow X \mid \varphi \text{ isomorphism}\} \neq \emptyset$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad \ni 1_X$

group wrt to composition : automorphisms of X

$\mathrm{Aut}(X)$

$X = \mathbb{A}^n \quad \mathrm{Aut}(\mathbb{A}^n) \supseteq \{\text{affinities}\}$

$\varphi : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ affinity or affine isomorph.

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longrightarrow Mx + b$ $\quad$ M $n \times n$ matrix $\in M(n \times n, K)$

$\det M \neq 0$

$b \in \mathbb{A}^n$

$\varphi = (\varphi_1, \dots, \varphi_n)$

$$\begin{cases} \varphi_1 = m_{11} x_1 + \dots + m_{1n} x_n + b_1 \\ \vdots \\ \varphi_n = m_{n1} x_1 + \dots + m_{nn} x_n + b_n \end{cases}$$

$y = \varphi(x) = Mx + b \quad\quad x = \varphi^{-1}(y) = M^{-1} y - M^{-1} b$

$Aut(\mathbb{A}^n)$: there is no description if $n > 1$

If $n = 1$: $Aut(\mathbb{A}^1) = Aff(\mathbb{A}^1)$

$\varphi : \mathbb{A}^1 \xrightarrow{\sim} \mathbb{A}^1$ $\quad \varphi(t) = F(t)$, $F \in K[t]$

$\exists \varphi^{-1} : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ $\quad \varphi^{-1}(t) = G(t)$, $G \in K[t]$

$\forall t$: $\underbrace{G(F(t))} = t$ ∴ if K is infinite $\Longrightarrow$ equality of polynomials

Take derivative : $G'(F(t))\underbrace{F'(t)} = 1$ in $K[t]$:

char $K = 0$

$F'(t)$ is invertible $\Longrightarrow F'(t) = c \neq 0$

$\Longrightarrow F(t) = ct + d$,

also $G(t)$ has deg 1

$n > 1$ $\exists$ automorphisms of $\mathbb{A}^n_K$ which are not linear

polynom.

$\varphi : (x_1, x_2) \longrightarrow (x_1, x_2 + F(x_1))$

$$\begin{array}{cc} x_1 & x_2 + F(x_1) \\ \| & \| \\ y_1 & y_2 \end{array}$$

$\varphi \begin{cases} y_1 = x_1 \\ y_2 = x_2 + F(x_1) \end{cases}$ $\quad \varphi^{-1} \begin{cases} x_1 = y_1 \\ x_2 = y_2 - F(y_1) \end{cases}$

φ^{-1} exists and is regular

Jacobian conjecture $\quad \mathbb{A}^n \quad \operatorname{char} K = 0$.

$\varphi : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ regular : it is an isom.

$\Longleftrightarrow \quad |J(\varphi)|$ is a constant $\neq 0$.

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3) $\operatorname{Aut}(\mathbb{P}^n_K) \supseteq \{\text{projectivities}\}$

$\mathbb{P}^n_K \qquad \mathbb{P}^m_K \qquad A \in M(m+1, n+1; K)$

$L(A) : K^{n+1} \longrightarrow K^{m+1} \Longrightarrow \varphi_A : \underline{\mathbb{P}^n_K \setminus \mathbb{P}(\operatorname{Ker} L(A))} \longrightarrow \mathbb{P}^m$

$x = \begin{pmatrix} x_0 \\ \vdots \\ x_{n+1} \end{pmatrix} \longrightarrow Ax$ regular map given

by homog. polynomials of deg 1

$A = (a_{ij}) \qquad \varphi_A = (F_0, \dots, F_m)$

$F_0 = a_{00} x_0 + \text{———} + a_{0n} x_n$

$\vdots$

$F_m = a_{m0} x_0 + \text{———} + a_{mn} x_n$

If $m = n$, A is invertible $\Longrightarrow \exists\, \varphi_A^{-1}$ and is associated to $A^{-1} \Longrightarrow \varphi_A$ is an isomorphism

A is not the only matrix which defines φ_A, $\forall \lambda \in K \setminus \{0\}$ λA also def. φ_A

$\{\text{projectivities}\} \simeq GL(n+1)/\sim \; = PGL(n+1)$

$Aut(\mathbb{P}^n_K) \supseteq PGL(n+1)$

Thm. Equality holds.

$X, Y \subseteq \mathbb{P}^n_K$

1) $X \simeq Y$ isomorphic if $\varphi : X \xrightarrow{\sim} Y$ isom.

2) X is projectively equivalent to Y if

$\exists\, \varphi_A : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ projectivity s.t. $\varphi_A(X) = Y$

Thm. Fundamental thm on projectivities.

$\mathbb{P}^n_K$ $\quad P_0, P_1, \dots, P_{n+1}$ $\quad n+2$ pts in general position

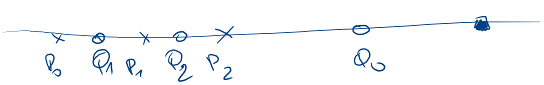
$\quad Q_0, Q_1, \dots, Q_{n+1}$ $\quad$ " " " "

then $\exists!\, \varphi : \mathbb{P}^n_K \longrightarrow \mathbb{P}^n_K$ s.t. $\varphi(P_i) = Q_i$ $\quad$ projectivity

$\forall\, i = 0, \dots, n+1$.

$n = 1$ $\quad P_0, P_1, P_2$ $\quad$ 2 by 2 distinct

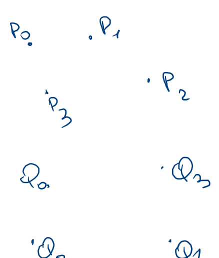
$\quad Q_0, Q_1, Q_2$



When $\exists$ a proj. φ s.t. $(P_0 P_1 P_2 P_3) \longrightarrow (Q_0 Q_1 Q_2 Q_3)$?

Def. of cross-ratio (birapporto).

$n = 2$ $\quad$ 3 by 3 not aligned



4) Veronese maps

$\mathbb{P}^n$ Fix $d \geq 1$: consider all the monomials of deg d in $x_0, \dots, x_n$; fix an ordering

$$\mathbb{P}^n \xrightarrow[\text{of deg } d]{[\text{monomials}]} \mathbb{P}^{N}$$

N

$n=1$ $d=3$ $x_0^3, x_0^2x_1, x_0x_1^2, x_1^3$

$$\mathbb{P}^1 \xrightarrow[{[x_0^3, x_0^2x_1, x_0x_1^2, x_1^3]}]{} \mathbb{P}^3$$ parametrization of skew cubic

$\forall n, d$ we get a regular map:

V_p (all monomials of deg d) $= \emptyset$ in $\mathbb{P}^n$

$x_0^d = x_1^d = \dots = x_n^d = 0$

$N = \#\{$monomials of deg d in $x_0, \dots, x_n\} - 1$

$$= \binom{n+1+d-1}{d} - 1 = \binom{n+d}{d} - 1$$

$n=1$ $\quad v_d : \mathbb{P}^1 \longrightarrow \mathbb{P}^d$ $\quad$ $d=3$ skew cubic

$[x_0^d, x_0^{d-1}x_1, \dots, x_1^d]$

$d=2$ $\quad v_2 : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ $\quad [y_0, y_1, y_2]$

$[x_0, x_1] \longrightarrow [x_0^2, x_0x_1, x_1^2]$ $\qquad y_0y_2 - y_1^2 = 0$

$\vdots$

$v_{d,n} : \mathbb{P}^n \longrightarrow \mathbb{P}^N$ $\quad$ Veronese map given by the mon. of deg d

$d=2, n=2$ $\quad v_{2,2} : \mathbb{P}^2 \longrightarrow \mathbb{P}^5$

$[x_0, x_1, x_2] \longrightarrow [x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2]$

Questions: what is the image? Is it closed? Irreducible?

$V_{2,2} = v_{2,2}(\mathbb{P}^2)$ $\qquad v_{2,2} : \mathbb{P}^2 \longrightarrow V_{2,2}$

Is $v_{2,2}$ invertible? $\qquad V_{2,2} \simeq \mathbb{P}^2$