k infinite

$n, d \geq 1$ $\quad \nu_{n,d} : \mathbb{P}^n_k \longrightarrow \mathbb{P}^N$, $\quad N = \binom{n+d}{d} - 1$

$[x_0, \dots, x_n] \longrightarrow [x_0^d, x_0^{d-1} x_1, \dots, x_0^{d-1} x_n, \dots \dots, x_n^d]$

Veronese map $\quad \nu_{n,d}(\mathbb{P}^n_k) = V_{n,d}$: irreducible

because $\nu_{n,d}$ is continuous (being regular) and $\mathbb{P}^n_k$ is irreducible.

Fix coordinates in $\mathbb{P}^N$: $\nu_{i_0 \dots i_n}$, with $i_0 + \dots + i_n = d$ and $i_0, \dots, i_n \geq 0$

Parametrically $\nu_{n,d}$ is given by

$* \left\{ \nu_{i_0 \dots i_n} = x_0^{i_0} x_1^{i_1} \dots x_n^{i_n} \right.$ $\quad \nu_{n,d} : [x_0 \dots x_n] \longrightarrow [\dots, \nu_{i_0 \dots i_n}, \dots]$

$\nu_{d0\dots0} = x_0^d$

$\nu_{d-1,1,0\dots0} = x_0^{d-1} x_1$

$\vdots$

$V_{n,d}$ is defined by binomial equations of degree 2

$$\begin{cases} v_{d0\ldots0} = x_0^d \\ v_{d-1,1,0\ldots0} = x_0^{d-1}x_1 \\ v_{d-1,0,1,0\ldots0} = x_0^{d-1}x_2 \\ \vdots \\ v_{i_0\ldots i_n} = x_0^{i_0}\cdots x_n^{i_n} \\ \vdots \\ v_{0\ldots0d} = x_n^d \end{cases}$$

$$\boxed{v_{i_0\ldots i_n} v_{j_0\ldots j_n} = v_{k_0\ldots k_n} v_{l_0\ldots l_n}}$$

$$x_0^{i_0}\cdots x_n^{i_n} x_0^{j_0}\cdots x_n^{j_n} = x_0^{i_0+j_0}\cdots x_n^{i_n+j_n}$$

if $i_0+j_0 = k_0+l_0$
$\vdots$
$i_n+j_n = k_n+l_n$

$\Rightarrow$ any point in $v_{n,d}(\mathbb{P}^n)$ satisfies all possible equations of this form

$n=d=2$ $\quad v_{22} : \mathbb{P}^2 \longrightarrow \mathbb{P}^5$

$[x_0\, x_1\, x_2] \longrightarrow [v_{200}, v_{110}, v_{101}, v_{020}, v_{011}, v_{002}]$

$$\begin{cases} v_{200} = x_0^2 \\ v_{110} = x_0x_1 \\ \vdots \\ v_{002} = x_2^2 \end{cases}$$

$x_0^2x_1^2 = (x_0x_1)^2$

$(x_0^2)x_1x_2 = (x_0x_1)(x_0x_2)$

$$(*)\begin{cases} v_{200}v_{020} = (v_{110})^2 \\ v_{200}v_{002} = (v_{101})^2 \\ v_{020}v_{002} = (v_{011})^2 \\ v_{200}v_{011} = v_{110}v_{101} \\ v_{020}v_{101} = v_{110}v_{011} \\ v_{002}v_{110} = v_{101}v_{011} \end{cases}$$

Let $[v] = [\ldots, v_{ijk}, \ldots]$ satisfying,
we look for $*[x_0\, x_1\, x_2]$ s.t. $v_{22}[x] = [v]$

Rmk $[v]$ satisfies$(*)$ $\Rightarrow$ $v_{200}, v_{020}, v_{002}$ one is $\neq 0$

Ass. $v_{200} \neq 0$ $\quad [v_{200}, v_{110}, \ldots] = [v_{200}^2, v_{200}v_{110}, v_{200}v_{101}, v_{200}v_{020}, v_{200}v_{011}, v_{200}v_{002}]$

$= [x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2]$

Guess. $\quad x_0 = v_{200}$
$\quad x_1 = v_{110}$
$\quad x_2 = v_{101}$

It is a good guess if
$v_{200}v_{020} = (v_{110})^2$
$v_{200}v_{011} = v_{110}v_{101}$
$(v_{101})^2 = v_{200}v_{002}$

$[v] = v_{22}[v_{200}, v_{110}, v_{101}]$

If $v_{020} \neq 0$ $\qquad [v] = v_{22}[v_{110}, v_{020}, v_{011}]$

$v_{22}(\mathbb{P}^2) \subseteq \mathbb{P}^5$ is closed projective variety

In gen. $v_{d0\ldots0}, v_{0d0\ldots}, \ldots, v_{0\ldots0d}$ at least one is $\neq 0$.

$v_{200} \neq 0$, $v_{020} \neq 0$, $v_{002} \neq 0$ $\qquad [v] = v_{22}$(any row of the matrix)

$$\begin{pmatrix} v_{200} & v_{110} & v_{101} \\ v_{110} & \underline{v_{020}} & v_{011} \\ v_{101} & v_{011} & \underline{v_{002}} \end{pmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \end{matrix}$$

The equations are the 2×2 minors

the rank is 1 $\qquad V_{2,2}$

Bijection: $\{3\times 3$ symmetric matrices of rk $1\}$

$\updownarrow$

$\{$points of $V_{22}\}$

V_{22} : the Veronese surface

$\mathbb{P}^2 \simeq V_{22}$

$v_{22} : \mathbb{P}^2 \longrightarrow V_{22}$

$v_{22}^{-1} : V_{22} \longrightarrow \mathbb{P}^2 \qquad [v] \longrightarrow \begin{cases} [v_{200}, v_{110}, v_{101}] \\ [-, v_{020}, -] \\ [-\ -\ v_{002}] \end{cases}$ regular

$v_{n,d} : \mathbb{P}^n \simeq V_{n,d}$

$v_{n,d}$: d-tuple embedding

$\shortparallel$

isomorphism with the image

$V_{n,d}$ $\quad H \subseteq \mathbb{P}^N$ hyperplane $\underline{\sum a_{i_0\ldots i_n} v_{i_0\ldots i_n} = 0}$

$V_{n,d} \cap H$ hyperplane section of $V_{n,d}$

$$\begin{aligned}
&\parallel\\
&\{[v]\in\mathbb{P}^N \mid \exists\, [x]\in\mathbb{P}^n \text{ s.t. } v_{i_0\ldots i_n} = x_0^{i_0}\cdots x_n^{i_n} \text{ and } \underline{\sum a_{i_0\ldots i_n} v_{i_0\ldots i_n} = 0}\}.\\
&= \{[v]\in\mathbb{P}^N \mid \text{— — - - - and } \sum a_{i_0\ldots i_n} x_0^{i_0}\cdots x_n^{i_n} = 0\}\\
&= \nu_{n,d}(X) \quad \text{where } X = \{[x] \mid \sum a_{i_0\ldots i_n} x_0^{i_0}\cdots x_n^{i_n} = 0\}
\end{aligned}$$

hypersurface in $\mathbb{P}^n$ def. by an equation of deg d

$$\mathbb{P}(K[x_0 \ldots x_n]_d) = K[x_0 \ldots x_n]_d \setminus \{0\} / \sim$$

The hyperplane sections of $V_{n,d}$ are in bijection with $\mathbb{P}(K[x_0 \ldots x_n]_d)$: it can be interpreted as "the space of hypersurfaces of deg d"

$F = F_1^{r_1}\cdots F_s^{r_s}$ $\qquad r_1 \deg F_1 + \cdots + r_s \deg F_s = d$

$V_p(F) = V_p(F_1) \cup \cdots \cup V_p(F_s)$ hypersurf. of deg $\deg F_1 + \cdots + \deg F_s \leq d$

Interpretation of elements of $\mathbb{P}$ as linear combinations:

$$r_1 V_p(F_1) + \cdots + r_s V_p(F_s) :$$

cycle of dim $n-1$ or divisor in $\mathbb{P}^n$ of deg d

Bijection between {hyperplane sections of $V_{n,d}$}
and {hypersurfaces of deg d}

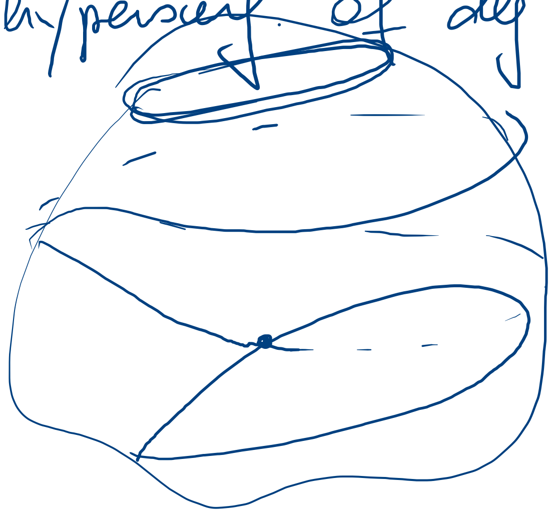
"Linearization process"

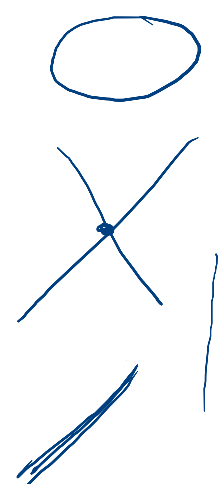
$$\nu_{22} : \mathbb{P}^2 \longrightarrow V_{2,2} \subset \mathbb{P}^5$$

{hyperplane sections of $V_{2,2}$}

$$\updownarrow \nu_{22}$$

{hypersurf. of deg 2 in $\mathbb{P}^2$} = {conics}





F irred

$F = L_1 L_2$

$F = L^2$

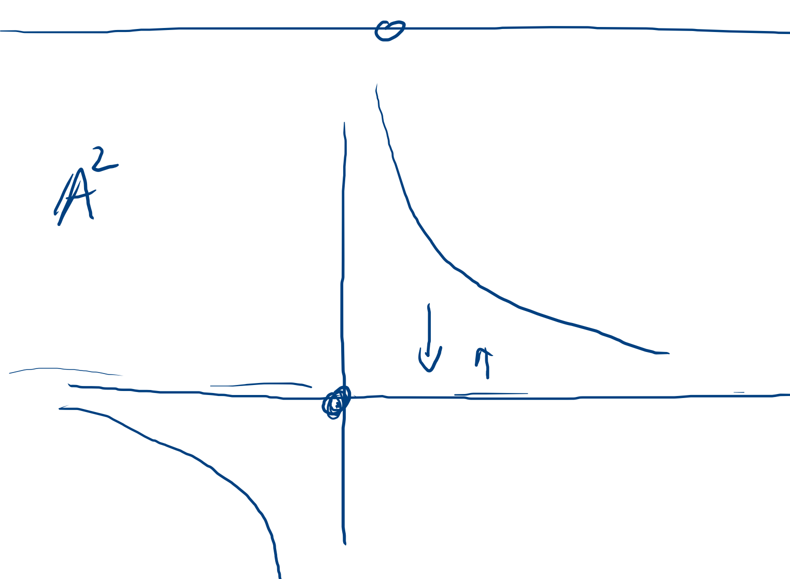
$X \subset \mathbb{A}^n$ affine variety $\quad I(X) = \langle G_1, \dots, G_r \rangle$

$X \setminus V(F) = X \cap (\mathbb{A}^n \setminus V(F))$ quasi-affine variety

locally closed

$X \setminus V(F)$ is isomorphic to an affine variety

Ex. $\mathbb{A}^1 \setminus V(x) = \mathbb{A}^1 \setminus \{0\}$



$V(xy-1)$

$\mathbb{A}^1 \setminus \{0\} \longleftrightarrow V(xy-1)$

$\mathbb{A}^1 \longleftrightarrow V(y)$

$xy - 1 = 0 \qquad \boxed{y} = \frac{1}{x}$

$$\mathbb{A}^{n+1}\ (x_1, \dots, x_n, x_{n+1})$$
$$\cup$$
$$Y = V(\underline{G_1, \dots, G_r}, x_{n+1}F - 1) = \underbrace{V(G_1 \dots G_r)}_{\substack{\text{cylinder} \\ \text{over } X}} \cap \underbrace{V(x_{n+1}F - 1)}_{x_{n+1} = \frac{1}{F}}$$

$$\boxed{Y \simeq X \setminus V(F) = X_F}$$

$\pi : Y \longrightarrow X_F$ regular

$(x_1 \dots x_{n+1}) \longrightarrow (x_1 \dots x_n)$

$x_{n+1}F(x_1 \dots x_n) - 1 = 0$

$F(x_1 \dots x_n) \neq 0$

$\varphi : X_F \longrightarrow Y$

$(x_1, \dots x_n) \longrightarrow \left(\underbrace{x_1 \dots x_n}, \underline{\frac{1}{F(x_1 \dots x_n)}}\right)$ regular

$F(x_1 \dots x_n) \neq 0$

φ, π are each one inverse $\Rightarrow$ isomorphism

Def. affine variety = locally closed in $\mathbb{P}^n$ which is isomorphic to a closed subset of some $\mathbb{A}^n_K$.

$X_F = X - V(F)$ is an affine variety

$$\mathcal{O}(X_F) \simeq \mathcal{O}(Y) = \frac{K[x_1 \dots x_{n+1}]}{\underset{\substack{\| \\ I(X) + \langle x_{n+1}F - 1 \rangle}}{I(Y)}} = \frac{\frac{K[x_1 \dots x_{n+1}]}{(G_1 \dots G_r)}}{\langle x_{n+1}F - 1 \rangle} =$$

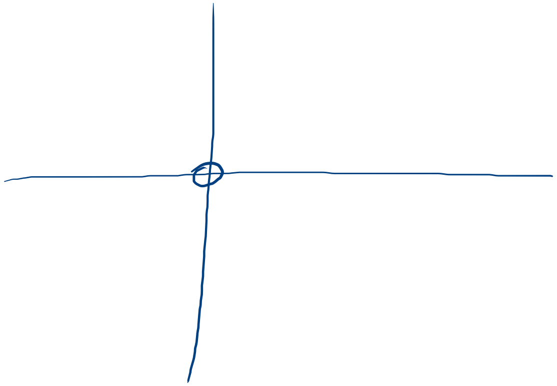
$$= \underbrace{K[t_1 \dots t_n}_{\text{coord. fns on } X}, \frac{1}{f}] = \mathcal{O}(X)\left[\frac{1}{f}\right] = \mathcal{O}(X)_f = S^{-1}\mathcal{O}(X)$$

f regular fn. on X induced by F

$$S = \{1, f, f^2, \dots, f^r, \dots\}$$

Example of a quasi-affine variety not affine

$X = \mathbb{A}^2 \setminus V(x,y)$ is not isom. to any closed subset of $\mathbb{A}^n$, $\forall n$



Idea: 1) $\mathcal{O}(X) = \mathcal{O}(\mathbb{A}^2)$

2) Over K alg. closed, the relative version of Nullstellensatz: if $\alpha \subsetneq \mathcal{O}(Y)$, if Y is an affine variety then $V(\alpha) \neq \emptyset$

If $X \simeq Y$ closed in $\mathbb{A}^n$ $\Longrightarrow$ $\mathcal{O}(X) \underset{u}{\simeq} \mathcal{O}(Y)$

proper ideals correspond to proper ideals

$\alpha = \langle x,y \rangle \subsetneq \mathcal{O}(X)$ $\qquad$ $V(x,y) \subseteq X$, $\quad V(x,y) = \emptyset$

$u(\alpha)$ proper ideal in $\mathcal{O}(Y)$. $\qquad$ $V(u(\alpha)) \neq \emptyset$ in Y

$u^\# : X \simeq Y$ $\qquad$ $u^\#(V(x,y)) = V(u(\alpha))$