Lezione del 30 Aprile 2021

Presenti in aula (su autodichiarazione)

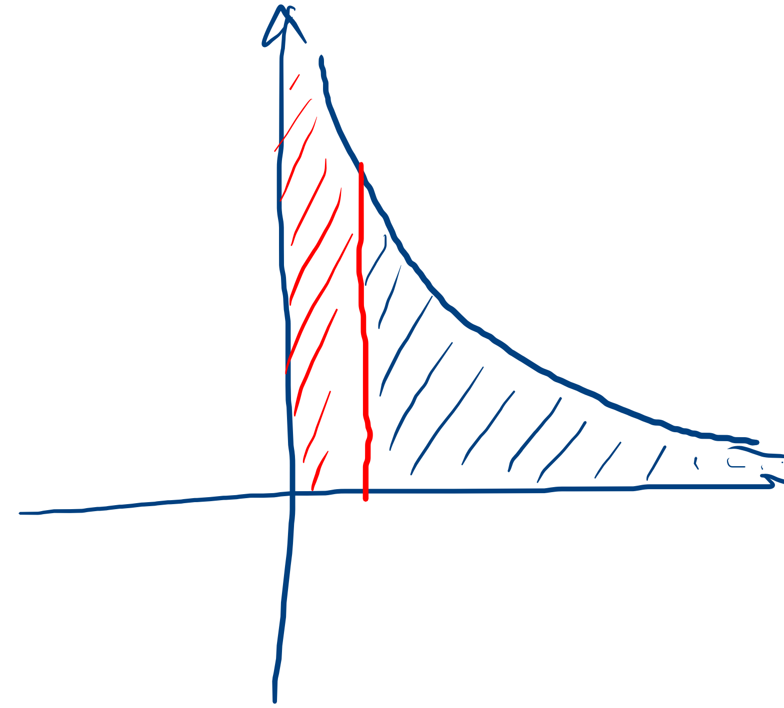
SM 6000695

SM 6000699

$$\lim_{t \to 0^+} \int_s^1 \frac{1}{x^2}\,dx = +\infty$$

$$\lim_{t \to +\infty} \int_1^t \frac{1}{x^2}\,dx = \dots < +\infty$$

$$\lim_{t \to +\infty} \int_1^t \frac{1}{x}\,dx = +\infty$$



$$\lim_{t\to+\infty} \int_1^t \frac{1}{x^p}\,dx$$

se $p=1$ non converge (diverge a $+\infty$)

se $p=2$ converge

$p \neq 1$

$$\int \frac{1}{x^p}\,dx = \left\{ \frac{1}{-p+1} x^{-p+1} + C \right\}$$

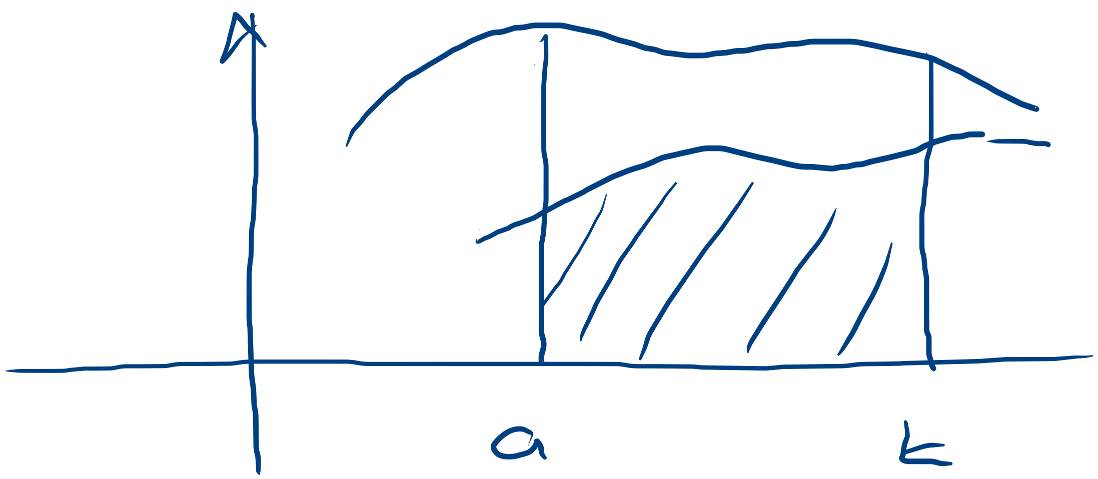
$$= \int x^{-p}\,dx$$

Siano f e g due funzioni continue positive in $[a, +\infty)$ e tali che

$$0 < f(x) \leq g(x) \qquad \forall x \in [a, +\infty)$$

$t > a$

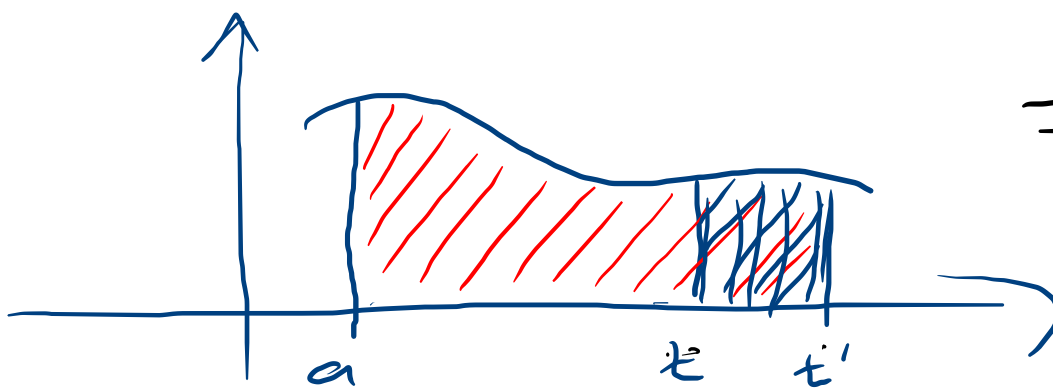
$$\int_a^t f(x)\,dx \leq \int_a^t g(x)\,dx \qquad \forall t$$



$$F(t) = \int_a^t \underline{f(x)\,dx} \qquad t \in [a, +\infty)$$

$$G(t) = \int_a^t \underline{g(x)\,dx}$$

$$F(a) = G(a) = 0 \qquad F(t) \le G(t) \quad \forall\, t \in [a, +\infty)$$



$$F(t') = F(t) + \underbrace{\int_t^{t'} f(x)\,dx}_{>0}$$

essendo $f > 0$

In altre parole essendo f positiva in $[a, +\infty)$ la funzione integrale $F(t) = \int_a^t f(x)\,dx$ è monotona crescente.

Dal Teorema fondamentale del Calcolo Integrale segue inoltre che F è derivabile

$$F'(x) = f(x)$$

(ossia F è una primitiva di f)

Pertanto se $0 < f(x) \leq g(x)$ $x \in [a, +\infty)$

con f e g funzioni continue in $[a, +\infty)$

segue che

$$0 < \int_a^t f(x)\,dx \leq \int_a^t g(x)\,dx$$

e quindi se

$$\lim_{t \to +\infty} \int_a^t g(x)\,dx \text{ esiste finito}$$

allora

$$F(t) = \int_a^t f(x)\,dx$$

è monotona crescente e limitata in $[a, +\infty)$

quindi esiste

$$\lim_{t \to +\infty} F(t) = \lim_{t \to +\infty} \int_a^t f(x)\,dx$$

Infatti da

$$0 < \underbrace{\int_a^t f(x)\,dx}_{\text{essendo monotono poiché } f \text{ è positiva}} \leq \underbrace{\int_a^t g(x)\,dx}_{} \xrightarrow{t \to +\infty} L$$

essendo monotono

poiché f è positiva

esiste certamente il limite

per $t \to +\infty$. Ossia $\lim_{t \to +\infty} \int_a^t f(x)\,dx = L' \leq$

$\leq L$.

Allo stesso modo se, nelle ipotesi di cui sopra, risulta

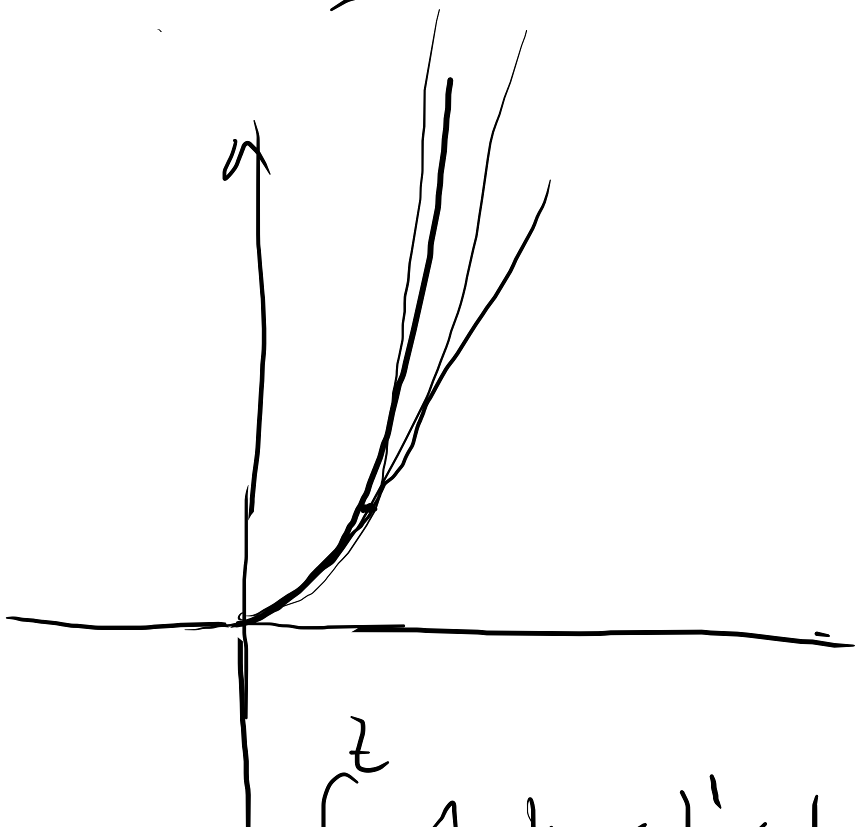
$$\lim_{t \to +\infty} \int_a^t f(x)\,dx$$

è $+\infty$ allora $G(t) = \int_a^t g(x)\,dx$

è divergente per $t \to +\infty$

Oss

$x^p \geq x^2$ se $x \geq 1$



$x \in [1, +\infty)$

$$\underbrace{\frac{1}{x^p}}_{= f(x)} \leq \underbrace{\frac{1}{x^2}}_{= g(x)}$$

$$= \int_1^{+\infty} g(x) = L < +\infty$$

$$\forall p \quad \lim_{t \to +\infty} \int_1^t \frac{1}{x^p}\,dx \leq L' < L \quad \Longleftarrow \quad \lim_{t \to +\infty} \int_1^t \frac{1}{x^2}\,dx$$

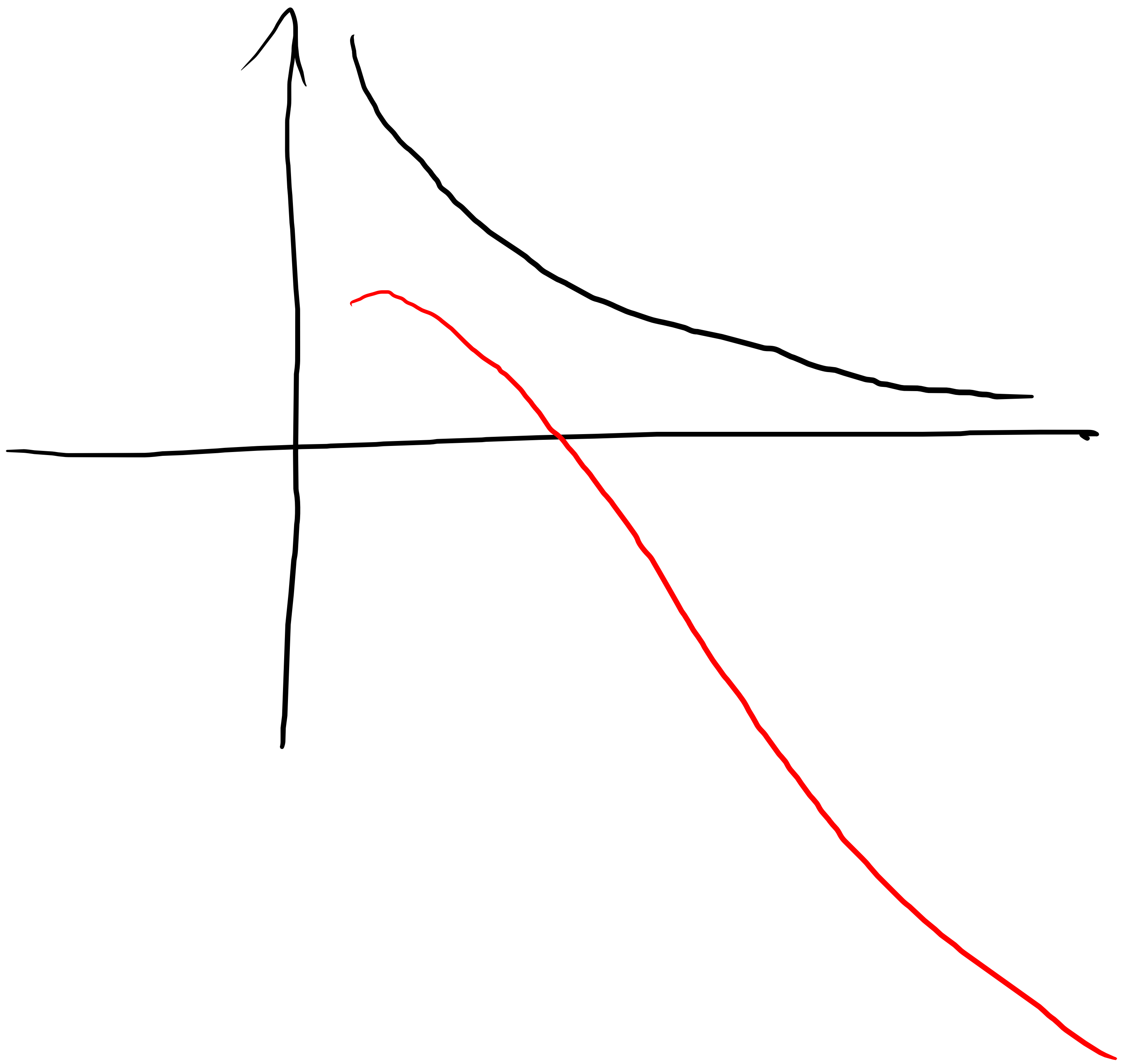
$x^p \leq x \qquad p \leq 1 \qquad x \geq 1$

$\Downarrow$

$\frac{1}{x} \leq \frac{1}{x^p}$

Essendo $\int_1^{+\infty} \frac{1}{x}\,dx = +\infty$

segue che $\lim_{t \to +\infty} \int_1^t \frac{1}{x^p}\,dx$ ~~co~~ diverge!



Esercizio

$$\int_0^{+\infty} x \cdot e^{-\sqrt{1+x^2}}\, dx = \lim_{t \to +\infty} \int_0^t x \cdot e^{-\sqrt{1+x^2}}\, dx$$

$$x \cdot e^{-\sqrt{1+x^2}} = \frac{x}{e^{\sqrt{1+x^2}}} > 0 \qquad \underline{x > 0}$$

Sostituzione

$$\sqrt{1+x^2} = u \qquad 1+x^2 = u^2$$

$$2x\,dx = 2u\,du$$

$$x\,dx = u\,du$$

$$\int x\, e^{-\sqrt{1+x^2}}\,dx = \int u\, e^{-u}\,du = \int \frac{u}{e^u}\,du$$

per parti

$$\int \underset{f}{u}\,\underset{g'}{e^{-u}}\,du = u(-e^{-u}) - \int 1\cdot(-e^{-u})\,du = -u\,e^{-u} + \int e^{-u}\,du$$

$$g(u) = -e^{-u}$$

$$\int u e^{-u}\, du = \left\{ -u e^{-u} - e^{-u} + C \right\} \qquad u = \sqrt{1+x^2}$$

$$\int x \cdot e^{-\sqrt{1+x^2}}\, dx = \left\{ -\sqrt{1+x^2} \cdot e^{-\sqrt{1+x^2}} - e^{-\sqrt{1+x^2}} + C \right\}$$

$$\lim_{t \to +\infty} \int_0^t x \cdot e^{-\sqrt{1+x^2}}\, dx = \lim_{t \to +\infty} \left[-\sqrt{1+x^2}\, e^{-\sqrt{1+x^2}} - e^{-\sqrt{1+x^2}} \right]_0^t$$

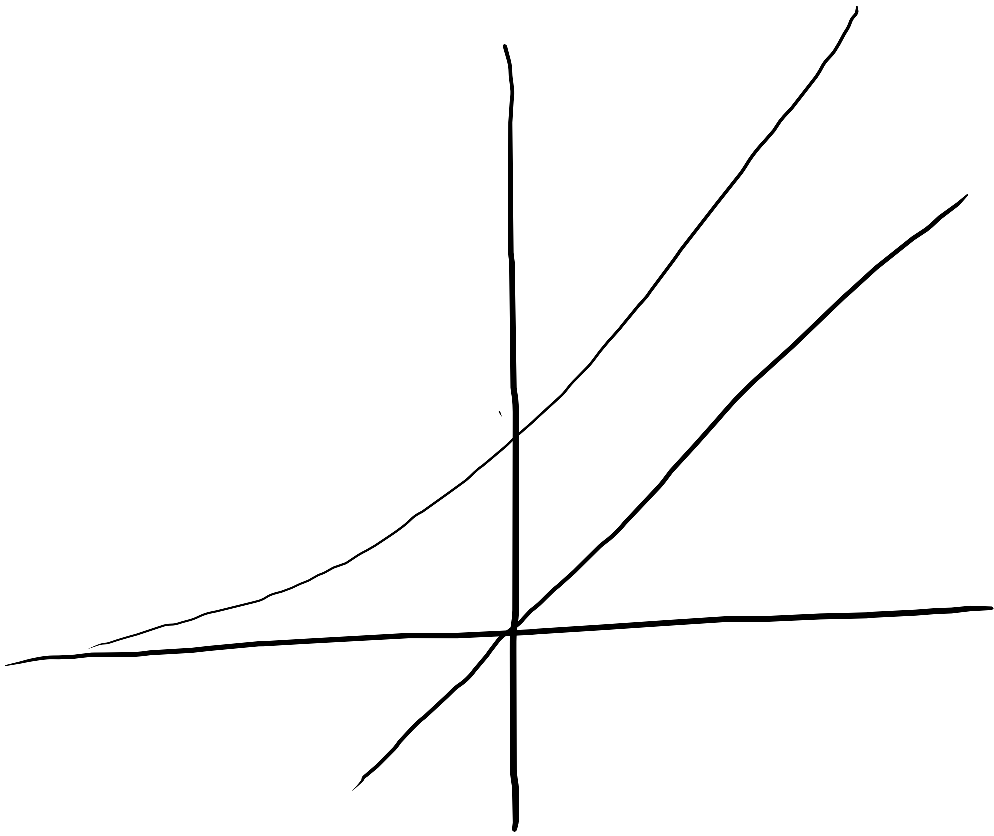
$$= \lim_{t\to+\infty}\left[-\sqrt{1+t^2}\, e^{-\sqrt{1+t^2}} - e^{-\sqrt{1+t^2}} + \sqrt{1}\cdot e^{-1} + e^{-1}\right]$$

$$= \lim_{t\to+\infty}\left[\frac{1}{e^{\sqrt{1+t^2}}}\left(-1-\sqrt{1+t^2}\right) + \underbrace{\frac{1}{e}+\frac{1}{e}}_{=\frac{2}{e}}\right] = \frac{2}{e}$$

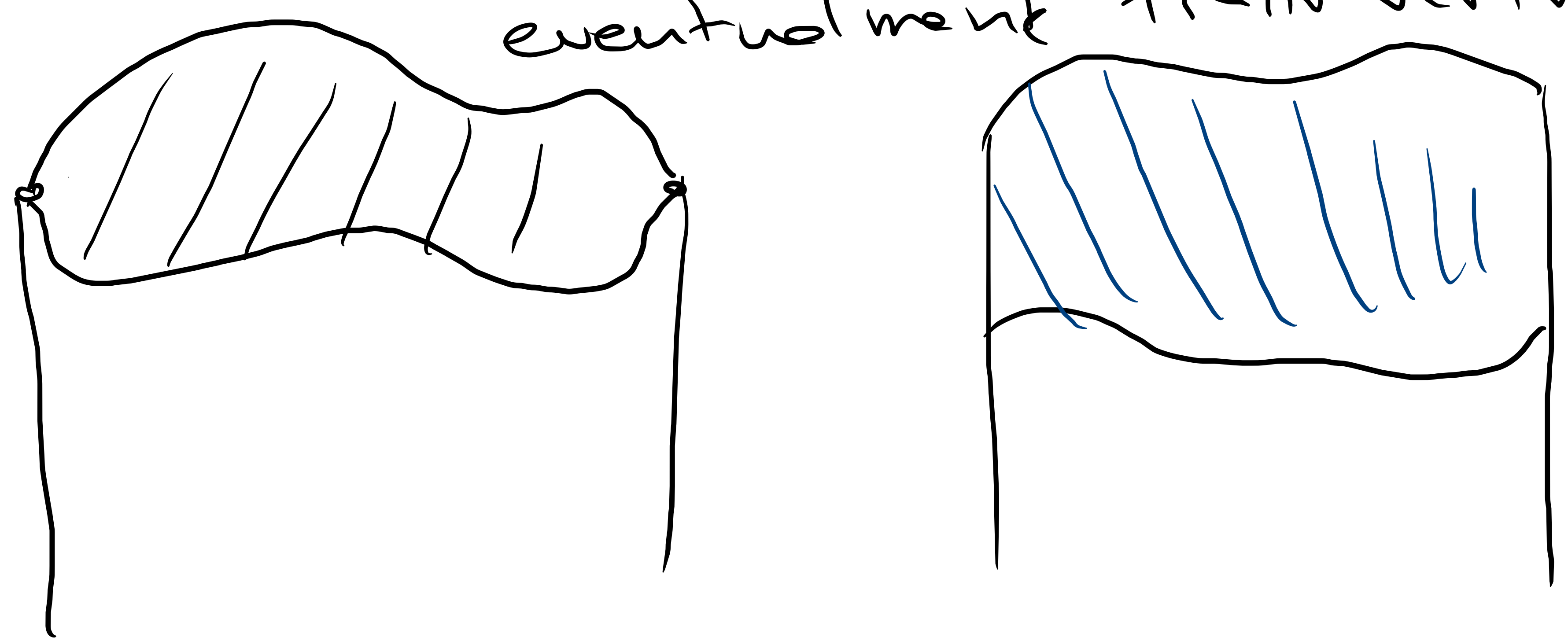
$$\lim_{s\to+\infty}\frac{-s}{e^s} \longrightarrow 0$$

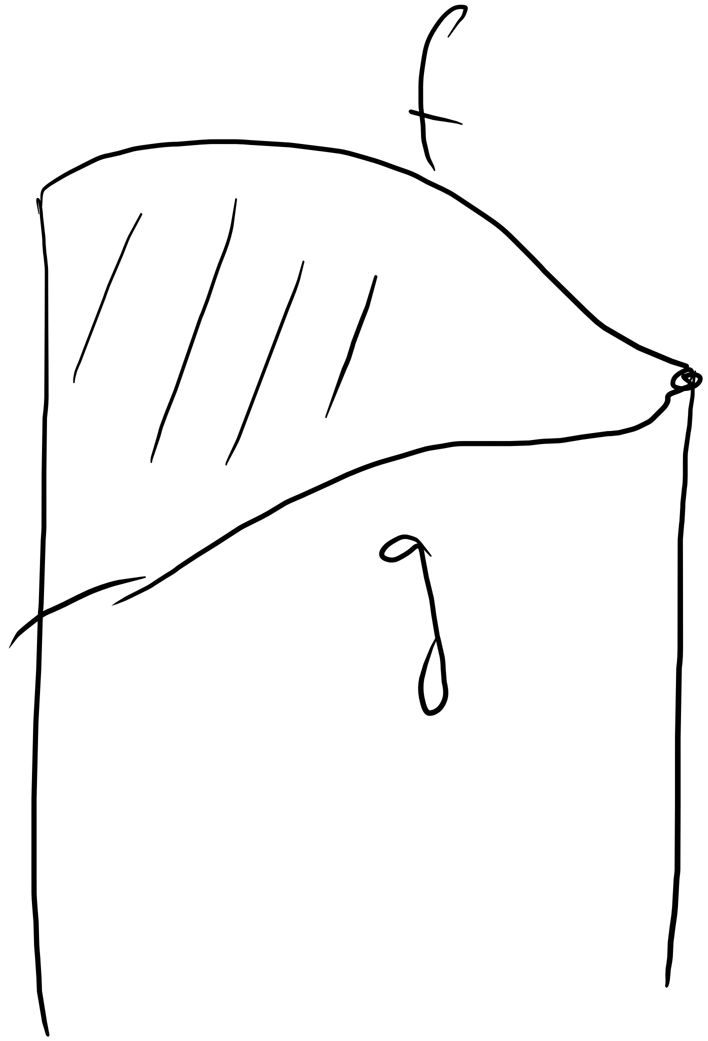
$$\int u e^{-u} \, du = \int \frac{u}{e^u} \, du$$

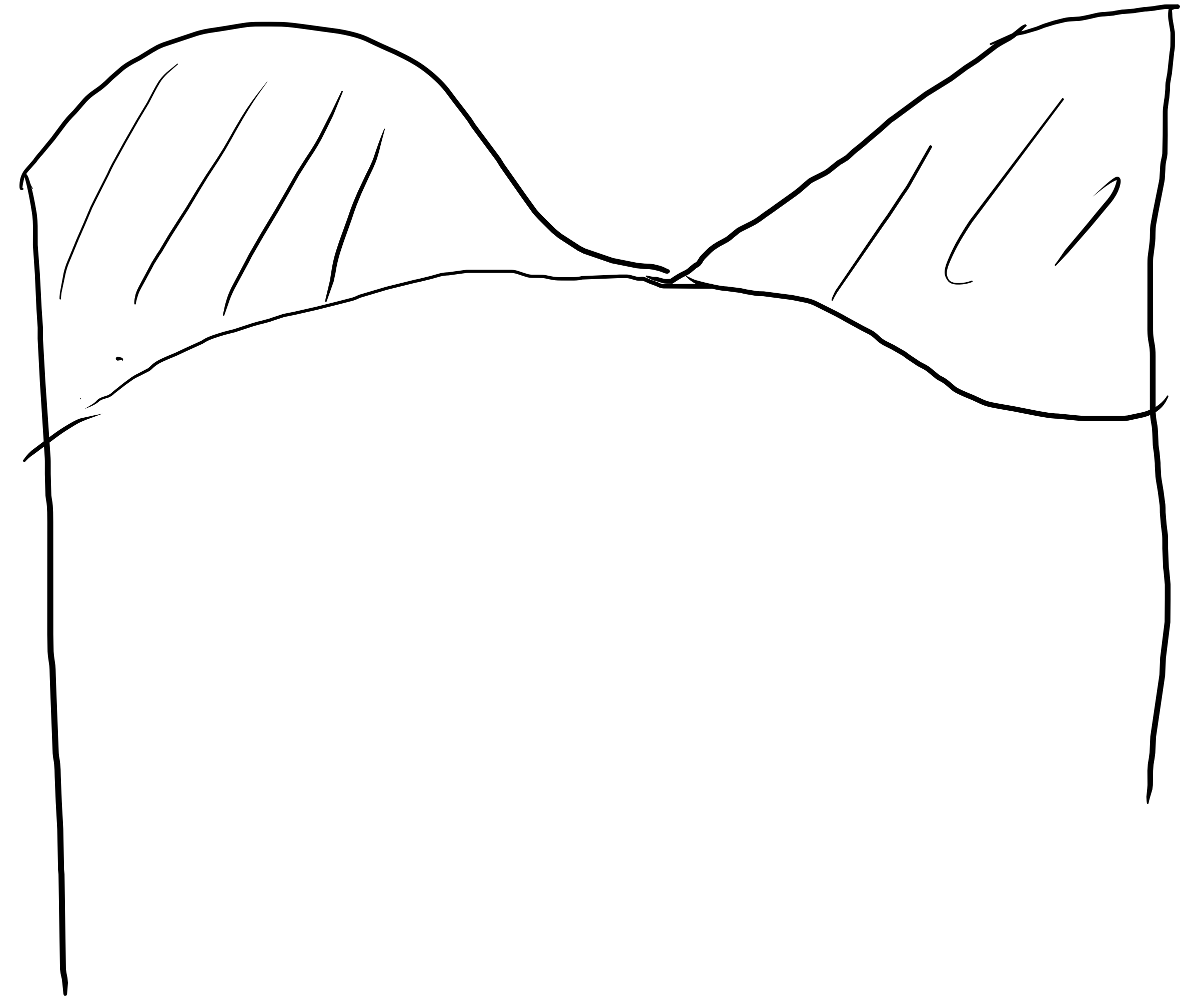
$$e^u \geq u$$



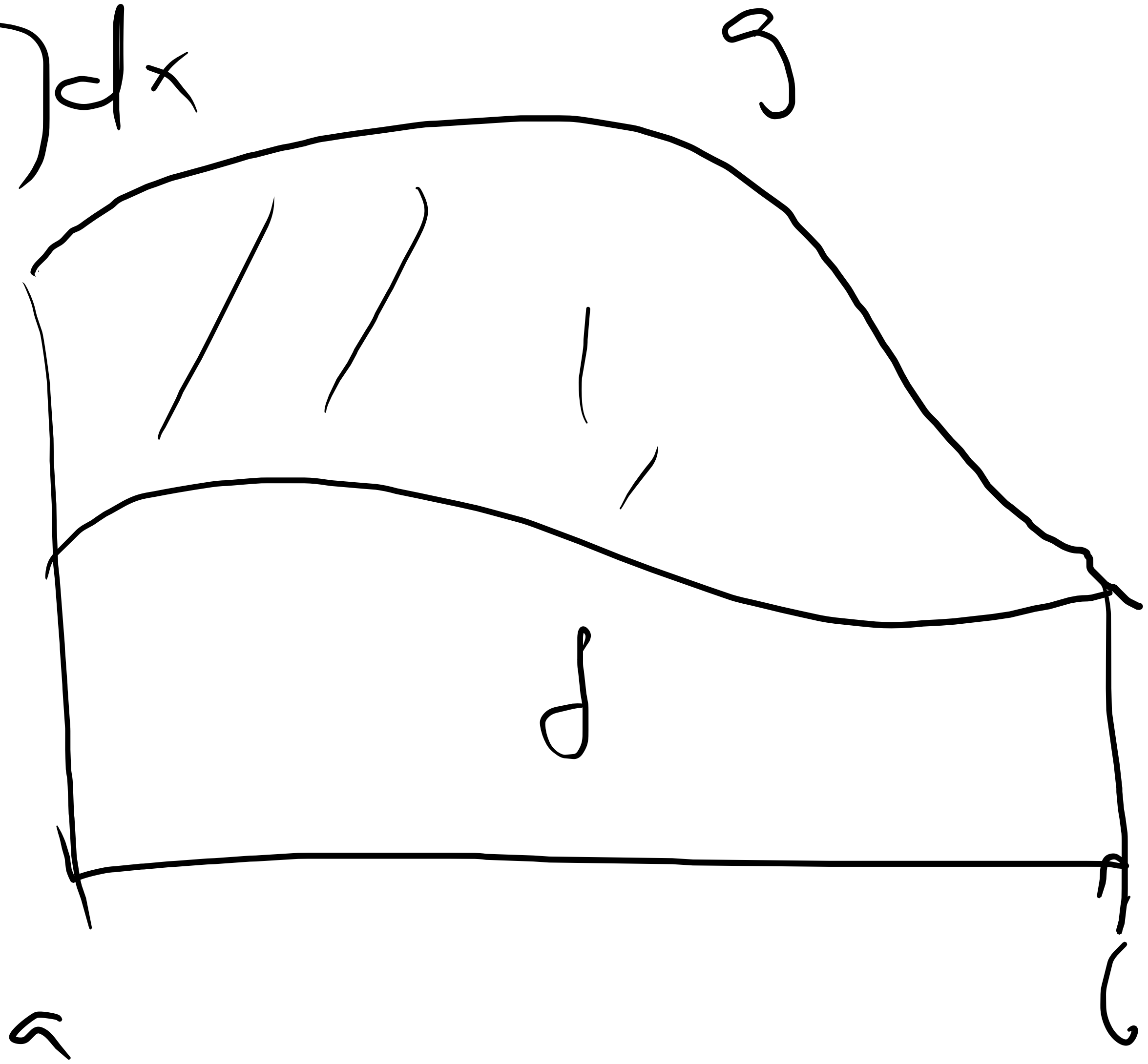
Se R è una regione piana delimitata dal grafico di due funzioni f e g (con f < g)
(ed eventualmente tratti verticali)




f
g

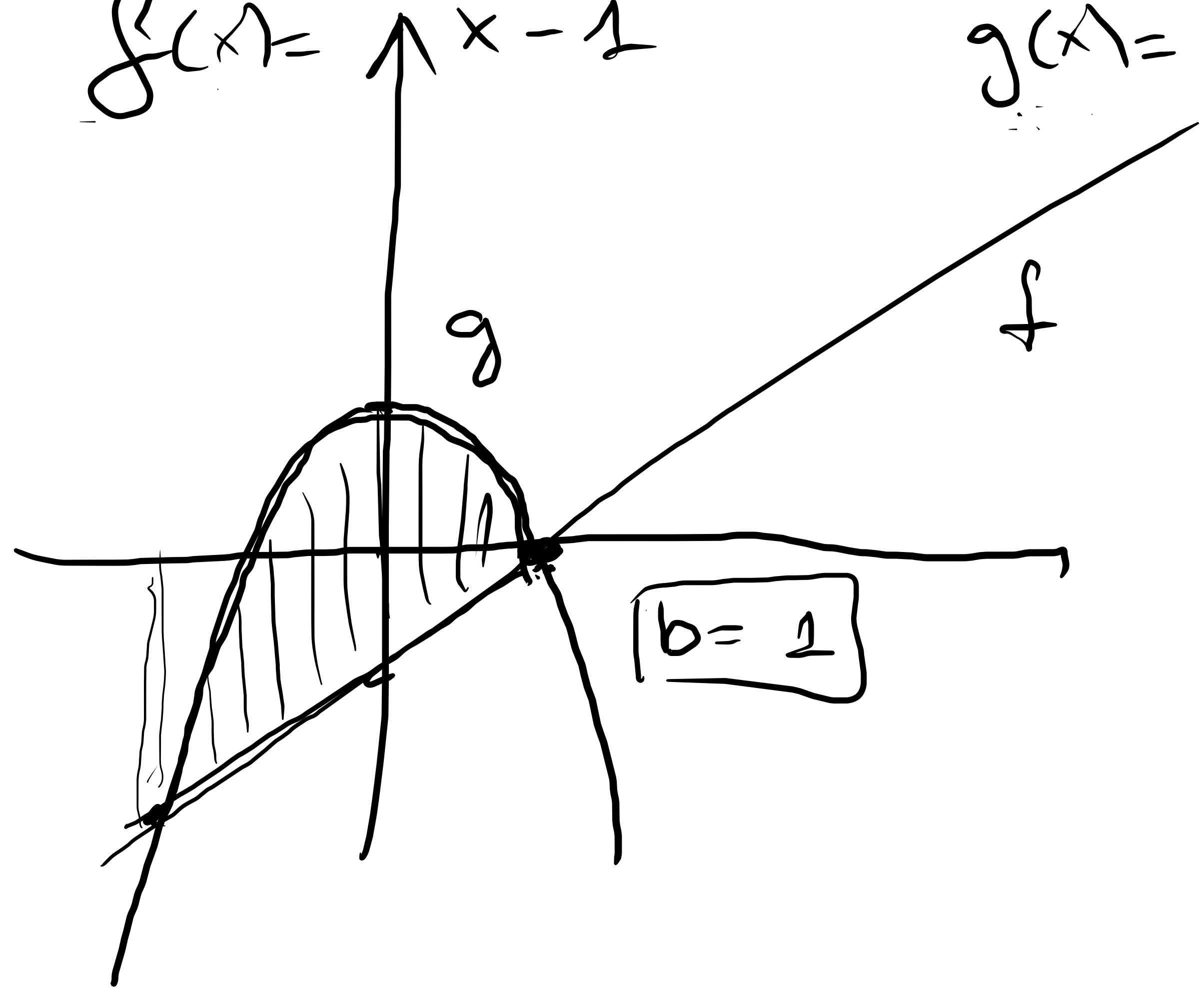


$$\int_a^b \left(g(x) - f(x)\right) dx$$



Es Determinare l'area della regione di piano compresa tra le curve di equazioni $y = x - 1$ e $y = -x^2+1$

$f(x) = x - 1$ $\qquad$ $g(x) = -x^2+1$



$$\begin{cases} y = -x^2 + 1 \\ y = x - 1 \end{cases} \longrightarrow x - 1 = -x^2 + 1$$

$$x^2 + x - 2 = 0$$

$$x^2 + x - 2 = (x-1)\cdot(x+2)$$

$$\boxed{\begin{array}{l} x_1 = 1 \\ x_2 = -2 \end{array}}$$

$$\Delta = 1 + 8 = 9$$

$$x_{1,2} = \frac{-1 \mp \sqrt{9}}{2} =$$

$$= \begin{cases} -2 \\ 1 \end{cases}$$

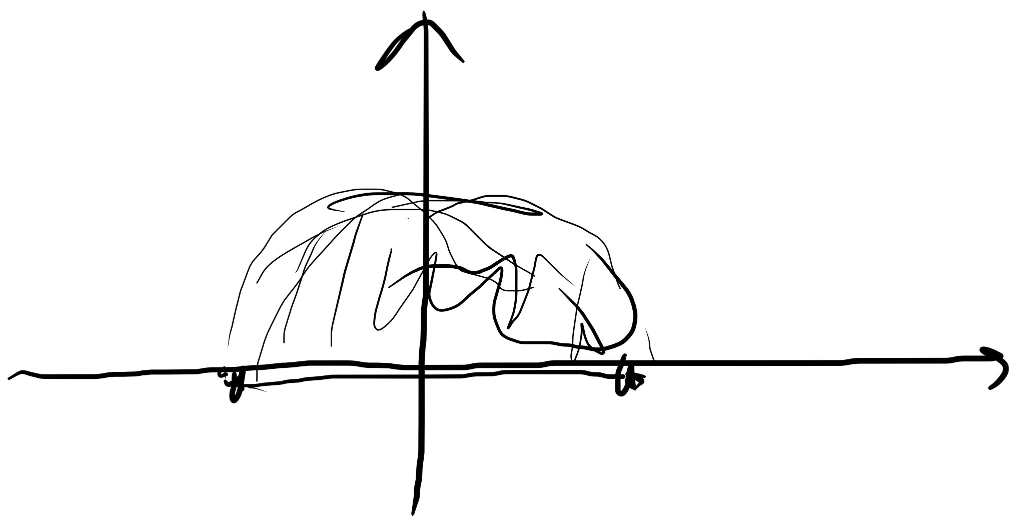
$$\int_{-2}^{1} \overset{g}{\left(-x^2+1\right)} - \overset{f}{\left(x-1\right)} \, dx =$$

$$= \int_{-2}^{1} \left(-x^2 - x + 2\right) dx = \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x + C\right]_{-2}^{1} =$$

$$= \left(-\frac{1}{3} - \frac{1}{2} + 2 + \left(-\frac{8}{3}\right) + 2 + 4\right)$$

$$= 5 - \frac{1}{2} = \frac{9}{2} \checkmark$$

Es Determinare l'area della regione di piano delimitata dall'asse x e dalla curva grafico di $g(x) = \sqrt{1-x^2}$



$$\int_{-1}^{1} \sqrt{1-x^2}\,dx$$

$\mathrm{dom}\, g = [-1, 1] =$

$= \{ x : -1 \le x \le 1 \}$

$g(-1) = g(1) = 0$

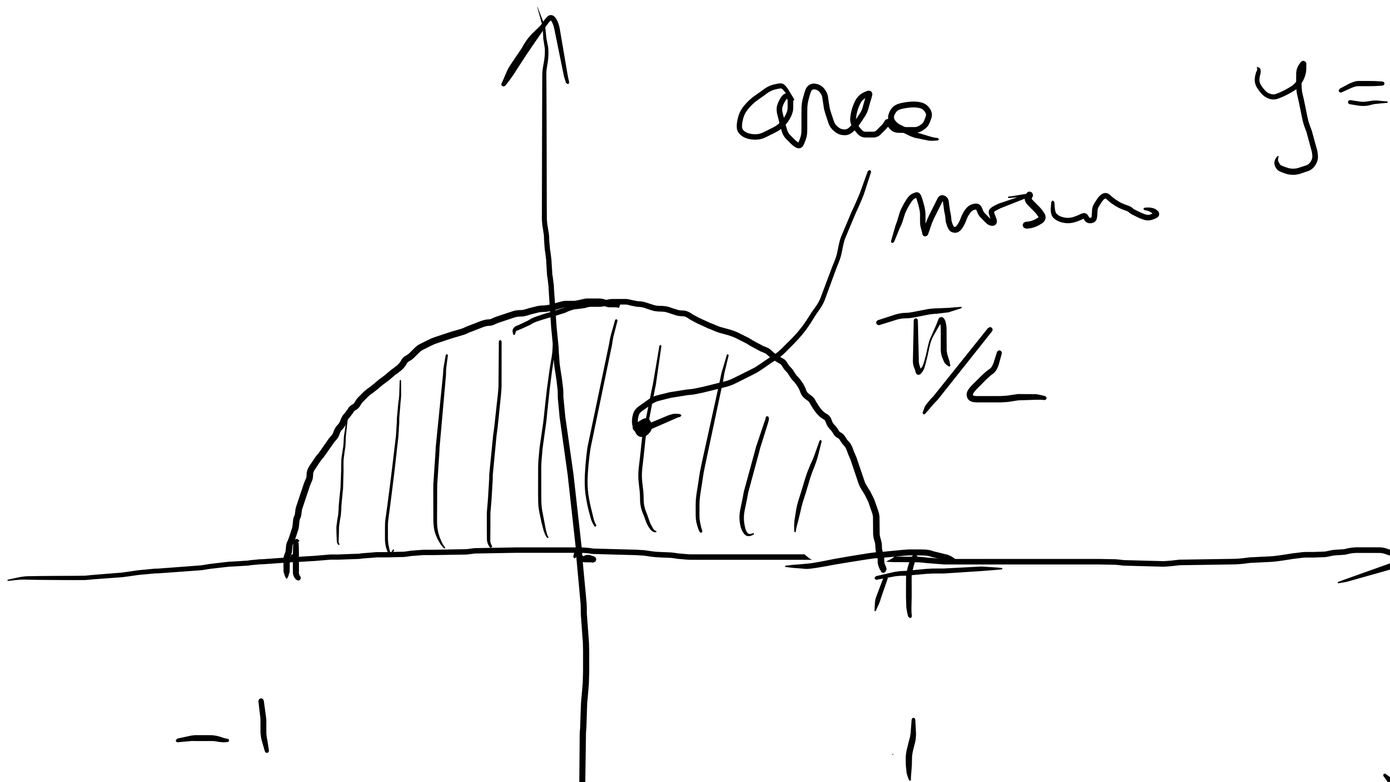
$$\int \sqrt{1-x^2}\,dx = \left\{ \frac{t - \operatorname{sen} t \cos t}{2} + C \right\}$$

$$\boxed{x = \cos t} \qquad t = \arccos x$$

L'intervallo $[-1, 1]$ diventa

l'intervallo $[-\pi, 0]$

$$\int_{-1}^{1} \sqrt{1-x^2}\,dx \overset{x=\cos t}{=} \left[\frac{t - \operatorname{sen} t \cos t}{2} \right]_{-\pi}^{0} = 0 + \pi/2$$



$$y = \sqrt{1-x^2}$$

$$y^2 = 1-x^2$$

$$\boxed{x^2+y^2=1}$$

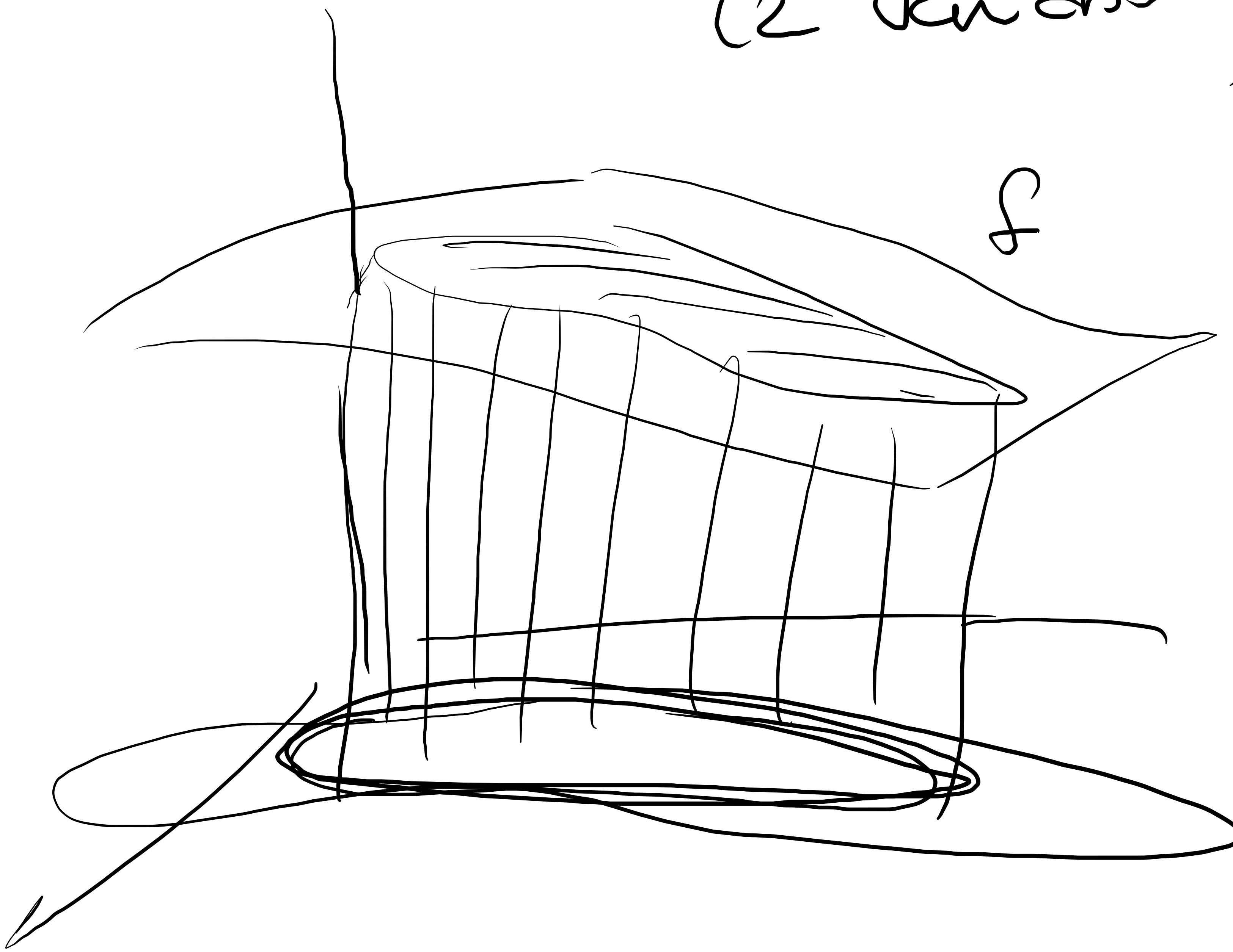
$$x^2+y^2=r^2$$

$$y=\sqrt{r^2-x^2}$$

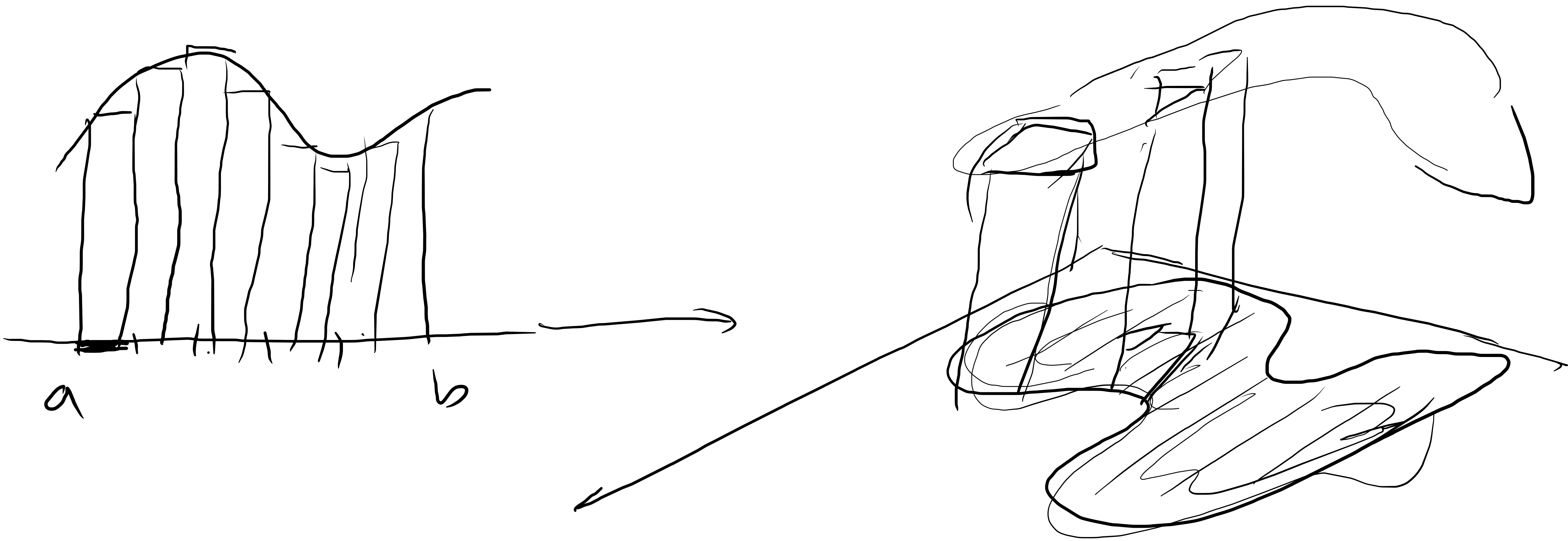
$$\boxed{\int_{-r}^{r}\sqrt{r^2-x^2}\,dx = \frac{\pi}{2}r^2}$$

Integrali di funzioni di più variabili

(2 variabili)



Provando a generalizzare la definizione
di integrale di Riemann per funzioni
di 2 o più variabili, appare naturale
immaginare di voler definire una
misura della porzione di spazio che
si trova sotto il grafico di una funzione di 2
o più variabili.


a
b

Presa una regione piana R limitata