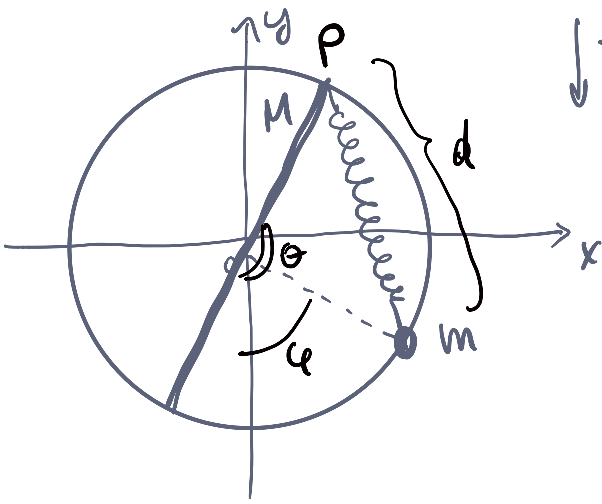
ES. 2 del 26.07.18



$\downarrow \bar{g}$

$$I = \frac{ML^2}{12} = M\frac{4R^2}{12} = \frac{MR^2}{3}$$

$$T_m = \frac{1}{2} m R^2 \dot{\varphi}^2 \qquad T_M = \frac{1}{2} I \dot{\theta}^2$$

$$V = mg y_m + \frac{1}{2} k d^2$$

$$y_m = -R\cos\varphi$$

$$d^2 = (x_P - x_m)^2 + (y_P - y_m)^2 = (R\sin\theta - R\sin\varphi)^2 +$$
$$+ (-R\cos\theta + R\cos\varphi)^2 =$$
$$= R^2(1 + 1 - 2\sin\theta\sin\varphi - 2\cos\theta\cos\varphi) =$$
$$= 2R^2(1 - \cos(\varphi - \theta))$$

$$L = T - U = \frac{1}{2} m R^2 \dot{\varphi}^2 + \frac{1}{2} I \dot{\theta}^2 + mgR\cos\varphi - kR^2(1 - \cos(\varphi - \theta))$$

2) Eq. del sist. e stab.

$$V(\theta, \varphi) = -mgR\cos\varphi + kR^2(1 - \overbrace{\cos(\varphi - \theta)}^{=\cos(\theta - \varphi)})$$

$$0 = \frac{\partial V}{\partial \theta} = kR^2 \sin(\theta - \varphi) \Rightarrow \sin(\theta - \varphi) = 0 \rightarrow \theta - \varphi = n\pi \quad n = 0, 1$$

$$0 = \frac{\partial V}{\partial \varphi} = mgR\sin\varphi + kR^2 \cancel{\sin(\varphi - \theta)} \rightarrow \varphi = \ell\pi \quad \ell = 0, 1$$

$(\theta^*, \varphi^*) = (0,0)\ ,\ (\pi, 0)\ ,\ (0, \pi)\ ,\ (\pi, \pi)$

$$B(\theta,\varphi) = \frac{\partial^2 V}{\partial q_i \partial q_j} = \begin{pmatrix} KR^2\cos(\theta-\varphi) & -KR^2\cos(\theta-\varphi) \\ -KR\cos(\theta-\varphi) & mgR\cos\varphi + kR^2\cos(\theta-\varphi) \end{pmatrix}$$

Pto eq. è stab. $\Leftrightarrow$ $B(\theta^*, \varphi^*)$ è def. pos.

$$B(0,0) = \begin{pmatrix} KR^2 & -KR^2 \\ -KR^2 & mgR + KR^2 \end{pmatrix} \to \det = KR^2(\cancel{KR^2} + mgR) - (\cancel{KR^2})^2 = mgKR^3 > 0$$

$\lambda_1 + \lambda_2 = \mathrm{Tr}\, B$ $\quad \mathrm{Tr} > 0 \Rightarrow$ DEF. POS.

$\to (0,0)$ è STAB

$$B(\pi,0) = \begin{pmatrix} -KR^2 & KR^2 \\ KR^2 & mgR - KR^2 \end{pmatrix}$$

Siccome $B_{11} < 0$, allora B non può essere def. pos.

[def. pos. $\Leftrightarrow$ $\forall u$ $\quad u^t B u > 0$. se scegliamo $u = \binom{1}{0}$ vediamo che $\exists$ vett. t.c. $u^t B u \not> 0$]

$\Rightarrow (\pi, 0)$ è INSTAB.

$B(0,\pi)$ non è def. $\Rightarrow$ $(0,\pi)$ è INSTAB.

$$B(\pi,\pi) = \begin{pmatrix} KR^2 & -KR^2 \\ -KR^2 & -mgR + KR^2 \end{pmatrix} \to \det = KR^2(-mgR + \cancel{KR^2}) - (\cancel{KR^2})^2 < 0$$

$\Rightarrow \lambda_1 \cdot \lambda_2 < 0 \Rightarrow$ matrice non è def. pos.

$\Rightarrow (\pi,\pi)$ è INSTAB.

3) $B = b(0,0) = \begin{pmatrix} kR^2 & -kR^2 \\ -kR^2 & kR^2 + mgR \end{pmatrix}$ $A = a(0,0) = \begin{pmatrix} I & \\ & mR^2 \end{pmatrix}$

$$\hat{L} = \underbrace{\frac{1}{2}(\dot\theta, \dot\varphi)\begin{pmatrix} I & \\ & mR^2 \end{pmatrix}\begin{pmatrix} \dot\theta \\ \dot\varphi \end{pmatrix}}_{\frac{1}{2}\dot{\bar q}\cdot A\dot{\bar q}} \underbrace{- \frac{1}{2}(\theta, \varphi)\begin{pmatrix} kR^2 & -kR^2 \\ -kR^2 & kR^2+mgR \end{pmatrix}\begin{pmatrix} \theta \\ \varphi \end{pmatrix}}_{-\frac{1}{2}\bar q\cdot B\bar q}$$

$$= \frac{1}{2} I\dot\theta^2 + \frac{1}{2} mR^2\dot\varphi^2 - \frac{1}{2} kR^2\theta^2 - \frac{1}{2}(kR^2+mgR)\varphi^2$$
$$- kR^2\theta\varphi$$

Eq. di Lagr. $A\ddot{\bar q} + B\bar q = 0$

$$\begin{pmatrix} I & \\ & mR^2 \end{pmatrix}\begin{pmatrix} \ddot\theta \\ \ddot\varphi \end{pmatrix} + \begin{pmatrix} kR^2 & -kR^2 \\ -kR^2 & kR^2+mgR \end{pmatrix}\begin{pmatrix} \theta \\ \varphi \end{pmatrix} = 0$$

4) $M = 3m \quad \to \quad I = \frac{MR^2}{3} = mR^2$

$\det(B - \lambda A) = 0$

$$\det\begin{pmatrix} kR^2 - \lambda mR^2 & -kR^2 \\ -kR^2 & kR^2 + mgR - \lambda mR^2 \end{pmatrix} =$$

$$= \det mR^2 \begin{pmatrix} \frac{k}{m} - \lambda & -\frac{k}{m} \\ -\frac{k}{m} & \frac{g}{R} + \frac{k}{m} - \lambda \end{pmatrix}$$

$$\to \left(\lambda - \frac{k}{m}\right)\left(\lambda - \frac{k}{m} - \frac{g}{R}\right) - \left(\frac{k}{m}\right)^2 = 0$$

$$\lambda^2 - \left(\frac{2k}{m} + \frac{g}{R}\right)\lambda + \frac{k}{m}\frac{g}{R} = 0$$

$$\underset{\omega_{1/2}^2}{\overset{\parallel}{\lambda_{1/2}}} = \left(\frac{k}{m} + \frac{g}{2R}\right) \pm \sqrt{\underbrace{\left(\frac{k}{m} + \frac{g}{2R}\right)^2 - \frac{k}{m}\frac{g}{R}}_{\left(\frac{k}{m}\right)^2 + \left(\frac{g}{2R}\right)^2}} > 0$$

5) Se pongo $g=0$, $\exists$ una cost. del moto?

$$L = T - U = \frac{1}{2}mR^2\dot{\varphi}^2 + \frac{1}{2}I\dot{\theta}^2 + \cancel{mgR\cos\varphi} - KR^2(1-\cos(\varphi-\theta))$$

$\downarrow$

Ora L è invariante sotto

$$\begin{aligned} \theta &\mapsto \theta + \alpha \\ \varphi &\mapsto \varphi + \alpha \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} \dot{\theta} &\mapsto \dot{\theta} \\ \dot{\varphi} &\mapsto \dot{\varphi} \end{aligned}$$

$\Rightarrow$ $\exists$ cost. del moto

Nöther

$$P = \sum_i \frac{\partial L}{\partial \dot{q}_i} \underset{\substack{\parallel \\ 1 \text{ in pto cas}}}{\frac{\partial \Phi_i}{\partial \alpha}} = \underbrace{I\dot{\theta} + mR^2\dot{\varphi}}_{\text{componente } M_z}$$

(invarianza per rotazioni attorno asse z)

Es. 2 del 19.02.20

$$V = \frac{\alpha}{m R(x,y,z)} - \frac{4\beta}{3m R(x,y,z)^{3/2}}$$

$R(x,y,z) = \sqrt{x^2+y^2+z^2}$

$\alpha, \beta > 0$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

1) Sist. inv. sotto rotazioni $\Rightarrow \bar{M}$ si conserva

$\bar{M} = \bar{r} \times m\dot{\bar{r}}$ $\qquad$ $\bar{r}, \dot{\bar{r}}$ giacciono sul piano $\perp \bar{M}$

2) Se moto avviene su piano xy, $z = 0$, $\dot{z} = 0$

$x = r\cos\varphi \qquad y = r\sin\varphi \qquad R = r$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\varphi}^2) - V$$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\varphi}^2) \underbrace{- \frac{\alpha}{mr} + \frac{4\beta}{3mr^{3/2}}}_{-V(r)}$$

3) φ è ciclica

cost. del moto è $\quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} = \ell \rightarrow \dot{\varphi} = \frac{\ell}{mr^2}$

$$L^* = L - p_\varphi \dot{\varphi} \Big|_{\dot{\varphi} = \frac{\ell}{mr^2}} =$$

$$= \frac{1}{2} m\dot{r}^2 + \frac{1}{2} m r^2 \left(\frac{\ell}{mr^2}\right)^2 - \frac{\alpha}{mr} + \frac{4\beta}{3mr^{3/2}} - \ell \frac{\ell}{mr^2}$$

$$= \frac{1}{2} m\dot{r}^2 - \frac{\ell^2}{2mr^2} - \frac{\alpha}{mr} + \frac{4\beta}{3mr^{3/2}}$$

4) $V_{eff} = \dfrac{\alpha}{mr} - \dfrac{4\beta}{3mr^{3/2}} + \dfrac{\ell^2}{2mr^2}$

$$V'_{eff} = -\frac{\alpha}{mr^2} + \frac{2\beta}{mr^{5/2}} - \frac{\ell^2}{mr^3} =$$

$$= -\frac{\alpha}{mr^3}\Big(\underbrace{r - \frac{2\beta r^{1/2}}{\alpha} + \frac{\ell^2}{\alpha}}_{=0}\Big) = 0$$

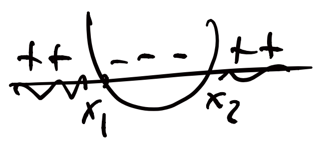
è eq. di 2° grado in $r^{1/2}$

$$\rightarrow \quad r_{\pm}^{1/2} = \frac{\beta}{\alpha} \pm \sqrt{\frac{\beta^2}{\alpha^2} - \frac{\ell^2}{\alpha}} = \frac{\beta}{\alpha} \pm \frac{1}{\alpha}\sqrt{\beta^2 - \alpha\ell^2}$$

$\Delta \geq 0$ se $\underline{\beta^2 \geq \alpha\ell^2}$

↳ solo in questo caso troviamo pti di equil.

Per stab. dovremmo calcolare V''_{eff} e valutarla in $r_{\pm}$



$x^2 + bx + c \geq 0$ $\quad x_1 < x_2$

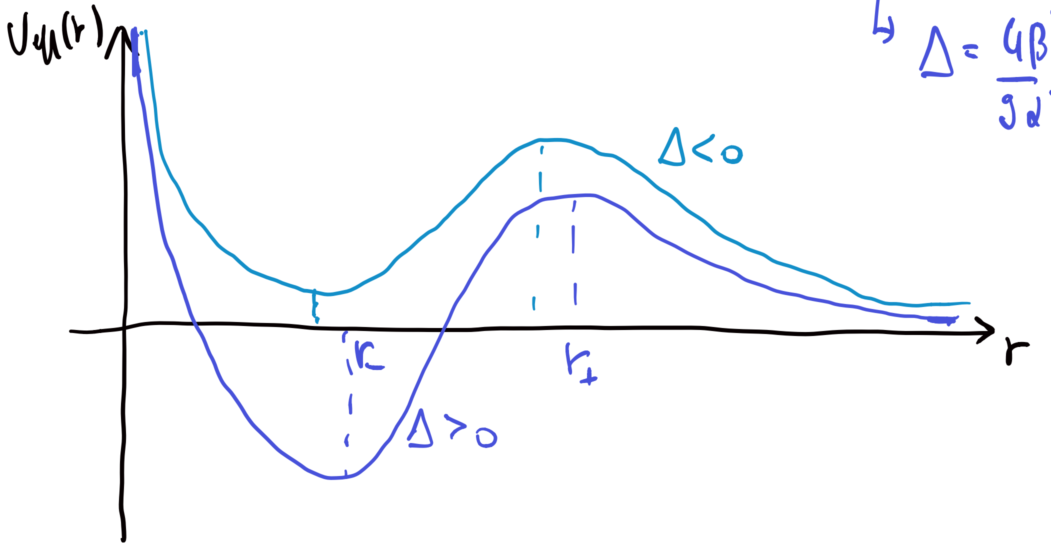
se $x \leq x_1$, $x \geq x_2$

$V'_{eff} < 0$ per $r < r_-$ $\quad r > r_+$

$V'_{eff} > 0$ per $r_- < r < r_+$

$$V_{eff}(r) = \frac{\alpha}{mr} - \frac{4\beta}{3mr^{3/2}} + \frac{\ell^2}{2mr^2} = \frac{\alpha}{mr^2}\left(r - \frac{4\beta}{3\alpha} r^{1/2} + \frac{\ell^2}{2\alpha} \right)$$

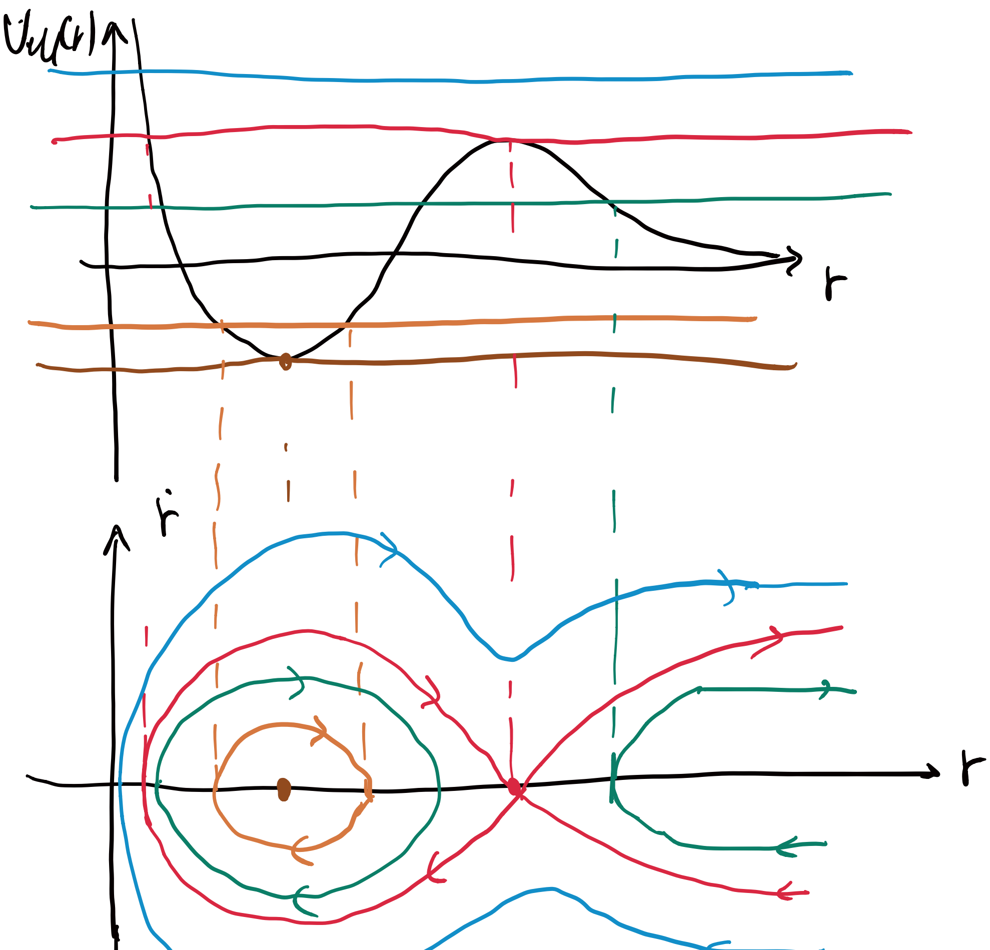
$$\hookrightarrow \Delta = \frac{4\beta^2}{9\alpha^2} - \frac{\ell^2}{2\alpha}$$

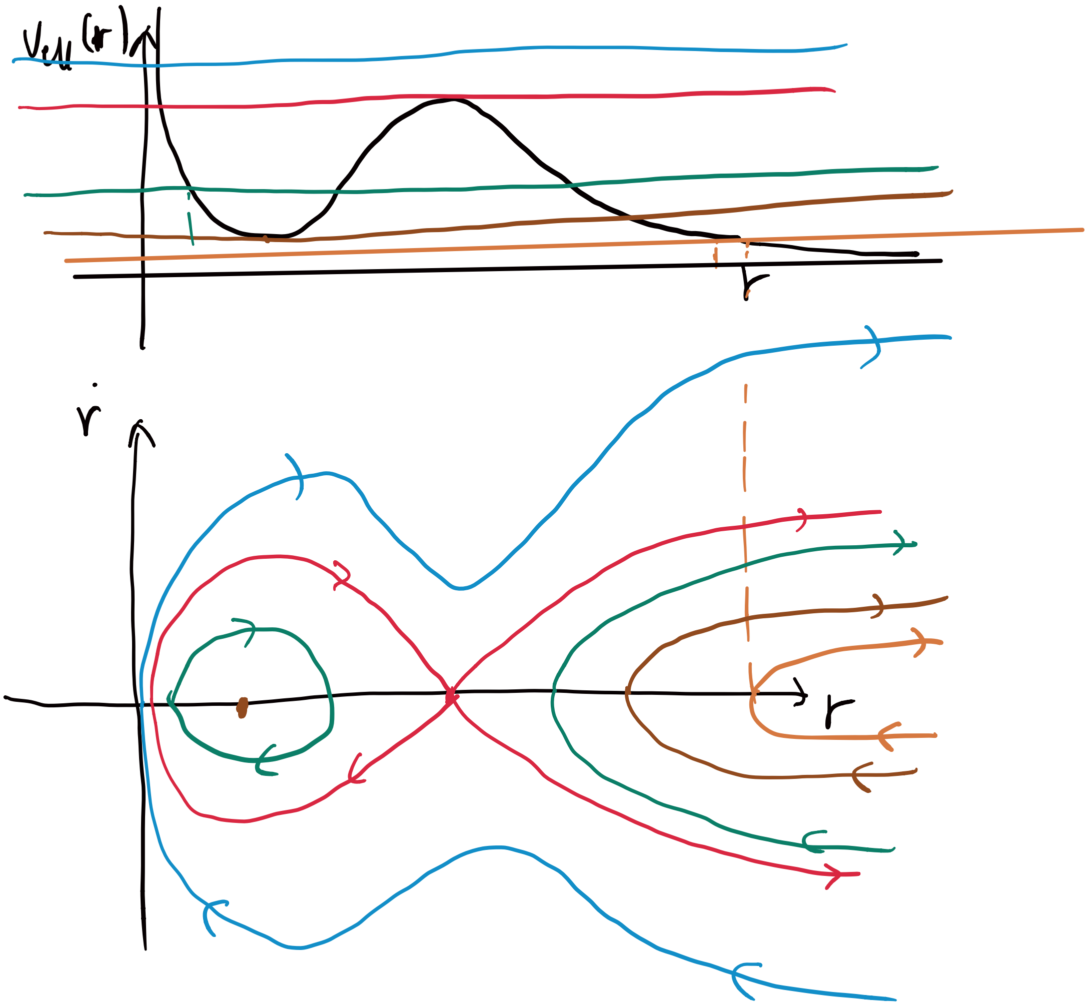


$$r_{MIN} = \frac{\beta}{\alpha} - \frac{1}{\alpha}\sqrt{\beta^2 - \alpha\ell^2} \quad \xleftarrow[\text{STAB.}]{} \quad \text{CIRCONF. sul piano}$$

$$r_{MAX} = \frac{\beta}{\alpha} + \frac{1}{\alpha}\sqrt{\beta^2 - \alpha\ell^2} \quad \xleftarrow[\text{INSTAB.}]{} \quad \text{CIRCONF. sul piano}$$

5)





$V_{eff}(r)$
r
$\dot{r}$
r