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=>
=>
=>
=> #EX3-UNCONSTRAINED OPTIMIZATION
=>
=> #Function
> f :=  $\frac{x^3 \cdot y}{3} + x^2 \cdot y - 3 \cdot x \cdot y + y^2$ ;
                                      $f := \frac{1}{3} x^3 y + x^2 y - 3 x y + y^2$  (1)
=>
=> #Gradient
> dfx := diff(f, x);
                                      $dfx := x^2 y + 2 x y - 3 y$  (2)
=>
> dfy := diff(f, y);
                                      $dfy := \frac{1}{3} x^3 + x^2 - 3 x + 2 y$  (3)
=>
=> #Critical Points
> CP := solve( {dfx=0, dfy=0}, {x, y}, allsolutions, explicit);
CP := {x=0, y=0}, {x=-3, y=-9/2}, {x=1, y=5/6}, {x=-3/2 + 3*sqrt(5)/2, y=0}, {x=
-3/2 - 3*sqrt(5)/2, y=0} (4)
=>
=>
> with(VectorCalculus) :
> Hf := Hessian(f, [x, y]);
                                      $Hf := \begin{bmatrix} 2 x y + 2 y & x^2 + 2 x - 3 \\ x^2 + 2 x - 3 & 2 \end{bmatrix}$  (5)
=>
> with(LinearAlgebra) :
> detHf := Determinant(Hf);
                                      $detHf := -x^4 - 4 x^3 + 2 x^2 + 4 x y + 12 x + 4 y - 9$  (6)
=>
> trHf := Trace(Hf);
                                      $trHf := 2 x y + 2 y + 2$  (7)
=>
> eigf := Eigenvalues(Hf);
eigf :=  $\begin{bmatrix} x y + y + 1 + \sqrt{x^4 + x^2 y^2 + 4 x^3 + 2 x y^2 - 2 x^2 - 2 x y + y^2 - 12 x - 2 y + 10} \\ x y + y + 1 - \sqrt{x^4 + x^2 y^2 + 4 x^3 + 2 x y^2 - 2 x^2 - 2 x y + y^2 - 12 x - 2 y + 10} \end{bmatrix}$  (8)
=>
=> #Detailed analysis for each point
=>
> CP[1];

```

$$\begin{aligned} &\{x=0, y=0\} \\ &x=0 \end{aligned} \quad (9)$$

> *Hf1* := *eval*(*Hf*, *CP*[1]);

$$Hf1 := \begin{bmatrix} 0 & -3 \\ -3 & 2 \end{bmatrix} \quad (10)$$

> *detHf1* := *Determinant*(*Hf1*);

$$detHf1 := -9 \quad (11)$$

> *trHf1* := *Trace*(*Hf1*);

$$trHf1 := 2 \quad (12)$$

> *eigf1* := *Eigenvalues*(*Hf1*); *evalf*(*eigf1*);

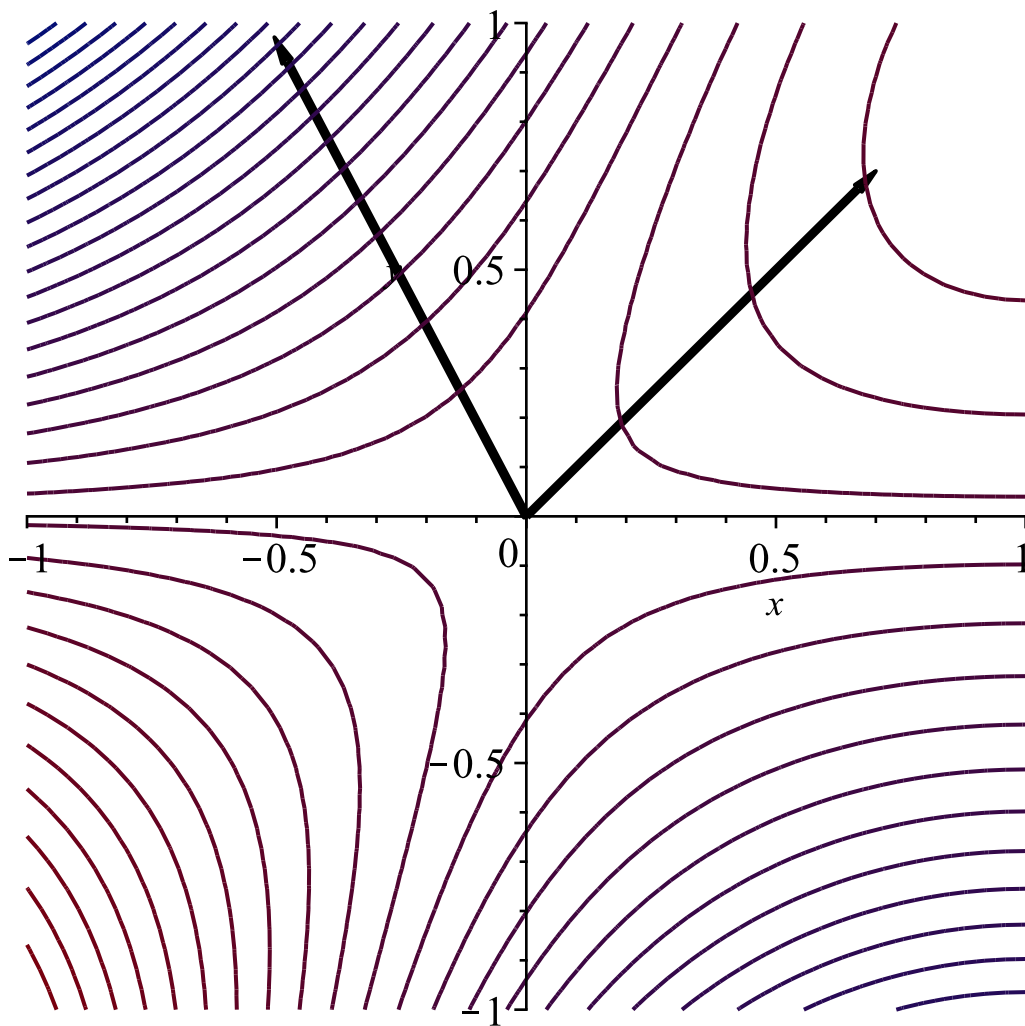
$$eigf1 := \begin{bmatrix} 1 + \sqrt{10} \\ 1 - \sqrt{10} \end{bmatrix} \quad (13)$$

$$\begin{bmatrix} 4.162277660 \\ -2.162277660 \end{bmatrix}$$

> *eiggf1* := *Eigenvectors*(*Hf1*);

$$eiggf1 := \begin{bmatrix} 1 + \sqrt{10} \\ 1 - \sqrt{10} \end{bmatrix}, \begin{bmatrix} -\frac{3}{1 + \sqrt{10}} & -\frac{3}{1 - \sqrt{10}} \\ 1 & 1 \end{bmatrix} \quad (14)$$

> *p1* := *contourplot*(*f*, *x*=0-1..0+1, *y*=0-1..0+1, *contours*=30) : *q1* := *arrow* $\left(\left\{ 0.7 \cdot \left[-\frac{3}{1 + \sqrt{10}}, -\frac{3}{1 - \sqrt{10}} \right], 0.7 \cdot [1, 1] \right\}, width = [0.01, relative], head_length = [0.05, relative], color = black \right)$: *display*({*p1*, *q1*});



```
> CP[2];
```

$$\left\{x = -3, y = -\frac{9}{2}\right\} \quad (15)$$

```
> Hf2 := eval(Hf, CP[2]);
```

$$Hf2 := \begin{bmatrix} 18 & 0 \\ 0 & 2 \end{bmatrix} \quad (16)$$

```
> detHf2 := Determinant(Hf2);
```

$$\det Hf2 := 36 \quad (17)$$

```
> trHf2 := Trace(Hf2);
```

$$\text{tr} Hf2 := 20 \quad (18)$$

```
> eigf2 := Eigenvalues(Hf2); evalf(eigf2);
```

$$\text{eigf2} := \begin{bmatrix} 18 \\ 2 \end{bmatrix} \quad (19)$$

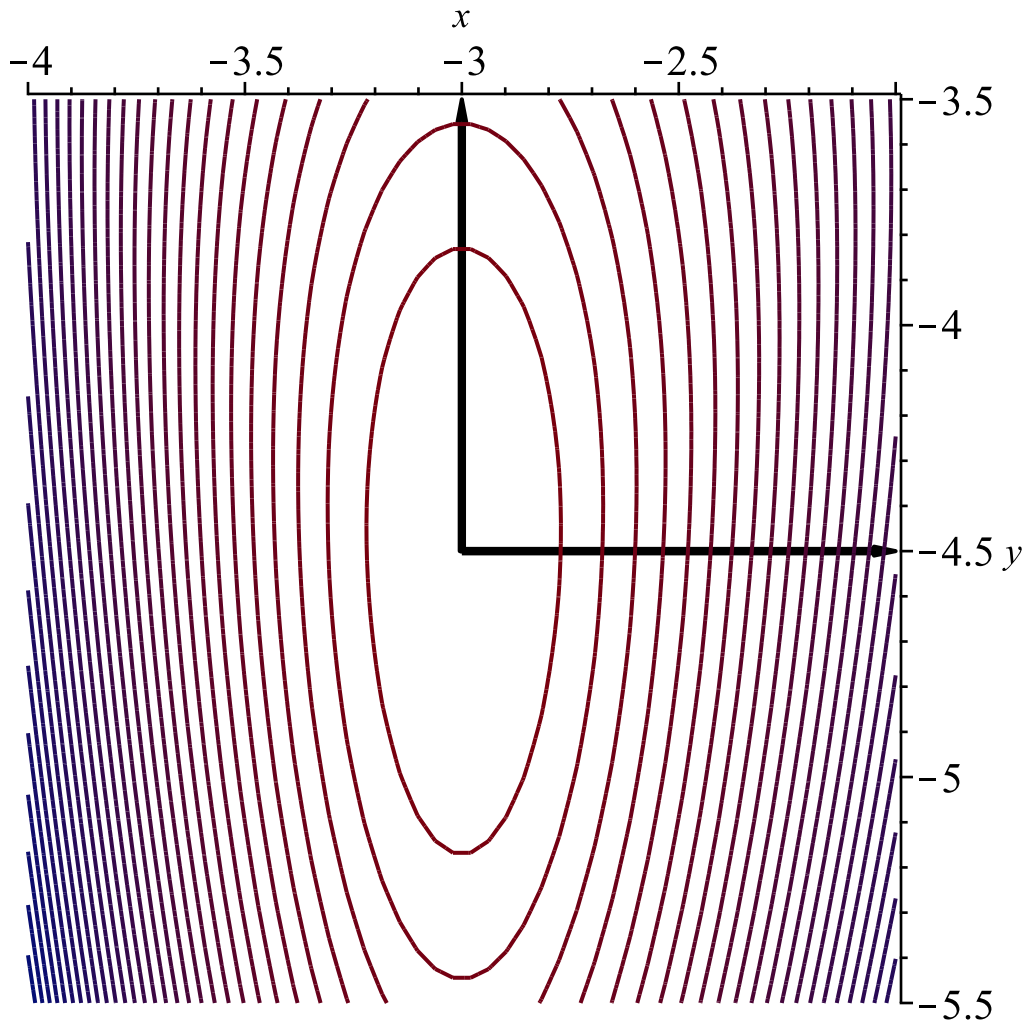
$$\begin{bmatrix} 18. \\ 2. \end{bmatrix}$$

```
> eiggf2 := Eigenvectors(Hf2);
```

$$\text{eiggf2} := \begin{bmatrix} 18 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(20)

```
> p2 := contourplot(f, x=-3-1..-3+1, y=-9/2-1..-9/2+1, contours=30) : q2 :=
arrow({[-3, -9/2]}, {[1, 0], [0, 1]}, width=[0.01, relative], head_length=[0.05,
relative], color=black) : display({p2, q2});
```



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>
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```
> CP[3];
```

$$\left\{ x = 1, y = \frac{5}{6} \right\}$$

(21)

```
> Hf3 := eval(Hf, CP[3]);
```

$$Hf3 := \begin{bmatrix} \frac{10}{3} & 0 \\ 0 & 2 \end{bmatrix}$$

(22)

```
> detHf3 := Determinant(Hf3);
```

$$\det Hf3 := \frac{20}{3} \quad (23)$$

```
> trHf3 := Trace(Hf3);
```

$$\text{tr} Hf3 := \frac{16}{3} \quad (24)$$

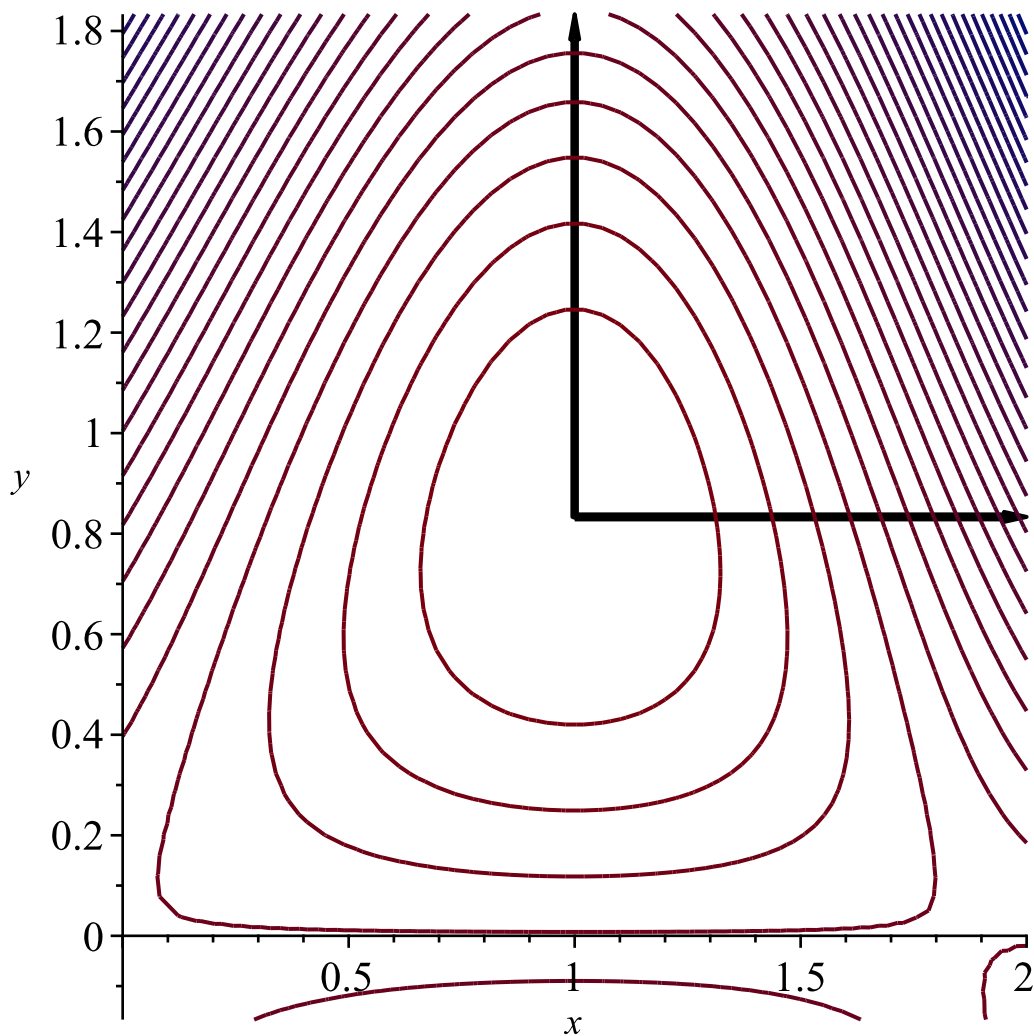
```
> eigf3 := Eigenvalues(Hf3); evalf(eigf3);
```

$$\text{eigf3} := \begin{bmatrix} \frac{10}{3} \\ 2 \\ 3.333333333 \\ 2. \end{bmatrix} \quad (25)$$

```
> eiggf3 := Eigenvectors(Hf3);
```

$$\text{eiggf3} := \begin{bmatrix} \frac{10}{3} \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (26)$$

```
> p3 := contourplot(f, x = 1 - 1..1 + 1, y = 5/6 - 1..5/6 + 1, contours = 30) : q3 := arrow(
    { [ 1, 5/6 ] }, { [ 1, 0 ], [ 0, 1 ] }, width = [ 0.01, relative ], head_length = [ 0.05, relative ], color
    = black ) : display( { p3, q3 } );
```



> CP[4];

$$\left\{x = -\frac{3}{2} + \frac{3\sqrt{5}}{2}, y = 0\right\} \quad (27)$$

> Hf4 := eval(Hf, CP[4]);

$$Hf4 := \begin{bmatrix} 0 & \left(-\frac{3}{2} + \frac{3\sqrt{5}}{2}\right)^2 - 6 + 3\sqrt{5} \\ \left(-\frac{3}{2} + \frac{3\sqrt{5}}{2}\right)^2 - 6 + 3\sqrt{5} & 2 \end{bmatrix} \quad (28)$$

> detHf4 := Determinant(Hf4);

$$\det Hf4 := -\frac{9(-5 + \sqrt{5})^2}{4} \quad (29)$$

> trHf4 := Trace(Hf4);

$$\text{tr} Hf4 := 2 \quad (30)$$

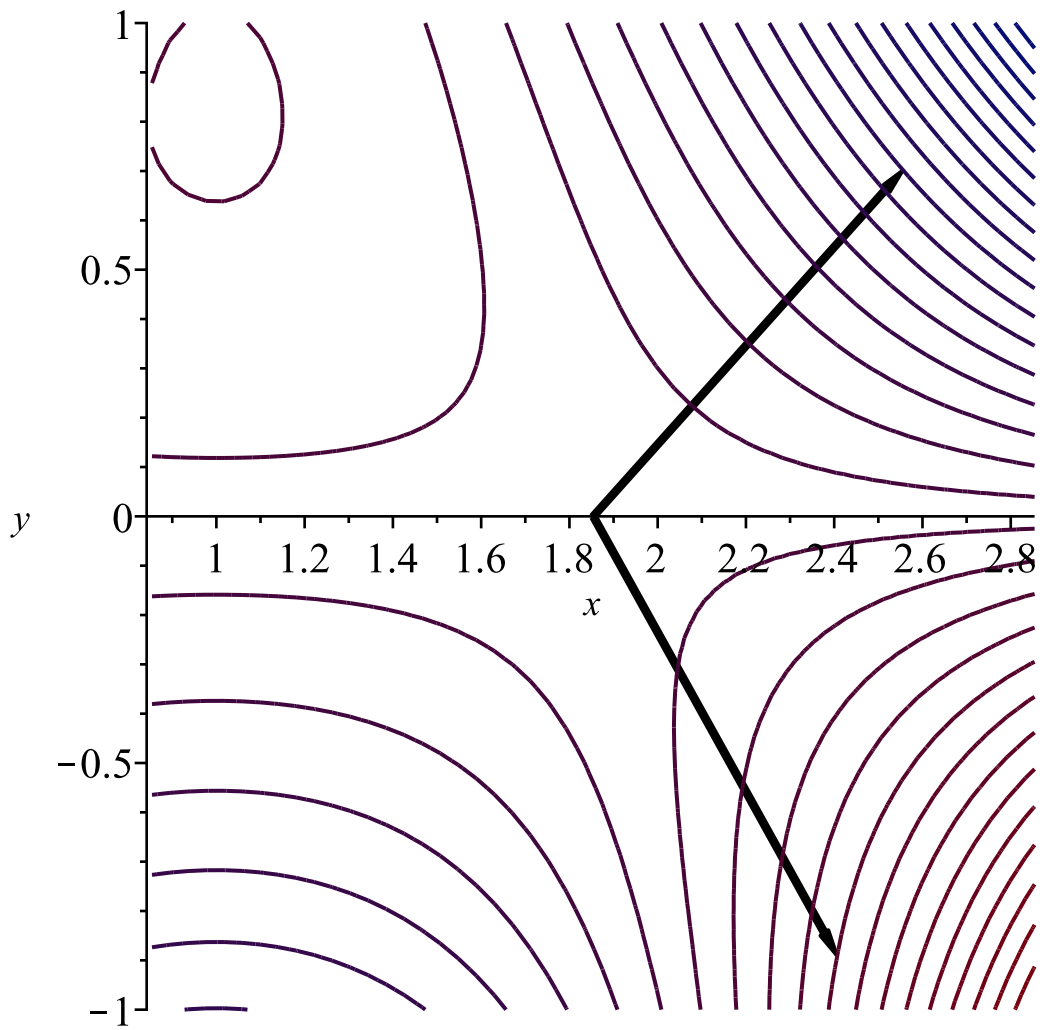
> eigf4 := Eigenvalues(Hf4); evalf(eigf4);

$$eigf4 := \begin{bmatrix} 1 + \frac{\sqrt{274 - 90\sqrt{5}}}{2} \\ 1 - \frac{\sqrt{274 - 90\sqrt{5}}}{2} \\ 5.264794312 \\ -3.264794312 \end{bmatrix} \quad (31)$$

> eiggf4 := Eigenvectors(Hf4);

$$eiggf4 := \begin{bmatrix} 1 + \frac{\sqrt{274 - 90\sqrt{5}}}{2} \\ 1 - \frac{\sqrt{274 - 90\sqrt{5}}}{2} \\ -\frac{3(-5 + \sqrt{5})}{2\left(1 + \frac{\sqrt{274 - 90\sqrt{5}}}{2}\right)} \\ -\frac{3(-5 + \sqrt{5})}{2\left(1 - \frac{\sqrt{274 - 90\sqrt{5}}}{2}\right)} \end{bmatrix}, \quad (32)$$

> p4 := contourplot(f, x = - $\frac{3}{2} + \frac{3\sqrt{5}}{2} - 1$.. - $\frac{3}{2} + \frac{3\sqrt{5}}{2} + 1$, y = 0 - 1 .. 0 + 1, contours = 30) : q4 := arrow($\left\{ \left[-\frac{3}{2} + \frac{3\sqrt{5}}{2}, 0 \right] \right\}$, $\left\{ 0.7 \cdot \left[-\frac{3(-5 + \sqrt{5})}{2\left(1 + \frac{\sqrt{274 - 90\sqrt{5}}}{2}\right)}, -\frac{3(-5 + \sqrt{5})}{2\left(1 - \frac{\sqrt{274 - 90\sqrt{5}}}{2}\right)} \right] \right\}$, 0.7 · [1, 1], width = [0.01, relative], head_length = [0.05, relative], color = black) : display({p4, q4});



> CP[5];

$$\left\{x = -\frac{3}{2} - \frac{3\sqrt{5}}{2}, y = 0\right\} \quad (33)$$

> Hf5 := eval(Hf, CP[5]);

$$Hf5 := \begin{bmatrix} 0 & \left(-\frac{3}{2} - \frac{3\sqrt{5}}{2}\right)^2 - 6 - 3\sqrt{5} \\ \left(-\frac{3}{2} - \frac{3\sqrt{5}}{2}\right)^2 - 6 - 3\sqrt{5} & 2 \end{bmatrix} \quad (34)$$

> detHf5 := Determinant(Hf5);

$$detHf5 := -\frac{9(5 + \sqrt{5})^2}{4} \quad (35)$$

> trHf5 := Trace(Hf5);

$$trHf5 := 2 \quad (36)$$

> eigf5 := Eigenvalues(Hf5); evalf(eigf5);

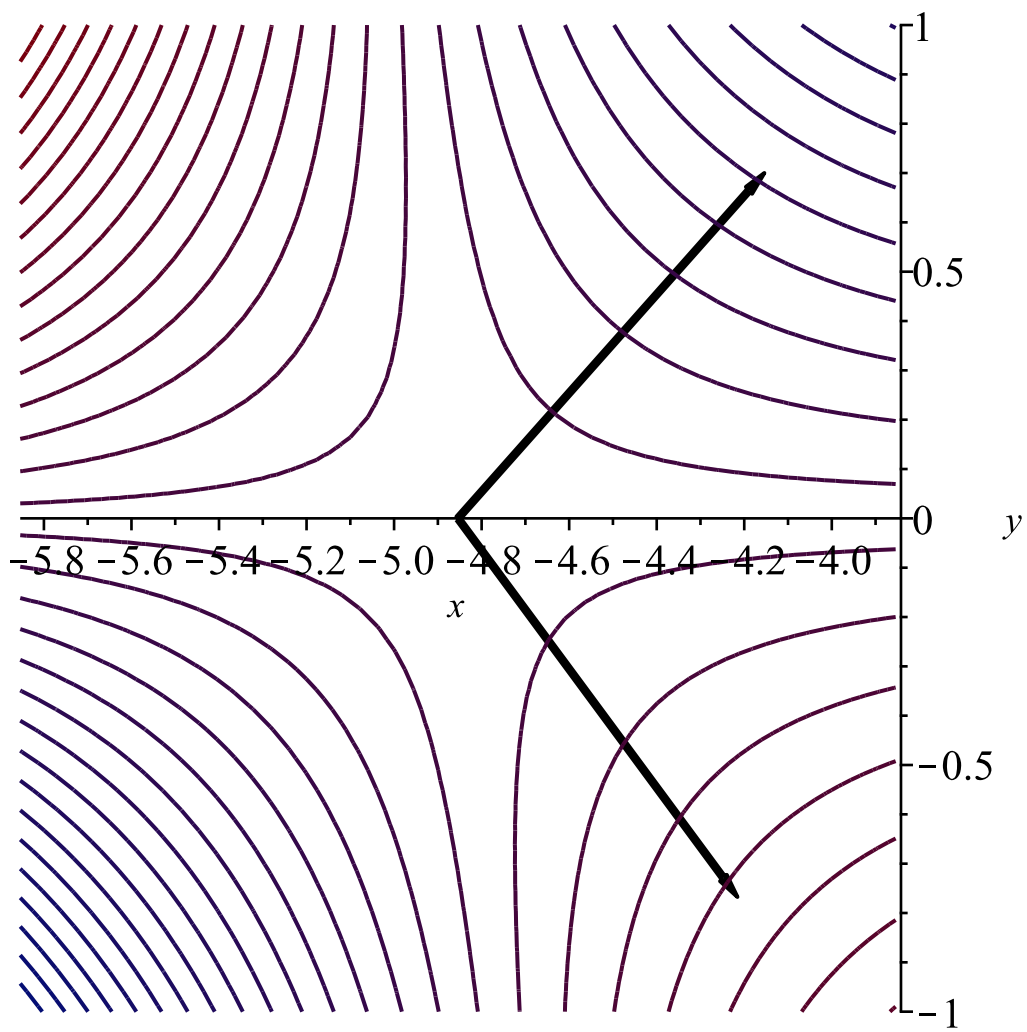
$$eigf5 := \begin{bmatrix} 1 + \frac{\sqrt{274 + 90\sqrt{5}}}{2} \\ 1 - \frac{\sqrt{274 + 90\sqrt{5}}}{2} \\ 11.90007016 \\ -9.90007016 \end{bmatrix} \quad (37)$$

> eiggf5 := Eigenvectors(Hf5);

$$eiggf5 := \begin{bmatrix} 1 + \frac{\sqrt{274 + 90\sqrt{5}}}{2} \\ 1 - \frac{\sqrt{274 + 90\sqrt{5}}}{2} \end{bmatrix}, \quad (38)$$

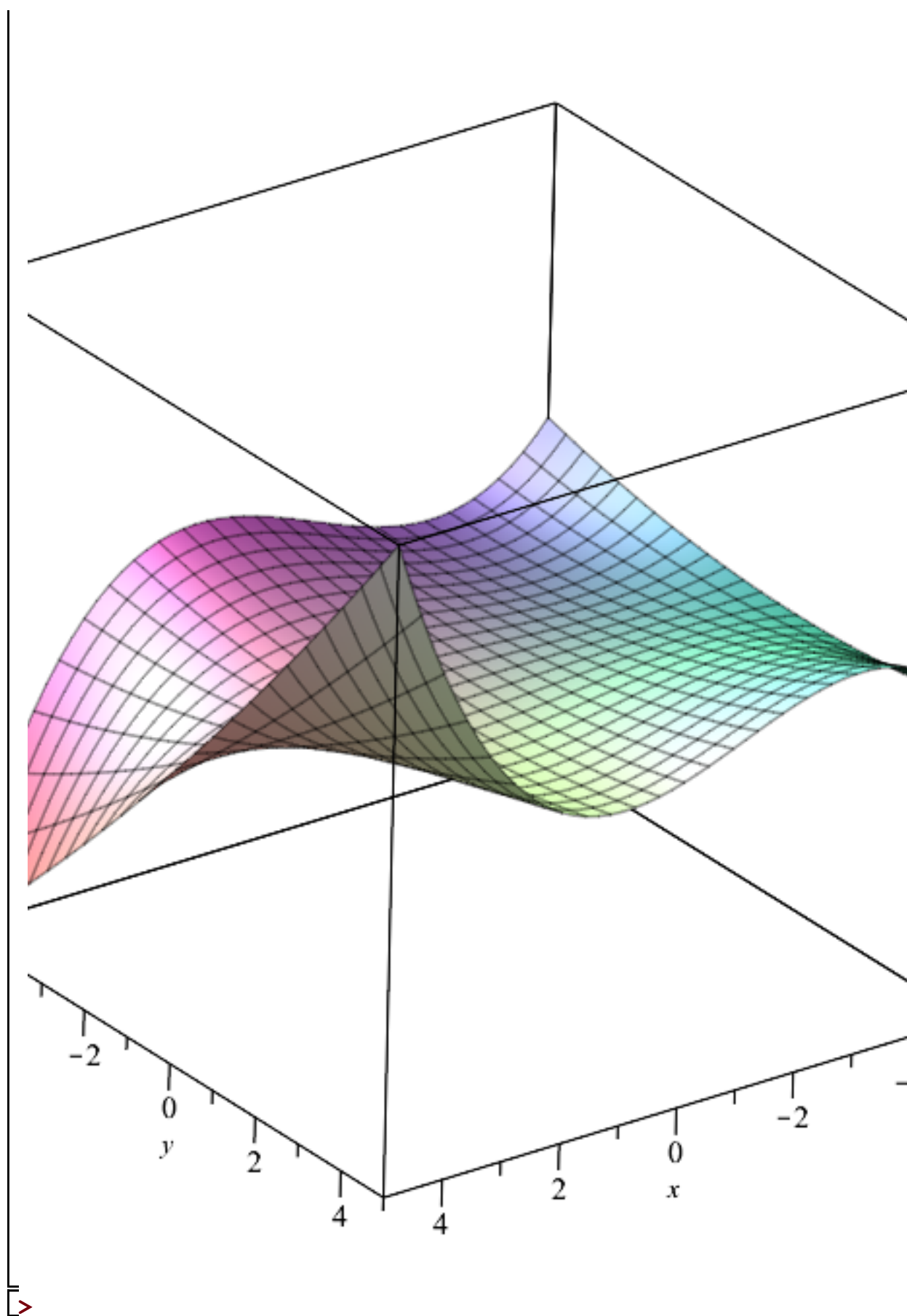
$$\begin{bmatrix} \frac{3(5 + \sqrt{5})}{2 \left(1 + \frac{\sqrt{274 + 90\sqrt{5}}}{2}\right)} & \frac{3(5 + \sqrt{5})}{2 \left(1 - \frac{\sqrt{274 + 90\sqrt{5}}}{2}\right)} \\ 1 & 1 \end{bmatrix}$$

> p5 := contourplot(f, x = - $\frac{3}{2}$ - $\frac{3\sqrt{5}}{2}$ - 1 .. - $\frac{3}{2}$ - $\frac{3\sqrt{5}}{2}$ + 1, y = 0 - 1 .. 0 + 1, contours = 30) : q5 := arrow($\left\{ \left[-\frac{3}{2} - \frac{3\sqrt{5}}{2}, 0 \right] \right\}$, $\left\{ 0.7 \cdot \left[\frac{3(5 + \sqrt{5})}{2 \left(1 + \frac{\sqrt{274 + 90\sqrt{5}}}{2}\right)}, \frac{3(5 + \sqrt{5})}{2 \left(1 - \frac{\sqrt{274 + 90\sqrt{5}}}{2}\right)} \right] \right\}$, 0.7 · [1, 1], width = [0.01, relative], head_length = [0.05, relative], color = black) : display({p5, q5});



```
> with(plots) :
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```
> plot3d(f, x=-5..5, y=-5..5, contours = 10);
```



```
> s := contourplot(f, x=-5..5, y=-5..5, contours = 70) : display( {s, q1, q2, q3, q4, q5} );
```

