Affine variety = locally closed of $\mathbb{P}^n$ isomorphic to an $X \subseteq \mathbb{A}^n$ closed

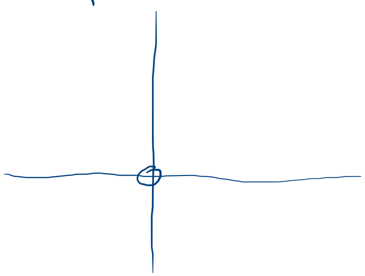
$X \subseteq \mathbb{A}^n$ closed, $F \in K[x_1 \dots x_n]$ $\quad X \setminus V(F)$ is affine

$$X \setminus V(F) \cong Y \subseteq \mathbb{A}^{n+1}$$

$$Y = V\big(I(X) + (x_{n+1}F - 1)\big)$$

$X = \mathbb{A}^2 \setminus \{(0,0)\}$ not affine $\qquad$ K alg. close

Pf. 1) $\mathcal{O}(X) = \mathcal{O}(\mathbb{A}^2) = K[x_1, x_2]$



$f \in \mathcal{O}(X) \qquad f : \mathbb{A}^2 \setminus \{(0,0)\} \longrightarrow K$

$\forall P \in X \quad \exists U_P \quad f|_{U_P} = \frac{F_P}{G_P}, \quad F_P, G_P \in K[x_1, x_2]$

$V(G_P) \cap U_P = \emptyset$, $\quad Q \neq P \quad f|_{U_Q} = \frac{F_Q}{G_Q}$, $\quad V(G_Q) \cap U_Q = \emptyset$

On $U_P \cap U_Q \neq \emptyset \qquad \frac{F_P}{G_P} = \frac{F_Q}{G_Q} \qquad F_P G_Q - F_Q G_P = 0$ on $\underline{U_P \cap U_Q \text{ open}}$

$\Rightarrow F_P G_Q - F_Q G_P = 0$ on $\mathbb{A}^2$ in $\underline{K[x_1, x_2]}$

We can assume $f=\frac{F_P}{G_P}$: F_P, G_P coprime : $k[x_1,x_2]$ is UFD

F_Q, G_Q coprime $\qquad \underline{F_P G_Q} = F_Q G_P$ in $k[x_1,x_2]$

$\Longrightarrow$ $F_P = F_Q$, $G_P = G_Q$ up to nonzero constants

f has a unique expression $f=\frac{F}{G}$ on $X \Longrightarrow$

$V(G)\cap X = \emptyset \qquad V(G)\subseteq\{(0,0)\} \Longrightarrow G$ is constant

$\Longrightarrow f$ is polynomial.

$\mathcal{O}(X) = \mathcal{O}(\mathbb{A}^2) = k[x_1,x_2] \supseteq \langle x_1,x_2\rangle = I$

$V(I) \subseteq X$, $V(I) = \emptyset$ $\qquad$ k alg. closed : the relative form of the Nullstellensatz holds :

if $Y\subseteq \mathbb{A}^n$ closed, $\quad \mathcal{O}(Y) = k[Y] \supsetneq \alpha$ proper $\Rightarrow V(\alpha)\neq\emptyset$

If by contrad. X is affine : $\exists\, Y\subseteq \mathbb{A}^n$ closed $\quad Y \xrightarrow[\sim]{\varphi} X$

$\varphi^*: \mathcal{O}(X) \xrightarrow{\sim} \mathcal{O}(Y)$, $\quad I \subsetneq \mathcal{O}(X)$, $\quad \varphi^*(I) \subsetneq \mathcal{O}(Y)$ $\qquad V(\varphi^*(I))\neq\emptyset$, $P \in V(\varphi^*(I))$

$\underline{\forall F\in I}$ $\quad \varphi^*(F)(P) = 0 = F(\underline{\varphi(P)})$: $\varphi(P)\in V(I)$

$\Longrightarrow V(I)\neq\emptyset$

So X cannot be affine.

Theorem $X \subseteq \mathbb{P}^n$ locally closed $\Longrightarrow$ X can be covered by affine varieties open in X.

If $X \subseteq \mathbb{P}^n$ is closed: $X = \bigcup_{i=0}^{n} \underbrace{(X \cap U_i)}_{\text{affine and open in } X}$

Pf $P \in X \subseteq \mathbb{P}^n$ we look for an open nbhd of P which is an affine variety

$\mathbb{P}^n = U_0 \cup U_1 \cup \dots \cup U_n \Longrightarrow P \in U_i$; we can ass. $i = 0$.

$P \in X \cap U_0$ locally closed in $\mathbb{A}^n = U_0$

$\overset{\shortparallel}{Z \setminus Z'}$ where Z, Z' are closed in $\mathbb{A}^n$

$P \in Z$ $P \notin Z'$ we look for F s.t.

$Z \setminus V(F) \subseteq \underset{\underset{P}{\Psi}}{Z \setminus Z'} = X \cap U_0 \subseteq Z$ $*$

$Z' \subsetneq Z' \cup \{P\}$ closed $\quad I(Z') \supsetneq I(Z' \cup \{P\})$

$\exists \; \underset{F}{\Psi}$ but $F \notin I(Z' \cup \{P\})$

$\Longrightarrow F(P) \neq 0$ $\quad \underline{V(F) \supseteq Z'}$

$\underline{Z \setminus V(F)} \subseteq \underset{\underset{P}{\Psi}}{Z \setminus Z'} \subseteq Z \Longrightarrow Z \setminus V(F)$ it is an open nbhd of P, affine, open in $\underset{X \cap U_0}{\overset{\shortparallel}{Z \setminus Z'}}$ open in X

$P \in X$ $\quad X \cap U_i = Z \setminus Z' \supseteq \underline{\underline{Z \setminus V(F)}} \ni P$

If we study local properties of X $\underline{\text{in } P}$ $\Longrightarrow$ we can replace X with an affine variety

$\boxed{\mathcal{O}_{X,P}}$: we can assume X is affine.

$\nu_{1,d} : \mathbb{P}^1 \longrightarrow \mathbb{P}^d$ $\qquad \nu_{1,d}(\mathbb{P}^1) = V(I)$

I generated by 2×2 minors of $\begin{pmatrix} x_0 & x_1 & \cdots & x_{d-1} \\ x_1 & x_2 & \cdots & x_d \end{pmatrix}$

where $x_0, \dots, x_d$ coord. on $\mathbb{P}^d$.

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

As a consequence: $P_0, \dots, P_d \in \nu_{1,d}(\mathbb{P}^1)$ distinct points : they are in general positions = linearly indep.

CATEGORY

To give a category $\mathcal{C}$ we must give two classes:

1) $ob(\mathcal{C})$ the class of the objects of $\mathcal{C}$

2) $Mor(\mathcal{C})$ the class of the morphisms of $\mathcal{C}$

$$\overset{\cdot}{\bigcup_{A,B \in ob(\mathcal{C})}} Hom(A,B)$$ where $Hom(A,B)$ is a set $\forall A, B \in ob(\mathcal{C})$

disjoint

$Hom_{\mathcal{C}}(A,B)$ $\quad \mathcal{C}(A,B)$: morphisms from A to B

$f \in Hom_{\mathcal{C}}(A,B)$ $\quad f : A \to B$ but the elements of $Hom_{\mathcal{C}}(A,B)$ are not necessarily maps of sets

If $f \in Mor(\mathcal{C}) \Rightarrow f : A \to B$ $\quad$ domain A, target B

We require:

3) morphisms can be composed: $\forall A, B, C \in ob(\mathcal{C})$
$\forall f \in Hom(A,B),\ g \in Hom(B,C) \Longrightarrow \exists\ g\circ f \in Hom(A,C)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ \ gf$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

4) $\forall A \in ob(\mathcal{C})$ $\quad Hom(A,A) \neq \emptyset$ and
$\exists\ 1_A : A \to A$ s.t. $1_A \circ f = f,\ \ g \circ 1_A = g$

A priori for $A \neq B$ $\quad Hom(A,B)$ could be $\emptyset$.

1_A identity morphism

5) associativity
$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \Longrightarrow h\circ(g\circ f) = (h\circ g)\circ f$

Examples 1) Set category of sets
$ob(Set) = \{sets\}$
$\forall A, B$ sets $\quad Hom_{Set}(A,B) = \{maps\ A \to B\}$

2) Algebraic structures as:
- Grp category of groups and homomorphisms
- AbGrp objects are only abelian groups
G, G' abelian groups $\quad Hom(G,G')$ is the same set
- Rings
- K-vector spaces
- R-modules, R a fixed rings

3) Topological spaces, continuous maps $\Big\}$ Top

4) Fixed X, topological space
$Cov(X) = \mathcal{C}$ st. $\quad ob(\mathcal{C})$ = coverings of X
$\qquad\qquad\qquad Hom(Y,Z)$ = covering maps

5) S poset = partially ordered set

$(S, \leq)$ $\rightsquigarrow$ $\mathcal{C}$: $ob(\mathcal{C}) = S$

$\forall a, b \in S$ $\quad Hom(a,b) = \begin{cases} \{*\} & \text{set with one element } 1_a \quad \text{if } a=b \\ \{*\} & \text{if } a \leq b \\ \emptyset & \text{if } a \nleq b \end{cases}$

$$\begin{tikzcd} a \arrow[r] \arrow[rr, bend right] & b \arrow[r] & c \end{tikzcd}$$

$a \leq b$, $b \leq c \implies a \leq c$: we can compose

$1_a \in Hom(a,a)$

$a \to b \to c \to d$ associativity holds

Particular case: X topological space

$Op(X) = \{ U \subseteq X \mid U \text{ open} \}$: partially ordered by inclusion $\implies$ category $Op(X)$

Morphisms: if $U \subseteq V$ $\quad U \xrightarrow{f} V$

we can interpret the only morphism in $Hom(U,V)$ as $f_{UV}: U \hookrightarrow V$

<u>Subcategory</u> $\mathcal{C}', \mathcal{C}$: $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ if

- $ob(\mathcal{C}') \subseteq ob(\mathcal{C})$
- $\forall A, B \in ob(\mathcal{C}')$ $\quad Hom_{\mathcal{C}'}(A,B) \subseteq Hom_{\mathcal{C}}(A,B)$

$\mathcal{C}'$ is a <u>full</u> subcategory of $\mathcal{C}$ if

$Hom_{\mathcal{C}'}(A,B) = Hom_{\mathcal{C}}(A,B)$, $\quad A, B \in ob(\mathcal{C}')$

FUNCTORS

covariant (contravariant) functor from $\mathcal{C}$ to $\mathcal{C}'$, F is:

1) $F : ob(\mathcal{C}) \longrightarrow ob(\mathcal{C}')$ $\quad A \rightsquigarrow F(A)$, $A \in ob(\mathcal{C})$, $F(A) \in ob(\mathcal{C}')$ ✓

2) $\forall A, B \in ob(\mathcal{C})$, $\forall f \in Hom_{\mathcal{C}}(A,B) \rightsquigarrow F(f) \in Hom_{\mathcal{C}'}(F(A), F(B))$ $\quad Hom_{\mathcal{C}'}(F(B), F(A))$

$$\begin{array}{ccc} A & & F(A) \\ f\downarrow & & \uparrow\ \downarrow F(f) \\ B & & F(B) \\ \text{in } \mathcal{C} & & \text{in } \mathcal{C}' \end{array}$$

$Hom_{\mathcal{C}}(A,B) \xrightarrow{F} Hom_{\mathcal{C}'}(F(A), F(B))$ $\quad (F(B), F(A))$

$f \longrightarrow F(f)$

functoriality properties:

- $F(1_A) = 1_{F(A)}$ $\qquad A \xrightarrow{1_A} A \qquad F(A) \xrightarrow{F(1_A)} F(A)$ ✓
- $A \xrightarrow{f} B \xrightarrow{g} C$, gf $\qquad F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$

$$\boxed{F(gf) = F(g) \circ F(f)}$$

$F(A) \xleftarrow{F(f)} F(B) \xleftarrow{F(g)} F(C)$, $F(gf)$

$$\boxed{F(f) \circ F(g) = F(gf)}$$

Examples

1) Forgetful functors

$U: Grp \longrightarrow Set \qquad G \longrightarrow U(G) =$ underlying set to G

$$\begin{array}{c} G \\ f \downarrow \\ G' \end{array} \qquad U(f) = f \text{ interpreted as map of sets}$$

2) Free functors

$$Set \longrightarrow AbGrp$$

$S \longrightarrow \mathbb{Z}[S]$ abelian free group generated by S

$\mathbb{Z}[f]$ map which extends f naturally

$$\begin{array}{ccc} S & \longrightarrow & \mathbb{Z}[S] \\ f\downarrow & & \downarrow \mathbb{Z}[f] \\ T & & \mathbb{Z}[T] \end{array}$$

3) Representable functors

$\mathcal{C}$, $A \in ob(\mathcal{C})$ fixed $\qquad h^A \qquad$ covariant from $\mathcal{C} \xrightarrow{h^A} Set$

$h^A(B) = Hom_{\mathcal{C}}(A, B)$

$$\boxed{h^A = Hom_{\mathcal{C}}(A, -)}$$

covariant repres. functor

$B \in ob(\mathcal{C})$

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr] & B \arrow[d, "f"] \\ & B' \end{tikzcd} \qquad \begin{array}{c} Hom_{\mathcal{C}}(A,B) \\ \downarrow h^A(f) \\ Hom_{\mathcal{C}}(A,B') \end{array} \qquad \begin{array}{c} g \\ \downarrow \\ f \circ g \end{array}$$

$h^A(1_B) = 1_{Hom(A,B)}$

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr] & B \arrow[d, "1_B"] \\ & B \end{tikzcd}$$

$$\begin{array}{c} A \xrightarrow{g} B \\ \downarrow f \\ B' \\ \downarrow f' \\ B'' \end{array} \qquad \begin{array}{c} Hom(A,B) \\ \downarrow h^A(f) \\ Hom(A,B') \\ \downarrow h^A(f') \\ Hom(A,B'') \end{array} \qquad \begin{array}{c} g \\ \downarrow \\ f \circ g \end{array}$$

$f' \circ (f \circ g) = (f' \circ f) \circ g$

Contravariant repr. functor $h_A = \mathrm{Hom}_{\mathcal{C}}(-, A)$