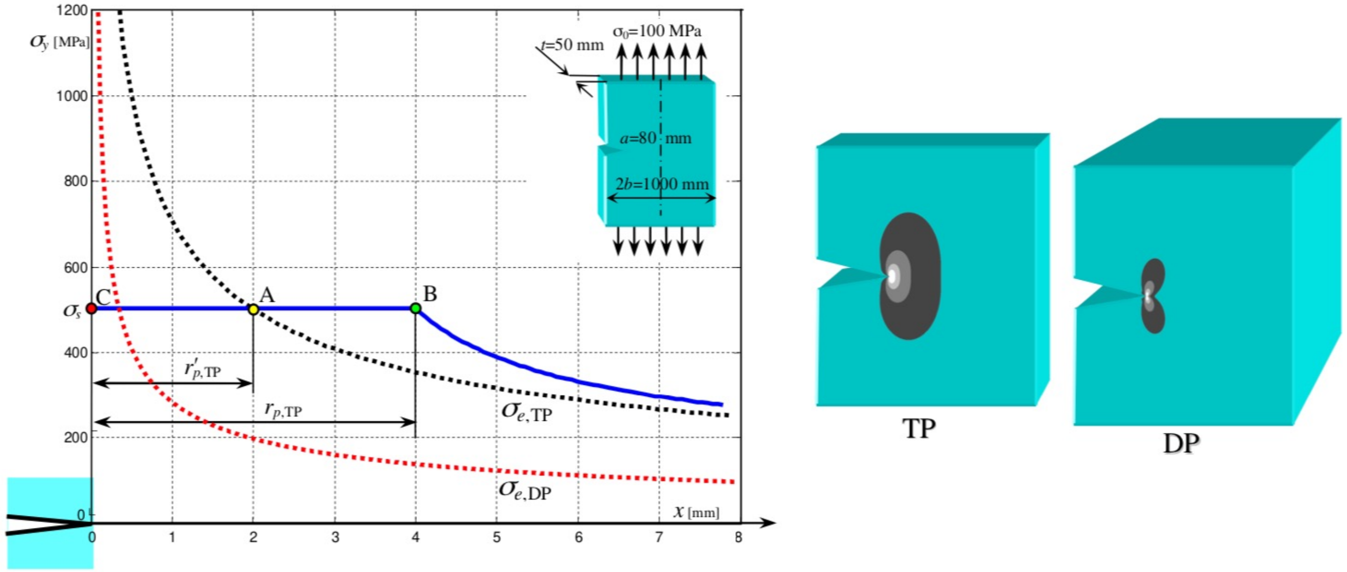
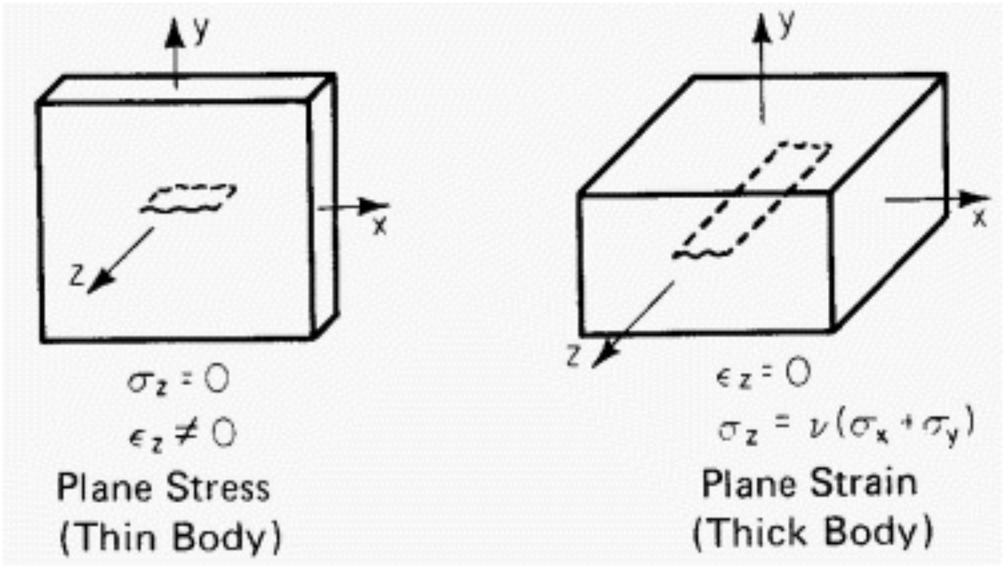
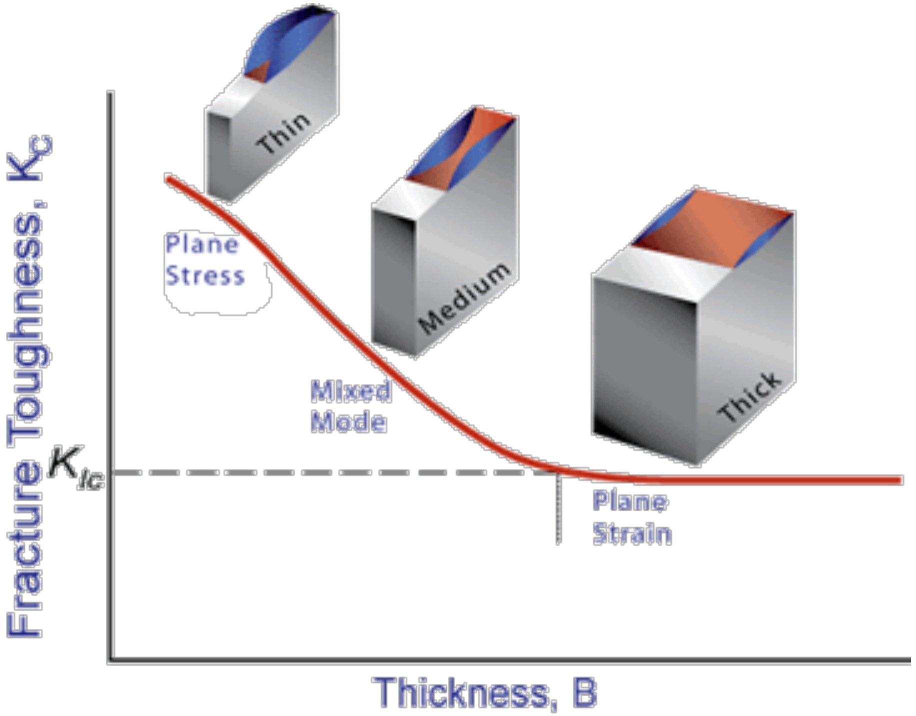


Fig.14.7 - a) Tensioni equivalenti di Von Mises lungo l'asse x per i casi di stato di tensione e deformazione piano (····TP/····DP). Estensione della zona plastica per il caso (TP). b) Forma della zona in cui σ_e=σ_s per i casi TP e DP.





If $b \to 0$ (a crack), σ_{max} tends to infinity at the crack tip, meaning that if one wants to limit the stress value to a threshold, as when designing to remain in the elastic regime, σ_∞ needs to tend to 0. However Griffith and Irwin have shown that a sample with a crack can still sustain a loading:

Griffith

$$\sigma_{TS}\sqrt{a} \div \sqrt{E 2\gamma_S}$$

Irwin

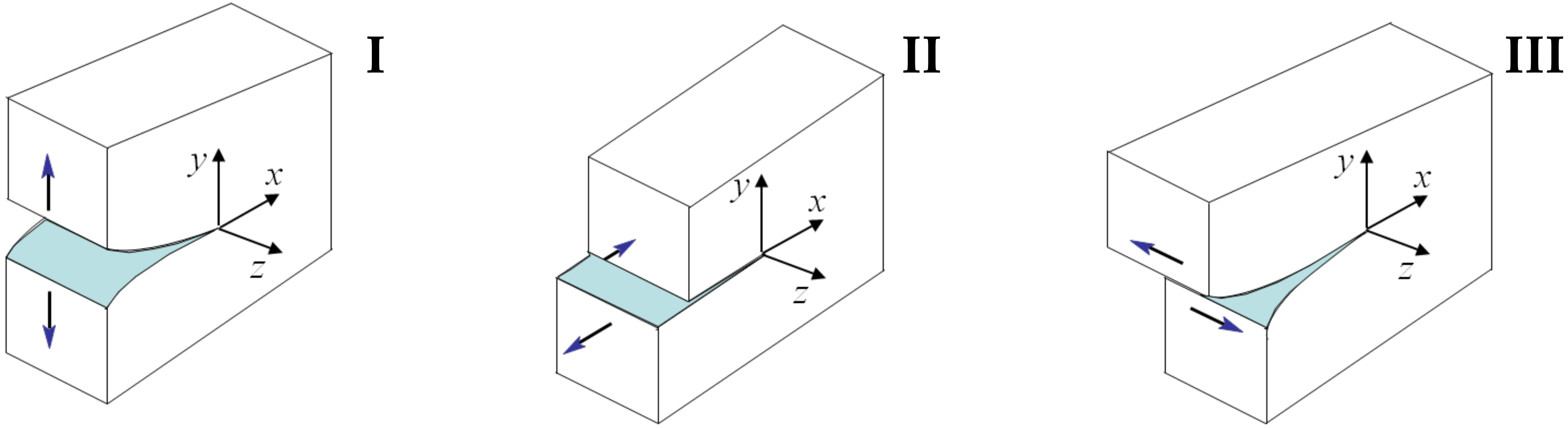
$$\sigma_{TS}\sqrt{a} \div \sqrt{E\left(2\gamma_s + W_{pl}\right)}.$$

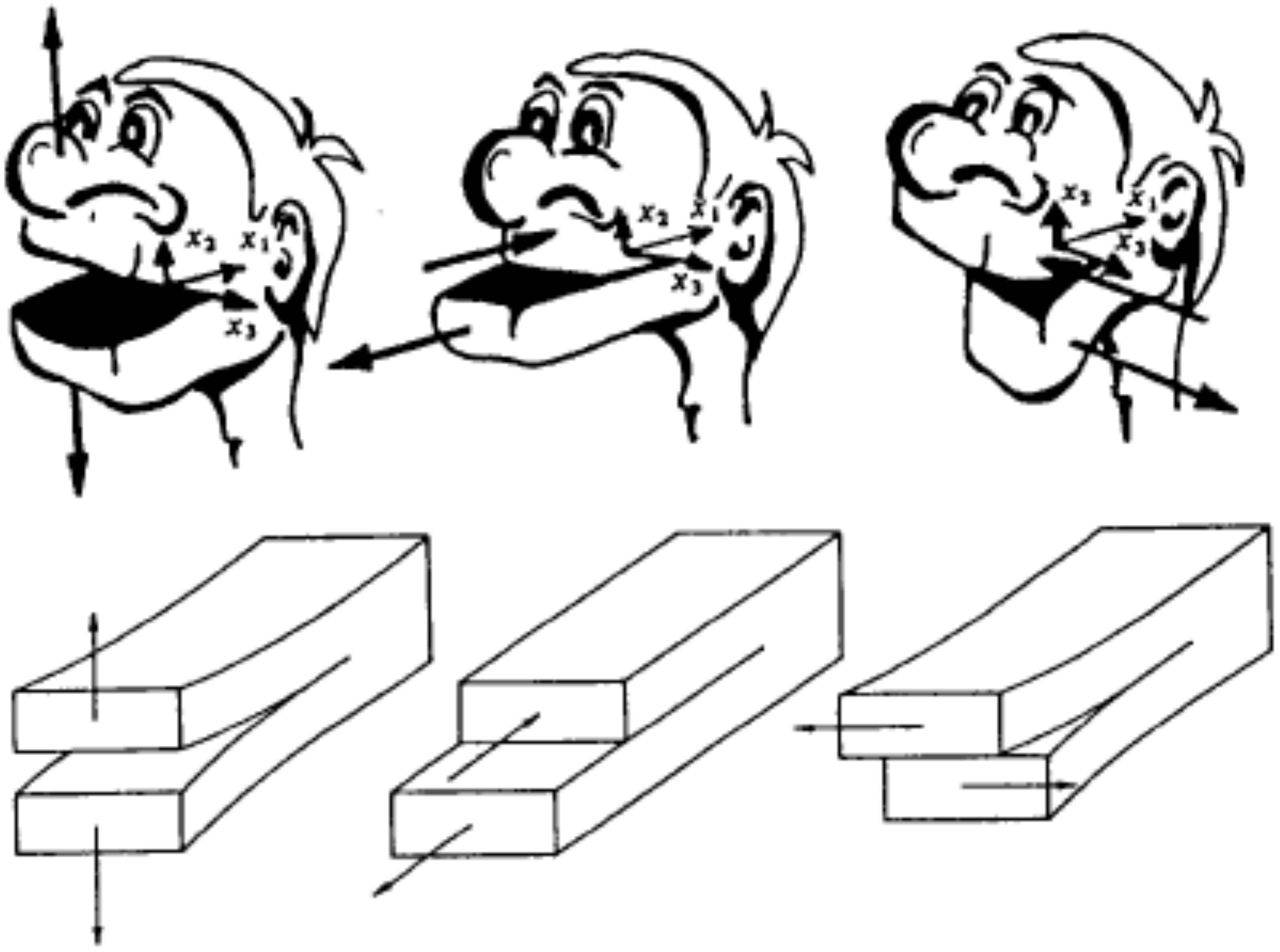
In order to solve the contradiction between the stress analysis and the experimental observations, fracture mechanics was developed to answer to the following questions:

- For a sample with an existing crack, knowing the crack size, what is the maximum loading leading to crack propagation?
- For a sample with an existing crack submitted to a cyclic loading, if the stress remains lower than the fracture stress, what is the life of the structure?

Linear Elastic Fracture Mechanics (LEFM)

Under this assumption, the laws of mechanics for elastic linear materials can directly be used. In order to solve the problem of a sample with a crack, **Irwin** has studied **three different loading modes**. There are two modes loading the crack in its plane, the **Mode I also called opening mode** as it tends to open the crack lips, and the **Mode II also called sliding mode** as it tends to make the two crack lips slide on each other, see. The **Mode III, also called shearing mode**, loads the crack out of its plane.

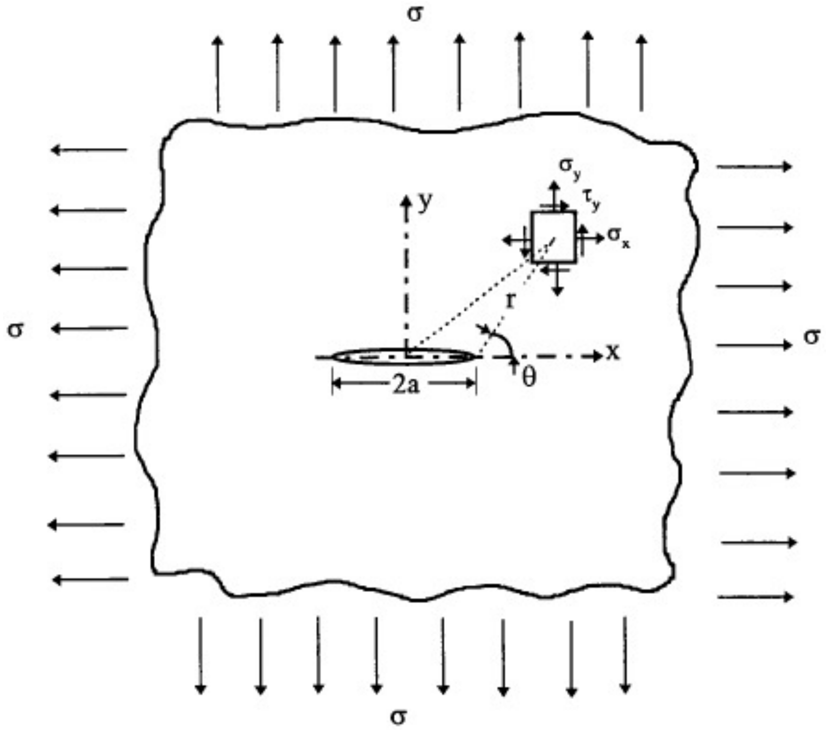



x2
x1
x3
x2
x1
x3
x2
x1
x3

In 1939, Harold M. Westergaard developed a solution for the stress field surrounding a crack that has two advantages over Inglis's solution.

First, Westergaard's solution applies directly to cracks, not to an ellipse that approaches a crack in the limit.

Second, the solution is expressed in rectangular coordinates rather than elliptical coordinates.



Westergaard chose to express the rectangular coordinates as complex numbers, $z = x + iy$

Westergaard's Solution

Westergaard found an Airy stress function of complex numbers that is the solution for the stress field in an infinite plate containing a crack.

The integral of Z is represented by a *bar* Z, and the integral of Z is represented by two bars Z. Finally, the derivative of Z is represented by Z' . In summary,

$$\frac{d\overline{\overline{Z}}}{dz} = \overline{Z} \qquad \frac{d\overline{Z}}{dz} = Z \qquad \frac{dZ}{dz} = Z'$$

Westergaard's choice for the Airy stress function, Φ, was $\phi = \operatorname{Re}\overline{\overline{Z}} + y\operatorname{Im}\overline{Z}$

$$Z(z) = \frac{\sigma_\infty}{\sqrt{1-\left(\frac{a}{z}\right)^2}}$$

where a is crack length

The expression for σ_{xx} is obtained by taking the derivatives of Φ with respect to y.

$$\frac{\partial \phi}{\partial y} = -\operatorname{Im} \overline{Z} + \operatorname{Im} \overline{Z} + y \operatorname{Re} Z \quad = y \operatorname{Re} Z$$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = \operatorname{Re} Z - y \operatorname{Im} Z'$$

The expression for σ_{yy} is obtained by taking the derivatives of Φ with respect to x.

$$\frac{\partial \phi}{\partial x} = \operatorname{Re} \overline{Z} + y \operatorname{Im} Z$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = \operatorname{Re} Z + y \operatorname{Im} Z'$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -y \operatorname{Re} Z'$$

Therefore

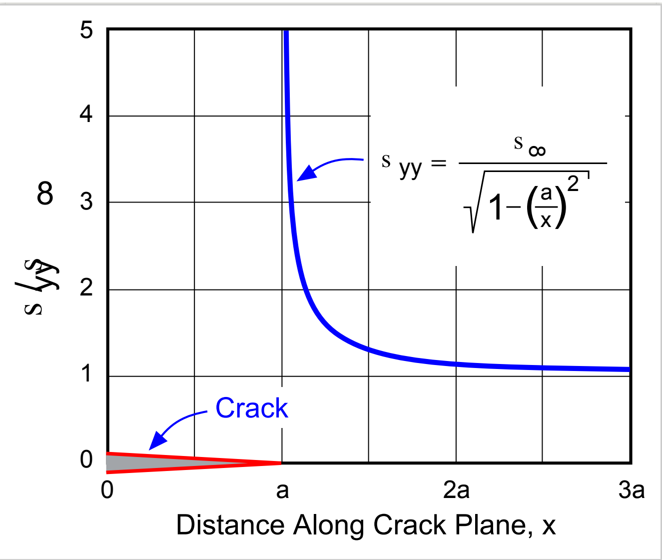
$$Z(z) = \frac{\sigma_\infty}{\sqrt{1 - \left(\frac{a}{z}\right)^2}}$$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = \operatorname{Re} Z - y \operatorname{Im} Z'$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = \operatorname{Re} Z + y \operatorname{Im} Z'$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -y \operatorname{Re} Z'$$

where a is the crack length and $z = x + iy$



For the configuration chosen by Westergaard

$$\sigma_{ij} = \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta)$$

$$u_i = \frac{K}{2\mu}\sqrt{\frac{r}{2\pi}}\, g(\theta)$$

Two decades later, Irwin showed that the solution could be simplified in the area immediately surrounding the crack tip, and invented the **stress intensity factor**

Table 2.1 Stress and displacement fields ahead a crack tip for modes I, II, III

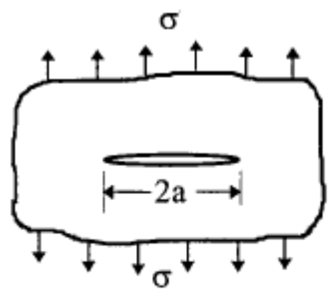
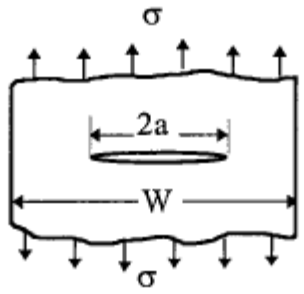
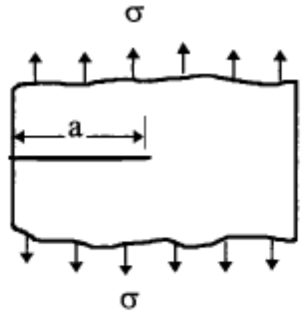
	Mode I	Mode II	Mode III
σ_{xx}	$\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$	$-\frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\left[2-\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right]$	0
σ_{yy}	$\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$	$\frac{K_{II}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}$	0
τ_{xy}	$\frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$	$\frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$	0
σ_{zz}	$\begin{cases} 0 & \text{plane stress} \\ \nu(\sigma_{xx}+\sigma_{yy}) & \text{plane strain} \end{cases}$	$\begin{cases} 0 & \text{plane stress} \\ \nu(\sigma_{xx}+\sigma_{yy}) & \text{plane strain} \end{cases}$	0
τ_{xz}	0	0	$-\frac{K_{III}}{\sqrt{2\pi r}}\sin\frac{\theta}{2}$
τ_{yz}	0	0	$\frac{K_{III}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}$
u_x	$\frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left[\kappa-1+2\sin^2\frac{\theta}{2}\right]$	$\frac{K_{II}}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left[\kappa+1+2\cos^2\frac{\theta}{2}\right]$	0
u_y	$\frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}\left[\kappa+1-2\cos^2\frac{\theta}{2}\right]$	$\frac{K_{II}}{2\mu}\sqrt{\frac{r}{2\pi}}\cos\frac{\theta}{2}\left[\kappa-1-2\sin^2\frac{\theta}{2}\right]$	0
u_z	$\begin{cases} -\frac{\nu z}{E}(\sigma_{xx}+\sigma_{yy}) & \text{plane stress} \\ 0 & \text{plane strain} \end{cases}$	$\begin{cases} -\frac{\nu z}{E}(\sigma_{xx}+\sigma_{yy}) & \text{plane stress} \\ 0 & \text{plane strain} \end{cases}$	$\frac{K_{III}}{\mu}\sqrt{\frac{r}{2\pi}}\sin\frac{\theta}{2}$

μ is the shear modulus, $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress

In general the stress intensity factor K, depends on the applied stress, crack size, and the geometry

$$K = Y\sigma\sqrt{\pi a}$$

Table 2.3 Stress intensity factors

Geometry	Stress Intensity Factor
1. Crack in an infinite body 	$K_I = \sigma\sqrt{\pi a}$
2. Centre crack in a strip of finite width 	$K_I = \sqrt{\sec\frac{\pi a}{W}}\,\sigma\sqrt{\pi a}$
3. Edge crack in a semi-infinite body 	$K_I = 1.12\,\sigma\sqrt{\pi a}$