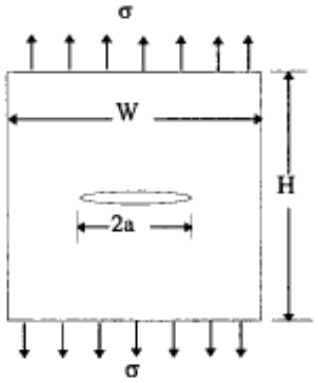
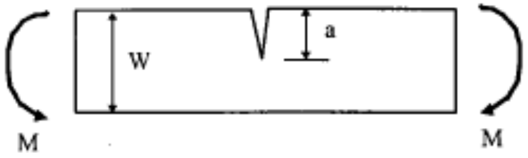
4. Centre crack in a finite width strip



$$K_I = f\left(\frac{a}{W}\right)\sigma\sqrt{\pi a}$$

a/W	f(a/W)	
	h/W = 1.0	h/W = ∞
0	1.12	1.12
0.2	1.37	1.21
0.4	2.11	1.35
0.5	2.83	1.46

5. Edge crack in a beam of width B subjected to bending



$$K_I = f\left(\frac{a}{W}\right)\sigma\sqrt{\pi a} \quad \text{where} \quad \sigma = \frac{6M}{BW^2}$$

a/W	f(a/W)
0.1	1.044
0.2	1.055
0.3	1.125
0.4	1.257
0.5	1.500
0.6	1.915

6. Thin-section (plane stress) double split beam

$$K_I = 2\sqrt{3}\,\frac{Pa}{c^{3/2}}$$

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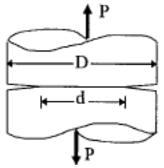
7. Circumferentially notched rod



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$$K_I = \frac{0.932P\sqrt{D}}{\sqrt{\pi d^2}} \quad \text{for} \quad 1.2 \leq \frac{D}{d} < 2.1$$

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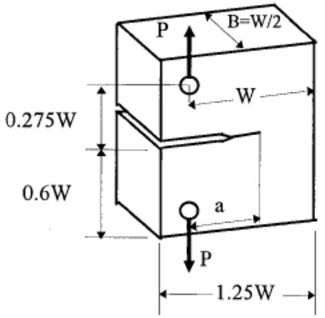
8. Compact tension specimen (CST)



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$$K_I = Y\frac{P\sqrt{\pi}}{B\sqrt{W}}$$

$$Y = 16.7\left(\frac{a}{W}\right)^{1/2} - 104.7\left(\frac{a}{W}\right)^{3/2} + 369.9\left(\frac{a}{W}\right)^{5/2} - 573.8\left(\frac{a}{W}\right)^{7/2} + 360.5\left(\frac{a}{W}\right)^{9/2}$$

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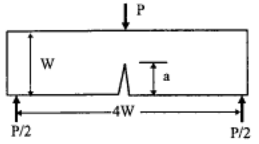
9. Single-edge notch bend (SENB), thickness B



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$$B = W/2$$

$$K_I = Y\frac{4P\sqrt{\pi}}{B\sqrt{W}}$$

$$Y = 1.63\left(\frac{a}{W}\right)^{1/2} - 2.6\left(\frac{a}{W}\right)^{3/2} + 12.3\left(\frac{a}{W}\right)^{5/2} - 21.3\left(\frac{a}{W}\right)^{7/2} + 21.9\left(\frac{a}{W}\right)^{9/2}$$

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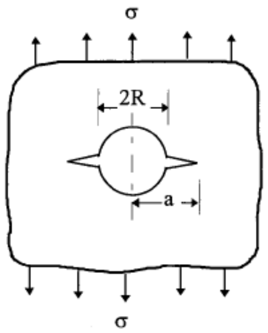
10. Crack emanating from a hole in an infinite body



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$$K_I = f\left(\frac{a}{R}\right)\sigma\sqrt{\pi a}$$

<table>
<tr><th>a/R</th><th>f(a/R)</th></tr>
<tr><td>1.01</td><td>0.3256</td></tr>
<tr><td>1.02</td><td>0.4514</td></tr>
<tr><td>1.04</td><td>0.6082</td></tr>
<tr><td>1.06</td><td>0.7104</td></tr>
<tr><td>1.08</td><td>0.7843</td></tr>
<tr><td>1.10</td><td>0.8400</td></tr>
<tr><td>1.20</td><td>0.9851</td></tr>
<tr><td>1.25</td><td>1.0168</td></tr>
<tr><td>1.30</td><td>1.0358</td></tr>
<tr><td>1.40</td><td>1.0536</td></tr>
<tr><td>1.80</td><td>1.0495</td></tr>
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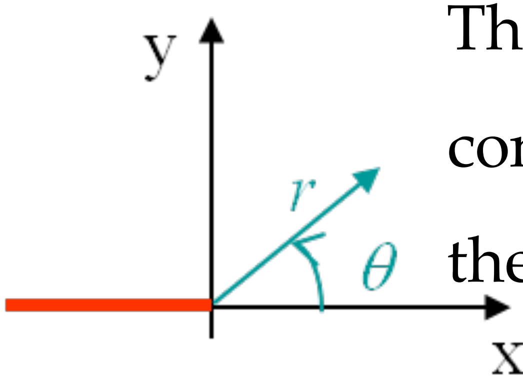


The only things that differentiate the three modes are the boundary conditions. The solution is obtained in $\sqrt{r}$, where r is the distance to the crack tip, and for Mode I:

$$\begin{cases} \sigma_{xx} = \frac{C}{\sqrt{r}}\cos\frac{\theta}{2}\left[1-\sin\frac{3\theta}{2}\sin\frac{\theta}{2}\right]+\mathcal{O}\left(r^0\right) \\ \boxed{\sigma_{yy} = \frac{C}{\sqrt{r}}\cos\frac{\theta}{2}\left[1+\sin\frac{3\theta}{2}\sin\frac{\theta}{2}\right]}+\mathcal{O}\left(r^0\right) \\ \sigma_{xy} = \frac{C}{\sqrt{r}}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\sin\frac{\theta}{2}+\mathcal{O}\left(r^0\right) \\ \mathbf{u}_x = \frac{C(1+\nu)}{E}\sqrt{r}\cos\left(\frac{\theta}{2}\right)\left[\kappa-1+2\sin^2\left(\frac{\theta}{2}\right)\right]+\mathcal{O}(r) \\ \mathbf{u}_y = \frac{C(1+\nu)}{E}\sqrt{r}\sin\left(\frac{\theta}{2}\right)\left[\kappa+1-2\cos^2\left(\frac{\theta}{2}\right)\right]+\mathcal{O}(r) \end{cases},$$

where κ is an expression of the Poisson ratio ν and is equal to ((3- ν)/(1+ ν)) for the plane stress state and to (3 - 4 ν) for the plane strain state.

Clearly the dominant term near the crack tip is in C/$\sqrt{r}$ meaning there is a singularity at the crack tip. Thus the value of the stress cannot be used to determine whether the crack will propagate or not. **The idea of Irwin was thus to consider "how fast the stress tends to infinity" near the crack tip**. **To do so he has defined the stress intensity factor (SIF) to circumvent the singularity of the solution**

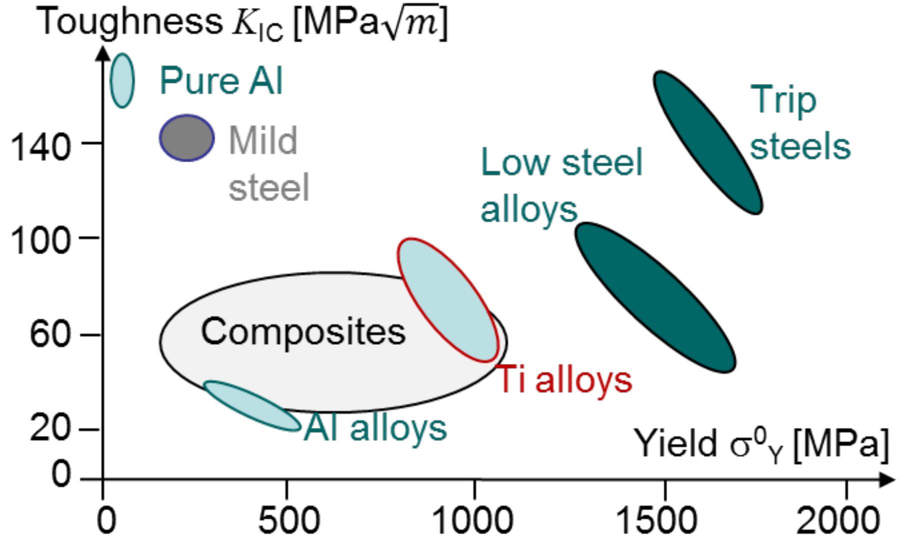
stress intensity factor (SIF)

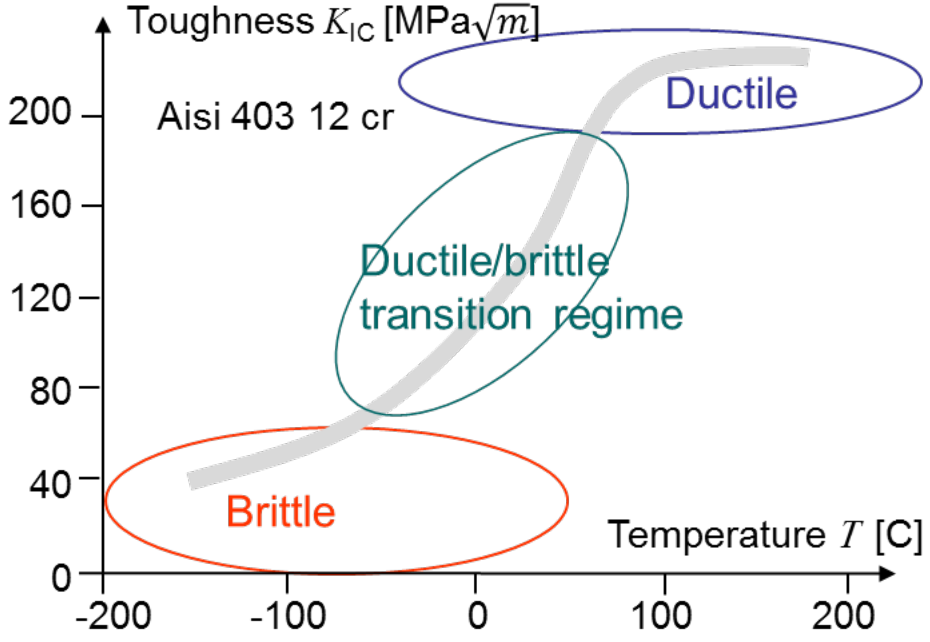
$$\begin{cases} K_I = \lim_{r\to 0}\left(\sqrt{2\pi r}\,\sigma_{yy}^{\text{mode I}}|_{\theta=0}\right) = C\sqrt{2\pi} \\ K_{II} = \lim_{r\to 0}\left(\sqrt{2\pi r}\,\sigma_{xy}^{\text{mode II}}|_{\theta=0}\right) = C\sqrt{2\pi} \\ K_{III} = \lim_{r\to 0}\left(\sqrt{2\pi r}\,\sigma_{yz}^{\text{mode III}}|_{\theta=0}\right) = C\sqrt{2\pi} \end{cases}.$$

For a given loading mode, this SIF, expressed in MPa √m, characterizes the stress evolution near the crack tip:

$$\begin{cases} \sigma^{\text{mode i}} = \dfrac{K_i}{\sqrt{2\pi r}}\mathbf{f}^{\text{mode i}}(\theta) \\ \mathbf{u}^{\text{mode i}} = K_i\sqrt{\dfrac{r}{2\pi}}\mathbf{g}^{\text{mode i}}(\theta) \end{cases},$$

where **f** and **g** are functions defined for each mode but independent of the loading and geometry (as long as we consider the asymptotic value). The loading and geometry effects are thus fully reported to the value of the SIF. Irwin has thus the idea to consider the value of the SIF to detect the crack propagation. *Indeed experiments have shown that for a given material, which obeys to the LEFM assumption, the crack propagates if the SIF reaches a threshold K_C called the toughness of the material.*





Therefore, one can write the crack propagation criterion in mode I as:

$K_I < K_{IC} \rightarrow$ the crack does not propagate,

$K_I > K_{IC} \rightarrow$ the crack does propagate,

where K_{IC} is the mode I toughness.

Remember, under the LEFM assumption: ***K_I*** **depends on the geometry and loading conditions only**, **K_{IC} depends on the material only**.

Materials for which the toughness is lower than 30 to 40 MPa √m are considered brittle materials.

Ductile materials have a higher toughness, but usually do not satisfy to the LEFM assumption as their behavior is no longer elastic. Some materials have a brittle behavior at low temperature and a ductile behavior at high temperature. For such materials the toughness depends on the temperature and there exists a transition region

Energy Release Rate G and Compliance

The energy release rate G provides a powerful tool for studying fracture problems of cracked bodies from a global view.

Let us consider the load displacement curve for a cracked specimen made of *linear elastic* media as shown

When the crack has length a , the specimen is less compliant than when the crack has length a + δa.

The compliance C of the specimen is the displacement per unit load, i.e the reciprocal of stiffness.

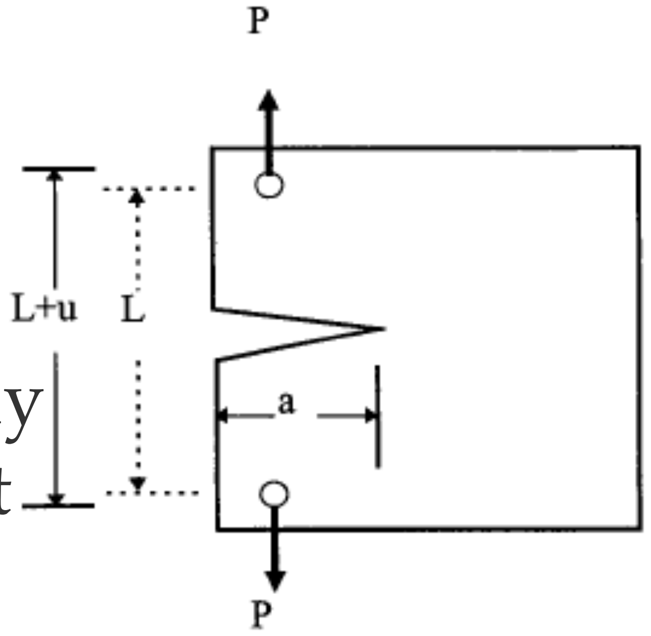
In general we may write

$C = u/P$

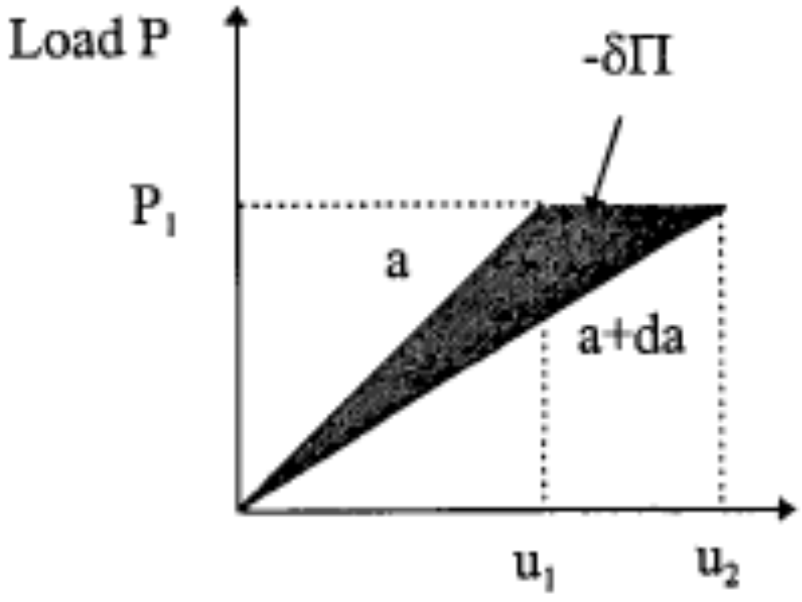
which is a geometry constant,

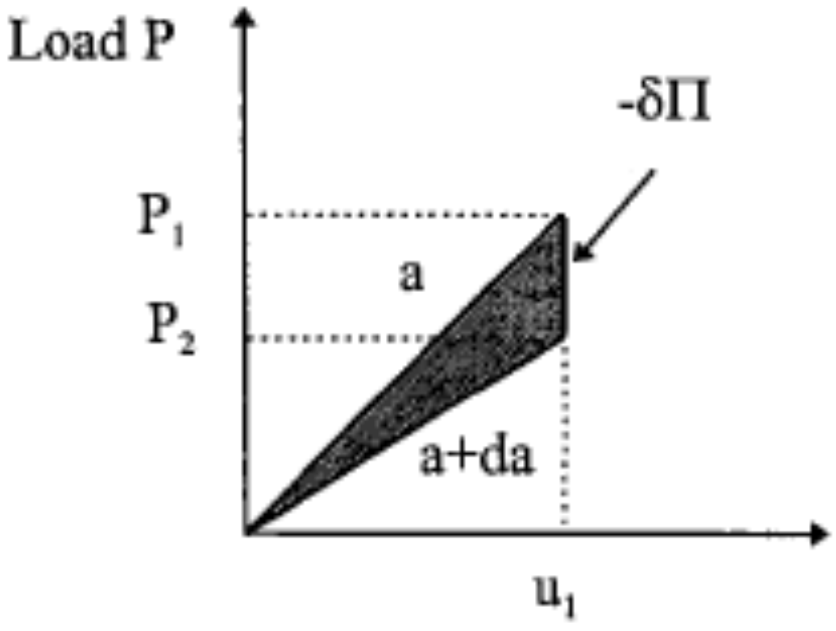
dependent on crack length and dimensions of the body

The displacement **u** refers to the relative displacement measured between the loading points.



A cracked body may be subjected to loads or displacement, or a combination of both. Let's consider two extreme cases: **constant load** and **constant displacement** or "fixed grip" condition, separately.



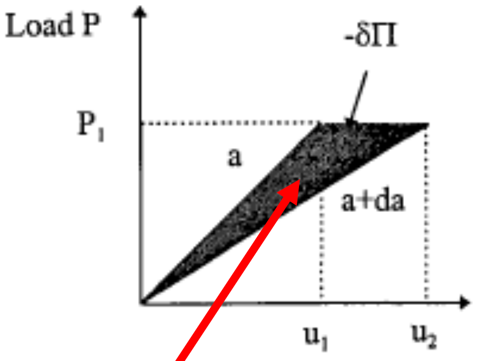


Constant Load Conditions

The potential energy in the specimen is the area above the load-displacement curve (the area below the load versus displacement curve is the strain energy stored in the specimen while the area of the rectangle is the work done by external force).

The potential energy change δΠ is the difference between the external work done and the stored but recoverable elastic strain energy. The energy stored in the specimen for a crack of length a + δa is greater than in the situation when the crack was length a , the increase being

$$\delta U_E = \frac{1}{2}P_1 u_2 - \frac{1}{2}P_1 u_1$$



However, to attain this stored energy the load has moved a distance $u_2 - u_1$ and so the work done by the external applied load is $\delta W = P_1(u_2 - u_1)$

The amount of the energy that appears to have "vanished" is equal to

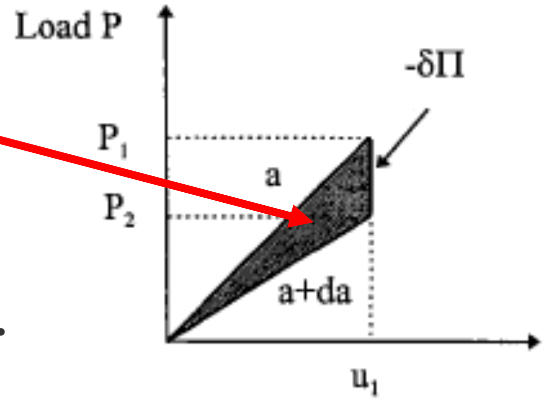
$$-\delta\Pi = \delta W - \delta U_E = P_1(u_2 - u_1) - \frac{1}{2}P_1(u_2 - u_1) = \frac{1}{2}P_1(u_2 - u_1) = \frac{1}{2}P_1\delta u$$

equal to the energy spent in increasing crack surfaces.

Constant Displacement Condition

Under fixed grip condition, an increase in crack length causes a decrease in stored elastic strain energy given by

$$\delta U_E = \frac{1}{2}(P_1 - P_2)u_1 = \frac{1}{2}u_1\delta P$$



Constant load requires a potential energy release rate of **1/2 P δu** .
Fixed-grip condition requires a potential energy release of **1/2 u δP** .

The compliance of C is the same for both cases, which is the same as stating that the difference between the two shaded areas tends to zero. In other words, δu = C δP , and the release of energy for crack extension in both cases is given by

$$\frac{1}{2}CP\delta P$$

Therefore the strain (or potential) energy release rate (with respect to crack length) for small crack extension δA can be found experimentally in a plate of uniform thickness B as

$$G = \frac{P}{2}\frac{\delta u}{\delta A} = \frac{P^2}{2}\frac{\delta C}{\delta A}$$

For a double cantilever beam (DCB) with a $\gg$ 2h and l $\gg$ 2h, determine the strain energy release rate G

The two arms of the DCB may be considered to a first approximation as cantilevers.

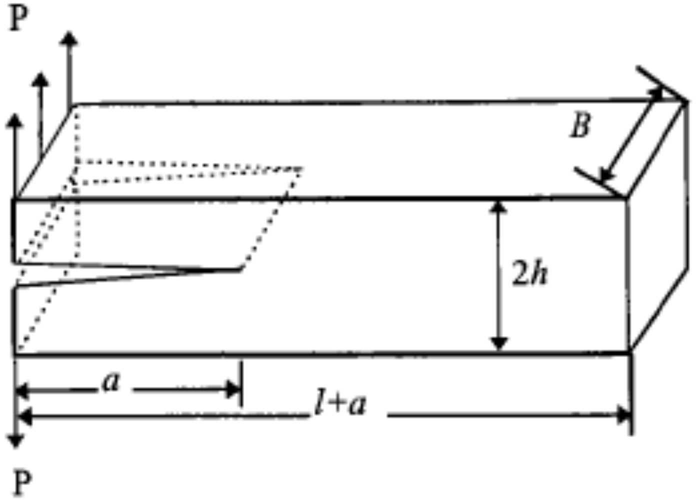


Fig.2.6 Double cantilever beam

Method 1: The displacement at the loading point is

$$u = \frac{Pa^3}{3EI} \quad \text{where} \quad I = \frac{Bh^3}{12}$$

hence the relative displacement of the two points of load application is

$$v = 2u = \frac{8Pa^3}{EBh^3}$$

thus the compliance of the specimen is $C = \frac{v}{P} = \frac{8a^3}{EBh^3}$

It follows that the energy release rate G is

$$G = \frac{P^2}{2B}\frac{\partial C}{\partial a} = \frac{12P^2a^2}{EB^2h^3}$$