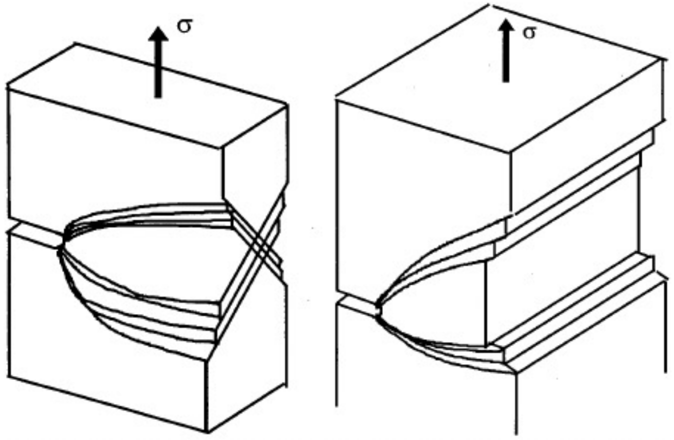
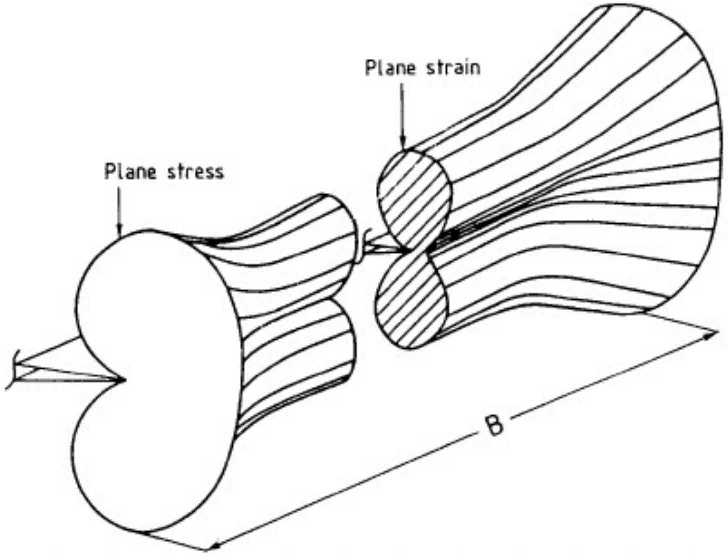
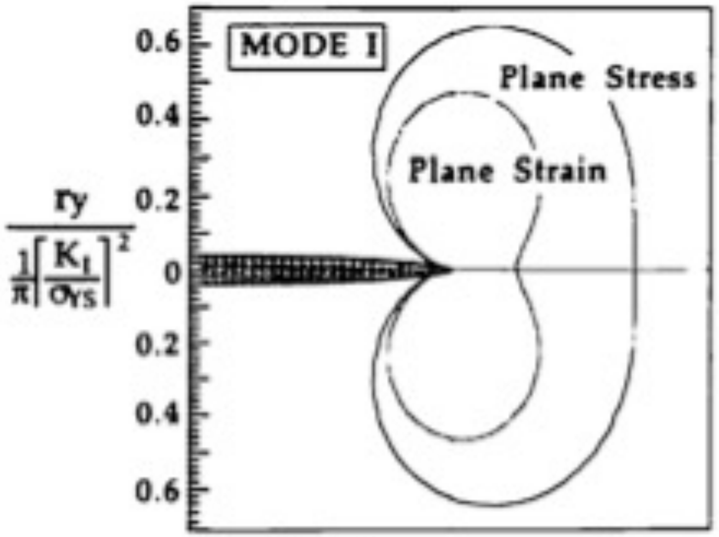
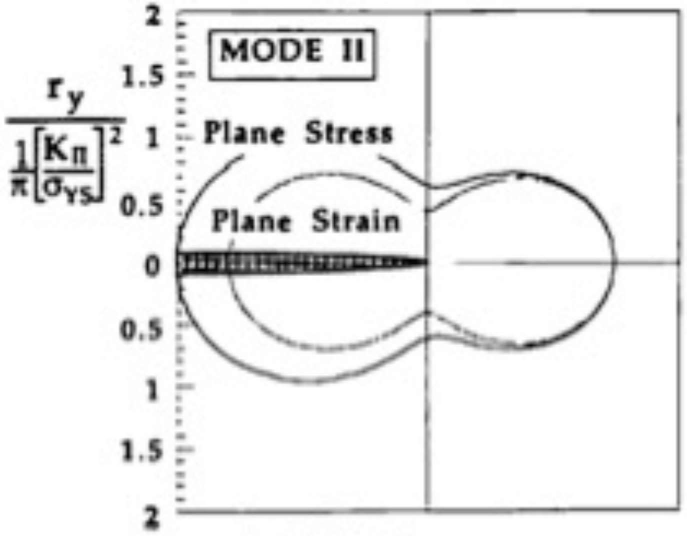
The three dimensional slip planes of a mode I crack are shown in Figure for plane stress and plane strain. For a finite plate, due the free surface effect, the plastic zone looks like a "dog-bone». Due to this thickness effect, the plastic constraint factor normally lies between 1 and 3, for example $\alpha = 1.7$. ***It is important to point out that although the plastic zone at the middle of the plate is smaller than that near the surface, the high triaxial stress that exists at the middle of the plate (this is sometimes called plastic constraint) causes crack growth to occur there first, under both static and fatigue conditions.***



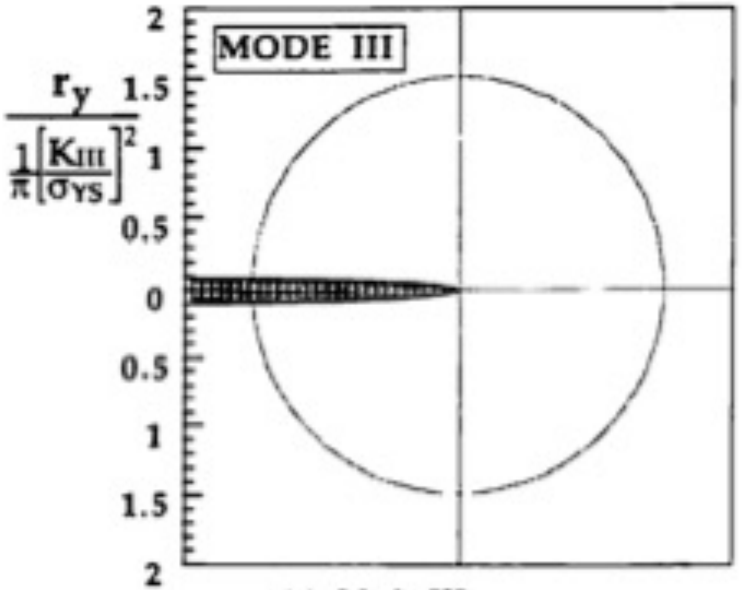




(a) Mode I



(b) Mode II



(c) Mode III

FIGURE 2.34 Crack tip plastic zone shapes estimated from the elastic solutions (Tables 2.1 and 2.3) and the von Mises yield criterion.

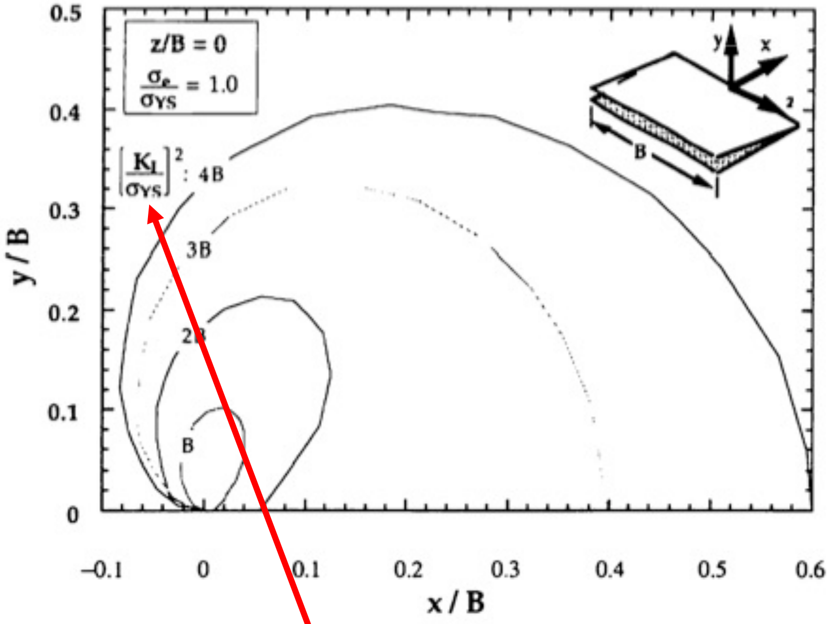


Figure 2.40 Effect of K_I, relative to thickness, of the plastic zone size and shape [29].

$$r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_{ys}}\right)^2$$

Residual Strength and Critical Crack Size

Failure will occur when K = K_c , the residual strength of a cracked component is therefore

$$\sigma_c = \frac{K_c}{Y\sqrt{\pi a}}$$

The size of the crack at this stress is called the "critical crack size".
It should be pointed that this equation is valid only when linear fracture mechanics is applicable, **that is the net stress level is far below the material's yield stress**. ***Otherwise the component will fail in a different mode: plastic collapse***.

Consider a centre cracked panel with a finite width W , the absolute highest load carrying capability is bounded by the plastic collapse strength: the stress level over the entire section exceeds the yield or ultimate tensile strength of the material. It is easy to show that the nominal stress at collapse is

$$\sigma_{pc} = \frac{W - 2a}{W}\sigma_{ys}$$

When this happens, fracture will occur, regardless of the fracture toughness.

Considering a centre cracked panel, there are three situations in which a plastic collapse failure would prevail:

(1) the toughness is very high;
(2) the crack is very small;
(3) the width W is very small.

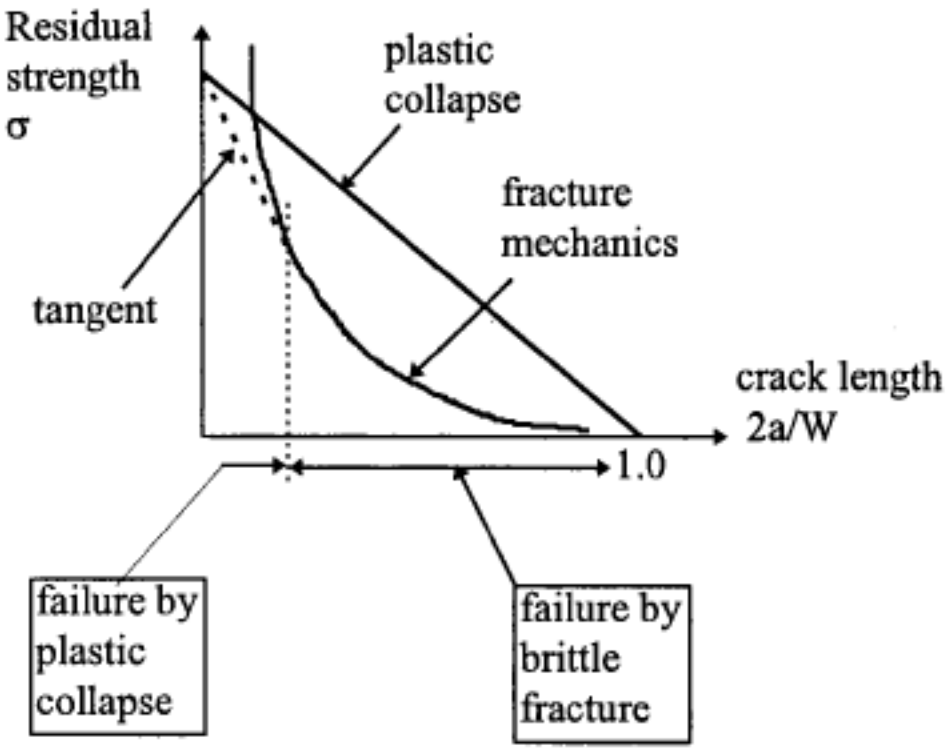


Fig.4.2 Competition between fracture and collapse

The intersection of the two curves is given by

$$\frac{W-2a}{W}\sigma_{ys} > \frac{K_c}{\sqrt{\pi a}\sqrt{\sec(\pi a/W)}}$$

Estimate the failure load under uniaxial tension for a centre-cracked panel of aluminium alloy of width W = 500 mm, and thickness B = 4 mm, for the following values of crack length 2a = 20 mm and 2a = 100 mm.

Yield stress σ_y = 350 MPa and fracture toughness K_{IC} = 70 MPa√m.

Solution

There are two possible failure modes: **plastic collapse** and **brittle fracture**. We will assess the load level required for each mode to prevail.

(i) **2a = 20 mm**.

Plastic collapse load $F_{pc} = \sigma_{ys} \cdot (W - 2a) \cdot B$ = **672 kN**

Fracture load $F_c = \sigma_c \cdot W \cdot B$ where $\sigma_c = K_{IC} / \sqrt{\pi a \sec(\pi a/W)}$ = 394.6 MPa

thus F_c = **790 kN**.

The actual failure load is the smaller of the above results, 672 kN.

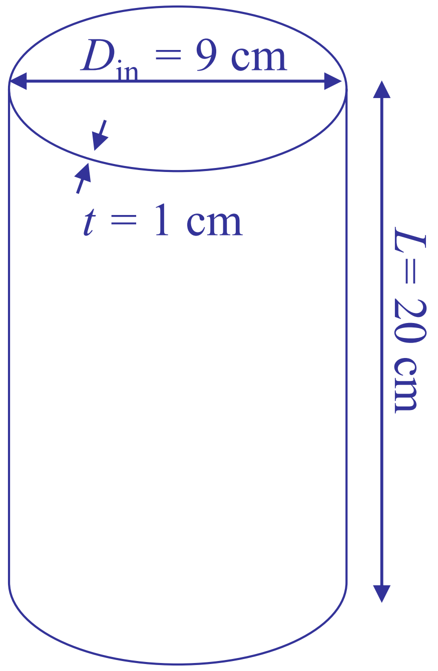
2a = 100 mm

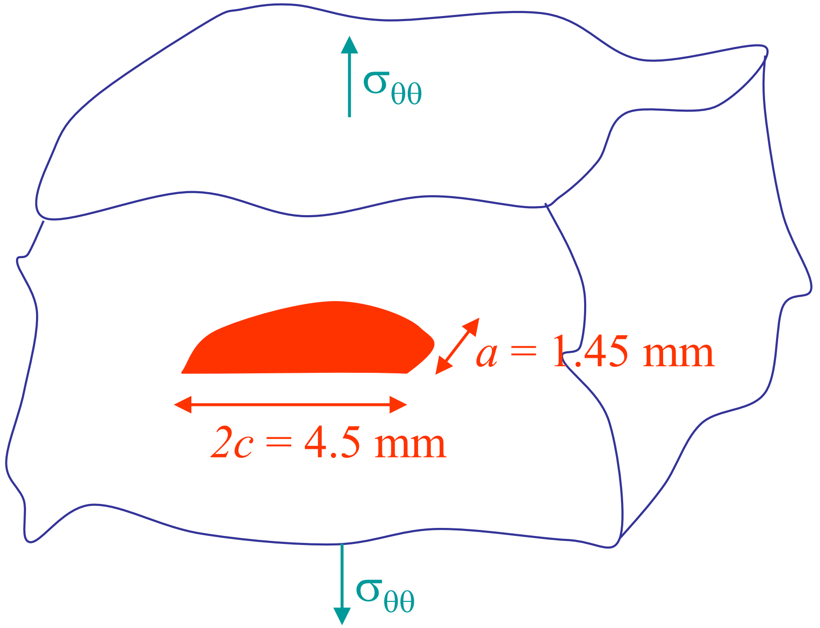
Plastic collapse load $F_{pc} = \sigma_{ys} \cdot (W - 2a) \cdot B$ = **560 kN**

Fracture load $F_c = \sigma_c \cdot W \cdot B$ where $\sigma_c = K_{IC} / \sqrt{\pi a} \sec(\pi a/W)$ = 172.2 MPa
thus F_c = **334.57 kN**

The actual failure load is the smaller of the above results, **334.57 kN.**

- Flawed cylinder
  - A piston is used to increase inner pressure
    - From 0 to 55 MPa
  - Cylinder made of
    - Peaked-aged aluminum alloy
      - 7075-T651
    - Yield σ_p^0 = 550 Mpa
    - Toughness K_{IC} = 30 MPa m$^{1/2}$
  - Malfunction
    - Cylinder burst
    - Post failure analyses
      - Initial elliptical flaw at inner wall
        - » 4.5 mm long
        - » 1.45 mm deep
        - » Normal to hoop stress
  - Origin of burst?



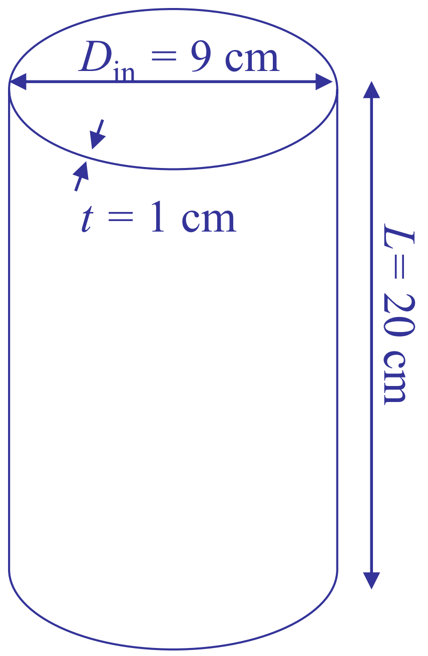


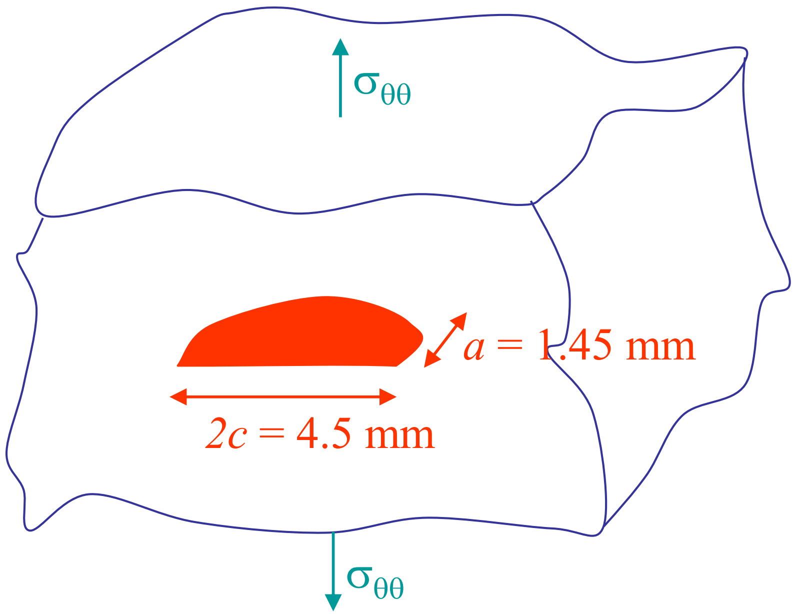
Exercise 1: Solution

- Stress field
  - Consider thick cylinder with
    - r_{in} = 0.045 m & r_{out} = 0.055 m
    - Inner pressure p

$$\begin{cases} \sigma_{rr}(r) = \dfrac{r_{\text{in}}^2 p}{r_{\text{out}}^2 - r_{\text{in}}^2}\left(1 - \dfrac{r_{\text{out}}^2}{r^2}\right) \\ \sigma_{\theta\theta}(r) = \dfrac{r_{\text{in}}^2 p}{r_{\text{out}}^2 - r_{\text{in}}^2}\left(1 + \dfrac{r_{\text{out}}^2}{r^2}\right) \end{cases}$$

$$\Rightarrow \begin{cases} \sigma_{rr}(r_{\text{in}} + a) = -0.808\, p \\ \sigma_{\theta\theta}(r_{\text{in}} + a) = 4.86\, p \end{cases}$$

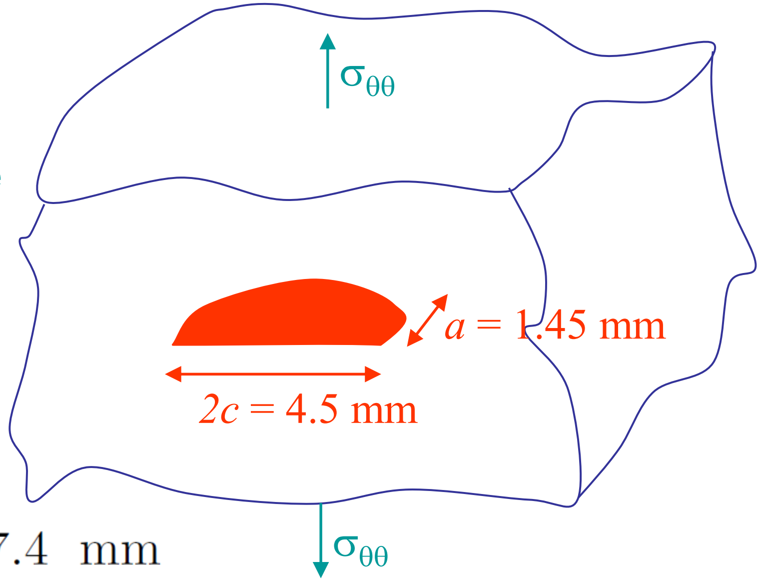




Exercise 1: Solution

- SIF
  - The wall is not perforated
    - Use SIF for semi-elliptical crack in large plate
    - See SIF handbook
    - Geometrical effect

$$\Psi \simeq \frac{3\pi}{8} + \frac{\pi}{8}\left(\frac{a}{c}\right)^2 = 1.3412$$



    - Plasticity correction
      - SSY criterion: $t, a > 2.5\left(\frac{K_{IC}}{\sigma_p^0}\right)^2 = 7.4\ \text{mm}$
      - Not fully satisfied ⟹ plasticity correction

$$Q = \Psi^2 - 0.212\frac{\sigma_{\theta\theta}^2}{\sigma_p^{0\,2}} = 1.7988 - 1.654\,10^{-5}\ \text{MPa}^{-2}\ p^2$$

    - SIF $K_I = 1.12\frac{\sigma_{\theta\theta}\sqrt{\pi a}}{\sqrt{Q}}$ with $\sigma_{\theta\theta}(r_{\text{in}} + a) = 4.86\ p$

$$\Longrightarrow K_I = 0.3672\sqrt{\text{m}}\ p\ \frac{1}{\sqrt{1.7988 - 1.654\,10^{-5}\ \text{MPa}^{-2}\ p^2}}$$

  - Rupture: $K_I\,(p = 104\ \text{MPa}) = 30\ \text{MPa}\sqrt{\text{m}} = K_{IC}$
    - For $r = r_{\text{in}}$, p would be 100 MPa, this is the critical value

Exercise 1: Solution

- Rupture mode
  - For $r = r_{in}$, p would be 100 Mpa, this is the critical value
  - This is out of the range of the piston activity
    - So rupture should come from fatigue
- Cyclic loading
  - p from 0 to 55 Mpa
    - Hoop stress from 0 to $\sigma_{\theta\theta}(r) = \frac{r_{in}^2 p}{r_{out}^2 - r_{in}^2}\left(1 + \frac{r_{out}^2}{r^2}\right)$

$$\Longrightarrow \sigma_{\theta\theta}(r = r_{in}, p = 55\,\text{MPa}) = 277.75\ \text{MPa}$$

  - SIF
    - Assuming a/c remains constant during crack propagation

$$\Psi \simeq \frac{3\pi}{8} + \frac{\pi}{8}\left(\frac{a}{c}\right)^2 = 1.3412$$

$$\Longrightarrow Q = \Psi^2 - 0.212\frac{\sigma_{\theta\theta}^2}{\sigma_p^{0\,2}} = 1.7988 - 0.054 = 1.745$$

$$\Longrightarrow K_{\max} = 1.12\frac{\boldsymbol{\sigma}_{\theta\theta}\sqrt{\pi a}}{\sqrt{Q}} = 417.42\ \text{MPa}\sqrt{a}$$

Exercise 1: Solution

- Cyclic loading
  - p from 0 to 55 Mpa

$$\Longrightarrow \quad K_{\min} = 0 \qquad \& \qquad K_{\max} = 1.12\frac{\sigma_{\theta\theta}\sqrt{\pi a}}{\sqrt{Q}} = 417.42\ \mathrm{MPa}\sqrt{a}$$

  - Due to initial flaw

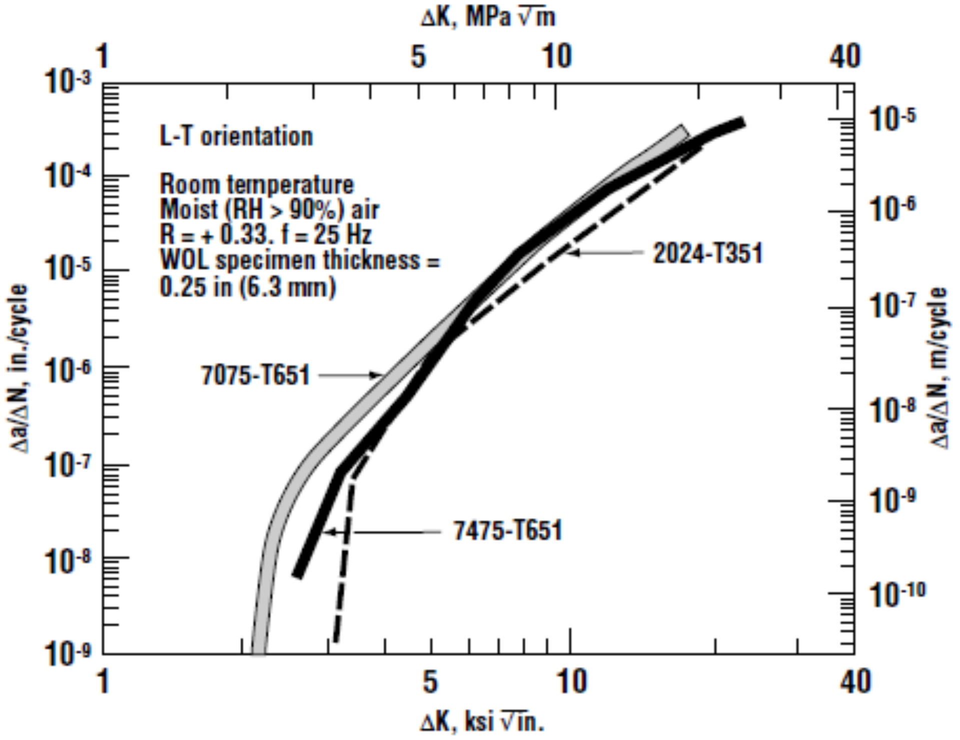
$$\Delta K\ (a = 0.00145\ \mathrm{m}) = 15.89\ \mathrm{MPa}\sqrt{\mathrm{m}}$$

    - Assuming curves are valid for R=0
    - We are in Paris regime

      $\Longrightarrow$ crack propagation

  - Rupture will happen for

$$K_{\max}\ (a = 0.00517\ \mathrm{m}) = 417.42\ \mathrm{MPa}\sqrt{0.00517}$$
$$= 30\ \mathrm{MPa}\sqrt{\mathrm{m}} = K_{IC}$$

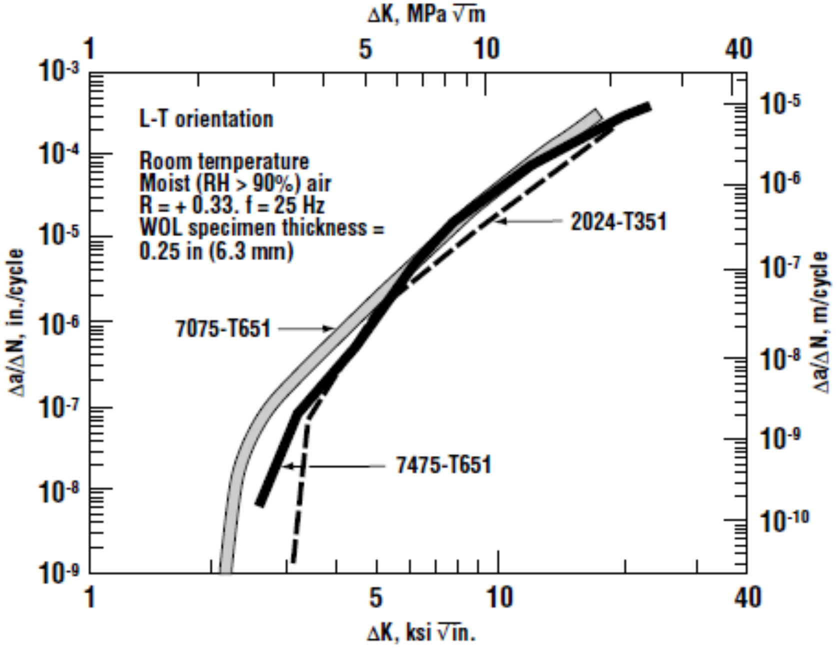


    - Number of cycles?

Exercise 1: Solution

Cyclic loading (2)

– As $\Delta K = 417.42 \text{ MPa}\sqrt{a}$



- Assuming curves are valid for R=0
- We are in Paris regime

$$\Longrightarrow \frac{da}{dN} = 3\ 10^{-11}\ \Delta K^{4.3}$$

$$\Longrightarrow \frac{da}{dN} = 3\ 10^{-11}\ 417.42^{4.3}\sqrt{a}^{4.3} = 5.567a^{2.15}$$

$$\Longrightarrow \frac{da}{a^{2.15}} = 5.567dN$$

$$\Longrightarrow -\frac{1}{1.15}\left(\frac{1}{a_{cr}^{1.15}} - \frac{1}{a_0^{1.15}}\right) = 5.567N_f$$

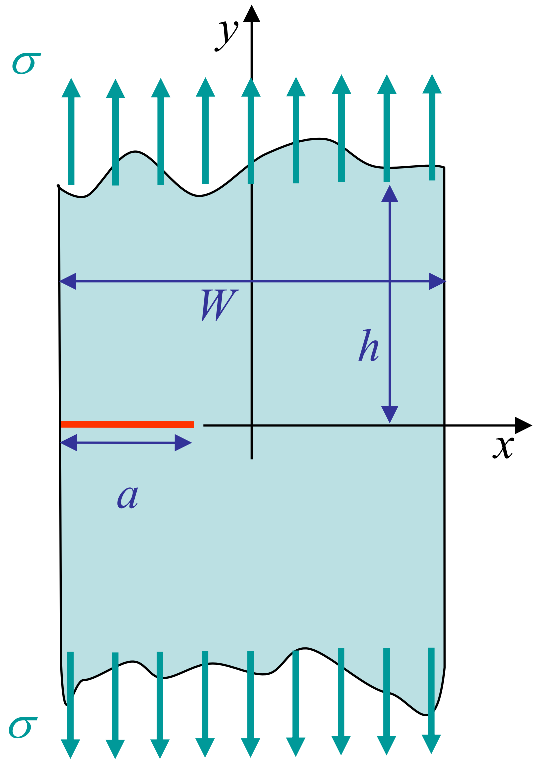
$$\Longrightarrow N_f = \frac{1}{1.15\ 5.567}\left(\frac{1}{0.00145^{1.15}} - \frac{1}{0.00517^{1.15}}\right) = 220 \text{ cycles}$$

- !!!Life strongly depends on the maximum pressure reached during accidents

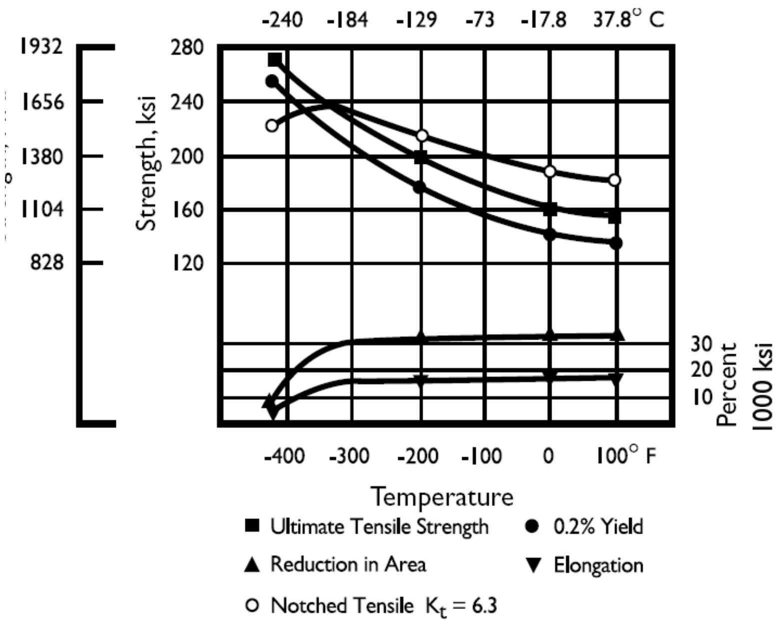
– Parameters in Paris law

Material	ΔK_{th} [MPa · m^{½}]	m [-]	C [m (MPa · m$^{1/2}$)$^{-m}$]
Mild steel	3.2-6.6	3.3	$0.24 \cdot 10^{-11}$
Structural steel	2.0-5.0	3.85-4.2	$0.07\text{-}0.11 \cdot 10^{-11}$
Structural steel is sea water	1.0-1.5	3.3	$1.6 \cdot 10^{-11}$
Aluminum	1.0-2.0	2.9	$4.56 \cdot 10^{-11}$
Aluminum alloy	1.0-2.0	2.6-2.9	$3\text{-}19 \cdot 10^{-11}$
Copper	1.8-2.8	3.9	$0.34 \cdot 10^{-11}$
Titanium alloy (6Al-4V, R=0.1)	2.0-3.0	3.22	$1 \cdot 10^{-11}$

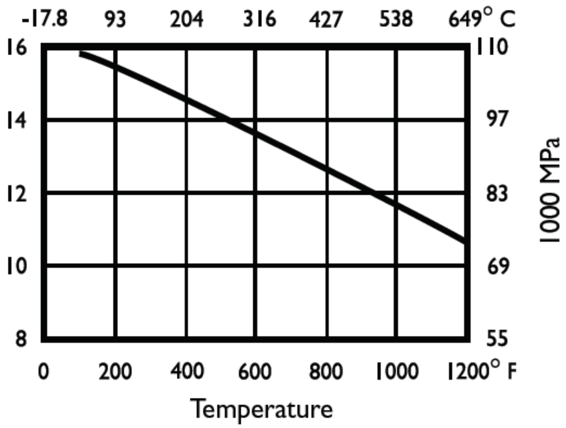
- Edge notch specimen under cyclic loading
  - Assume titanium alloy 6%Al - 4%V
    - See figures below
  - Assume a remains << W and << h
  - Initial crack size a = 1.5 cm
  - Cyclic loading between
    - Minimum value: 8 MPA
    - Maximum value: 80 MPa
  - What is the life of the structure?



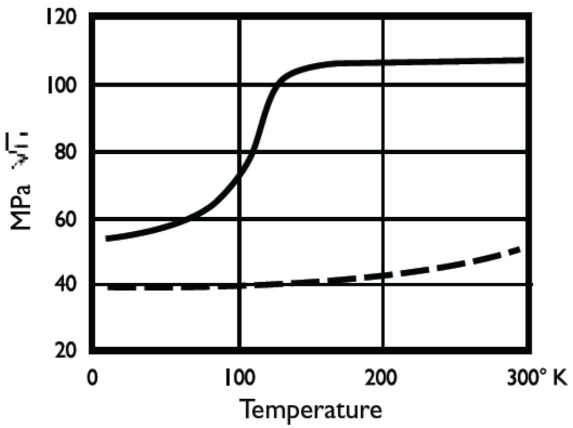
Tensile Properties vs. Temperature



Modulus of Elasticity



Fracture Toughness vs Temperature



Exercise 2: Solution

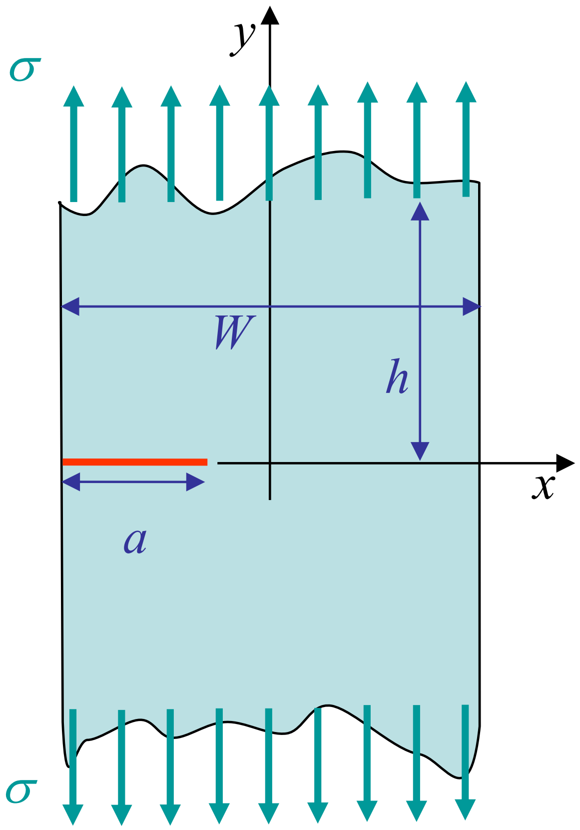
- Edge notch specimen under cyclic loading
  - Material properties at room temperature
    - Yield: 830 MPa
    - Toughness: 55 $\text{MPa} \cdot \text{m}^{1/2}$
  - SIF if a remains < 2% of W
    - $K_I = 1.122\, \sigma\, (\pi\, a)^{1/2}$
  - Plane strain & elastic fracture?
    - Yes if specimen thick enough

$$t > 2.5 \left(\frac{K_C}{\boldsymbol{\sigma}_p^0} \right)^2 = 2.5 \left(\frac{55}{830} \right)^2 \text{m} = 1.1\text{cm}$$

    - Crack is large enough

$$a > 2.5 \left(\frac{K_C}{\boldsymbol{\sigma}_p^0} \right)^2 = 2.5 \left(\frac{55}{830} \right)^2 \text{m} = 1.1\text{cm}$$

    - Moreover the applied stress << the yield stress



Exercise 2: Solution

- Edge notch specimen under cyclic loading (2)
  - Is the crack critical?

$$K_{I\,\mathrm{min},0} = 1.122\sigma_{\mathrm{min}}\sqrt{\pi a_0} = 1.95 \;\; \mathrm{MPa}\cdot\sqrt{\mathrm{m}} < K_C$$

$$K_{I\,\mathrm{max},0} = 1.122\sigma_{\mathrm{max}}\sqrt{\pi a_0} = 19.5 \;\; \mathrm{MPa}\cdot\sqrt{\mathrm{m}} < K_C$$

    - The crack will not lead to static failure
  - Fatigue?

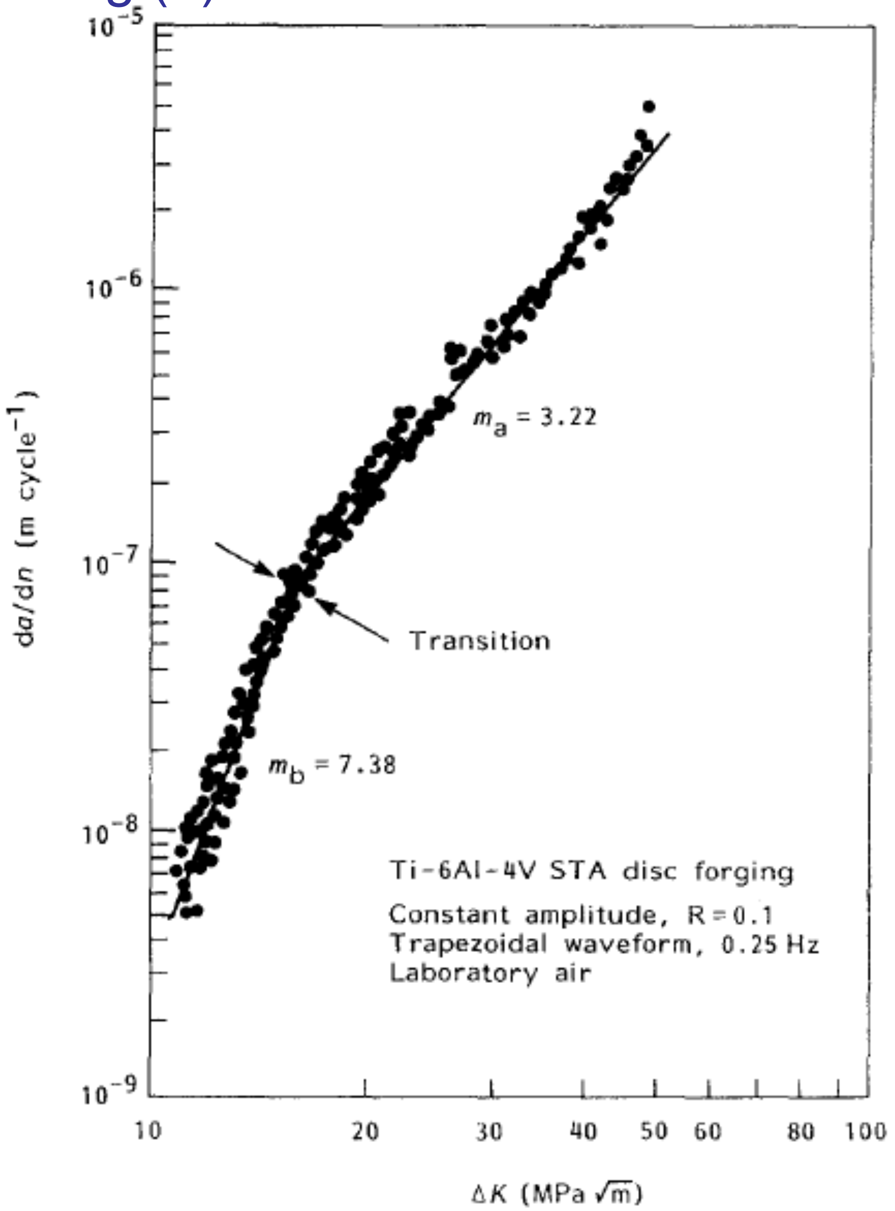
$$\Delta K_0 = 17.5 \; \mathrm{MPa}\cdot\sqrt{\mathrm{m}} > \Delta K_{\mathrm{th}}$$

    - As we are above the threshold there will be fatigue
    - $R = 0.1$, so crack experiences closing effect
  - What is the critical crack length leading to static failure?

$$a_f = \frac{1}{\pi}\left(\frac{K_C}{1.122\sigma_{\mathrm{max}}}\right)^2 = \frac{1}{\pi}\left(\frac{55}{1.122\,80}\right)^2 \mathrm{m} = 11.95\,\mathrm{cm}$$

Exercise 2: Solution

- Edge notch specimen under cyclic loading (3)
  - Parameters in Paris' law
    - There is a phase transition around $\Delta K = 17\ \text{MPa} \cdot \text{m}^{1/2}$ but we are above
    - Paris' coefficients:
      - $C = 10^{-11}\ \text{m}\ (\text{MPa} \cdot \text{m}^{1/2})^{-m}$
      - $m = 3.22$



Exercise 2: Solution

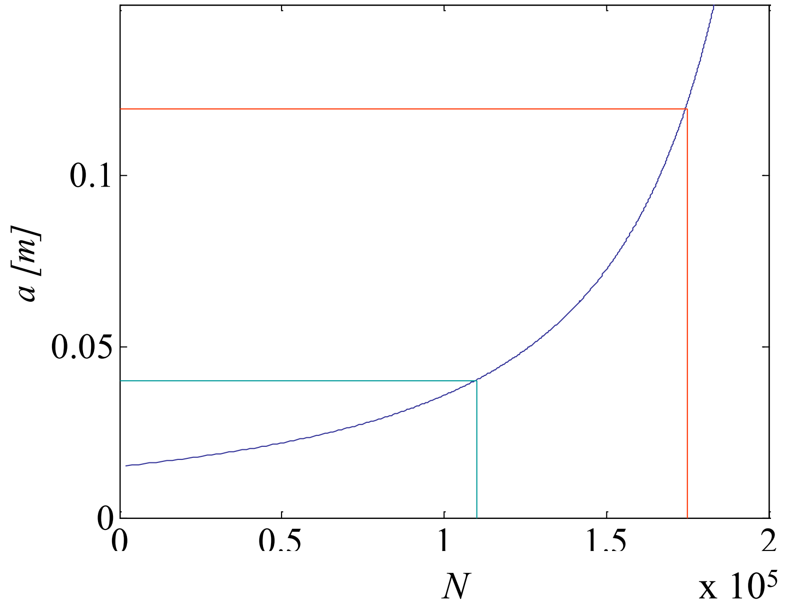
- Edge notch specimen under cyclic loading (4)
  - Paris' law $$\int_{a_0}^{a} \frac{da'}{C\left(\sigma_{\max}-\sigma_{\min}\right)^m 1.122^m \pi^{\frac{m}{2}} \left(a'\right)^{\frac{m}{2}}} = \int_0^N dN'$$
    - Can be integrated explicitly, and, for $m \neq 2$, it yields

$$N_f = \frac{1}{C\left(\sigma_{\max}-\sigma_{\min}\right)^m 1.122^m \pi^{\frac{m}{2}}} \left[\frac{2}{2-m}\left(a'\right)^{\frac{2-m}{2}}\right]_{a_0}^{a}$$

    - So the number of cycles in terms of the crack size is

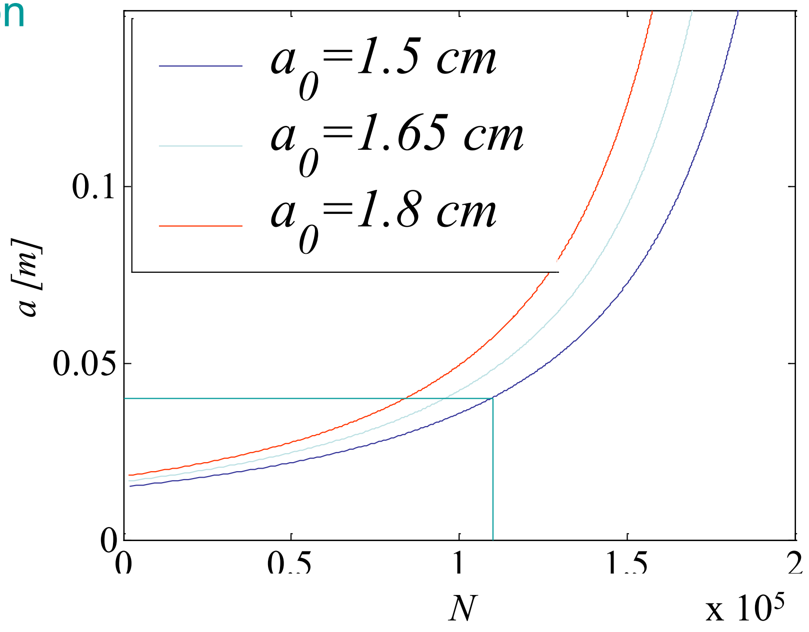
$$N_f = \frac{-1}{10^{-11}\, 72^{3.22}\, 1.122^{3.22}\, \pi^{1.61}\, 0.61} \left[(a)^{-0.61} - (a_0)^{-0.61}\right]$$

  - Critical size is reached after 1.74 10^5 cycles, but this value cannot be used as
    - Paris' law is not valid is zone III
    - After 1.5 10^5 cycles the crack is growing too quickly for allowing inspections
    - 1.1 10^5 is a conservative time life



Exercise 2: Solution

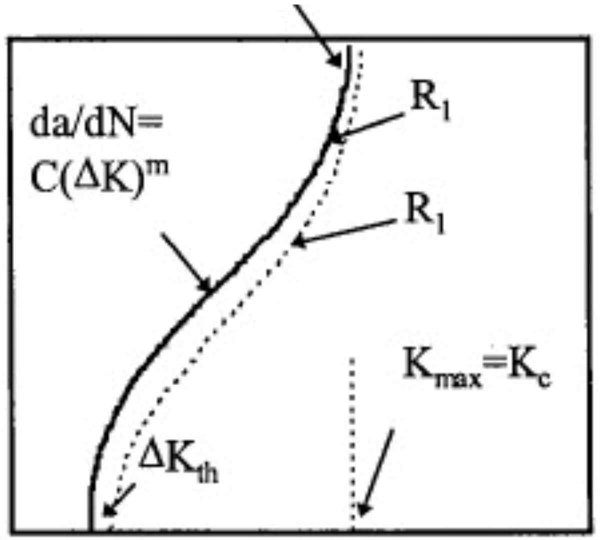
- Edge notch specimen under cyclic loading (5)
  - Some remarks $N_f = \frac{-1}{10^{-11}\, 72^{3.22}\, 1.122^{3.22}\, \pi^{1.61}\, 0.61}\left[(a)^{-0.61} - (a_0)^{-0.61}\right]$
    - If there is an error of 10% in the estimation of the initial crack length size
      - Life of the structure is reduced by 15%
      - Using a 1.5 10^5-cycle life prediction would actually lead to a crack size located in zone III
    - We have assumed $a < 0.02\ W$, which corresponds to a six-meter wide specimen. In practice
      - Crack size cannot be considered infinitely small
      - SIF must then be evaluated using
        - » Either FEM simulations
        - » Or SIF handbooks
      - Paris' law
        - » Cannot be integrated in a closed form
        - » Integration has to be performed numerically



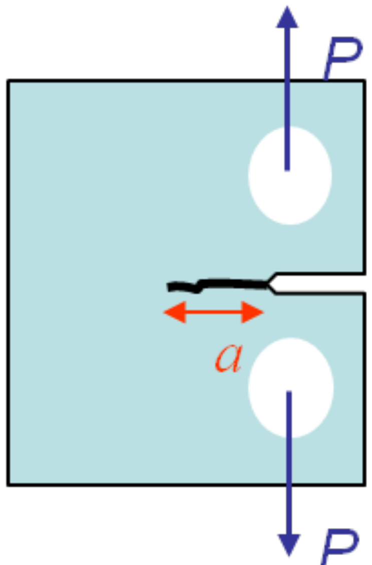
Fatigue Crack Growth

When a constant range of cyclic stress, $\Delta\sigma$ (= σ_{max} - σ_{min}), is applied to a cracked structure, stable fatigue crack growth can occur **at stress levels well below the yield stress**. In fact, the range of the stress intensity factor ΔK , where $\Delta K = K_{max} - K_{min}$ in a cycle may also be well below the materials fracture toughness K_{IC} . The reason for this is simple: the material near the crack tip is under severe plastic deformation.

Since the stress-strain field near a crack tip is uniquely determined by the stress intensity factor, fatigue crack growth rates can be correlated to ΔK and the figure shows a typical plot which can be divided into three zones; **threshold, stable crack growth and instability**.



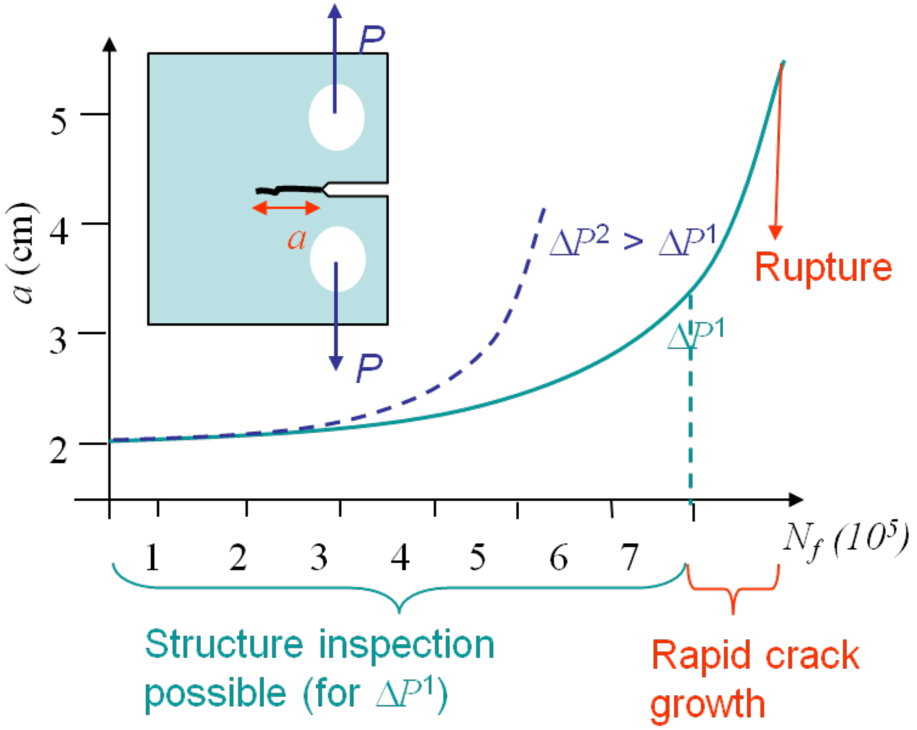
Stress intensity factor range



Let us consider a specimen under cyclic loading. The value of P ranges from P_{min} to P_{max} so that $\Delta P = P_{max} - P_{min}$. Due to the cyclic loading a crack nucleates at the stress concentration and propagates until failure of the specimen, although at the beginning the SIF remains lower than K_{IC}

Crack nucleation

The crack nucleates in a stress concentration area. During the loading of the sample, dislocations move along slip planes until reaching a free surface (or a bulk defect). **During the unloading phase, dislocations can move in the opposite direction but usually it happens on other slip planes**. After a few cycles one can observe the formation of **Persistent Slip Bands** (PSBs). These PSBs are the locations from which the crack can nucleate.



Fatigue crack growth: Stage I

Once the crack has nucleated, under the cyclic loading condition, it starts to growth along a slip plane of the crystal. The crack thus growths in a direction allowed by the crystallographic orientation.

Fatigue crack growth: Stage II

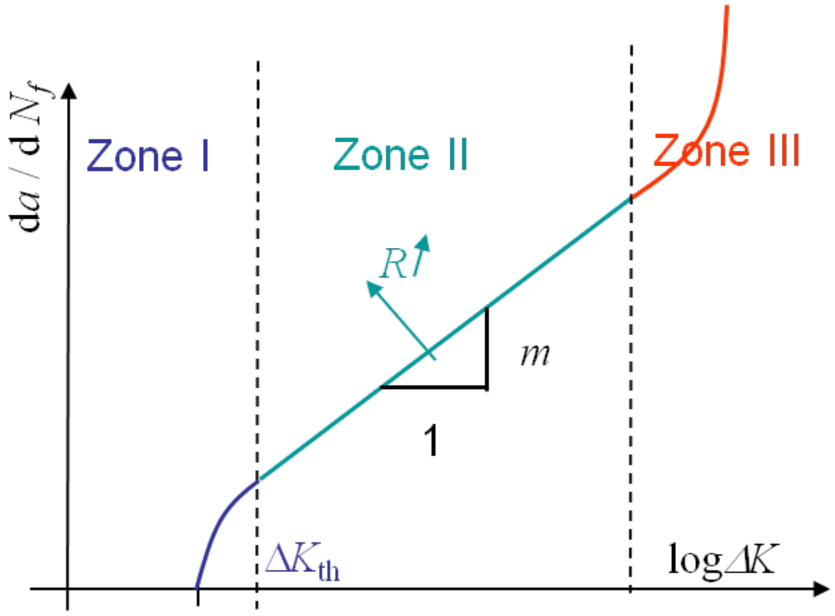
During the second stage in each grain the crack has to follow their crystallographic orientations. However when the crack size is larger than the size of a few grains, the crack appears as propagating at the macroscopic level in a direction governed by the maximum stress (straight for pure mode I).

This crack propagation stage is thus a macroscopic propagation stage.

Fatigue crack growth: Stage III

As the crack propagates under the cyclic loading its size increases. While at the beginning of the cyclic loading the SIF remains small compared the material toughness, as a increases the SIF also increases and, at maximum loading P_{max}, becomes close to the toughness K_{IC}. A fatigue fracture surface thus exhibits two surfaces: striation by fatigue in Stage II and brittle (or ductile) fracture in Stage III

Prediction of the structural life



As the crack loading remains small during fatigue problems, the **SSY assumption** usually holds during some intervals of the crack growth or even until failure, depending on the case.
As at the macro-scale the life of the structure with an initial crack size a is observed to depend on the loading ΔP and on $P_{max}P_{min}$ only, the SSY assumption allows us to say that the conditioning parameters are ΔK and $R = K_{max}K_{min}$.

Indeed from the SIFs equations we observed a linearity in ΔP of ΔK. Therefore the evolution of the crack size obeys to $(da/dN) = f\ (\Delta K, R)$
With the knowledge of this curve it is possible to determine for the number of cycles a structure can sustain before being replaced and to schedule the inspection intervals.

Crack propagation in Stage I

Experimentally, it has been observed that if $\Delta K < \Delta K_{th}$, **such a crack is considered as dormant.** **The value of ΔK_{th} is the fatigue threshold and depends on the material but also on the loading ratio R**.

If $\Delta K > \Delta K_{th}$, the crack will propagate until reaching the stage II. For steel, ΔK_{th} is between 2 and 5 MPa√m, but for steel in sea water, ΔK_{th} is between 1 and 1.5 MPa√m. **This means that the environmental conditions in which the material is considered are very important**.

Crack propagation in Stage II

This corresponds to the stage during which we can observe the striations. **The crack rate curve is linear with ΔK in a logarithmic scale.**

This is the Paris-Erdogan (1963) law:

$$da/dN = C\,\Delta K^m$$

This law is characterized by two parameters C and m which depend on the material and on the loading ratio R.

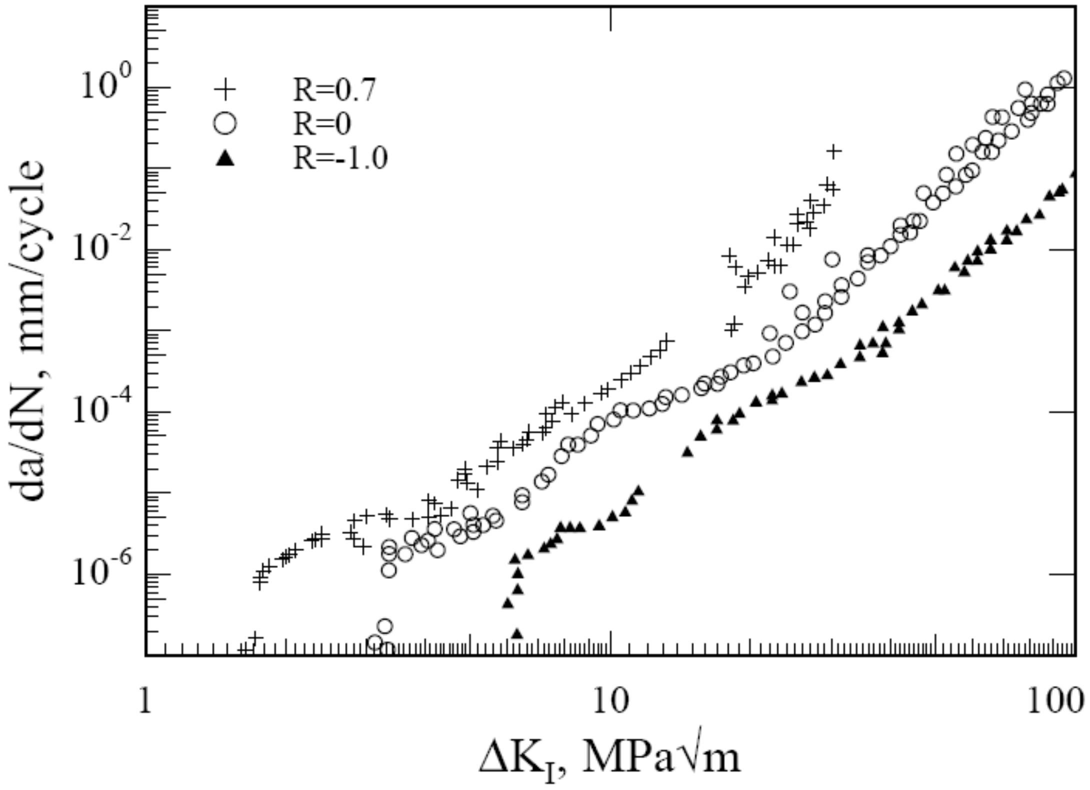
For steel, $C \approx 0.1 \times 10^{-11}\ m \cdot (MPa \cdot \sqrt{m})^{-m}$ and $m \approx 4$. For steel in sea water, these values change a lot: $C \approx 1.6 \times 10^{-11}\ m \cdot (MPa \cdot \sqrt{m})^{-m}$ and $m \approx 3.3$. **This means that the environmental conditions in which the material is considered are very important**. Note that the units of C are cumbersome as they depend on the coefficient m.

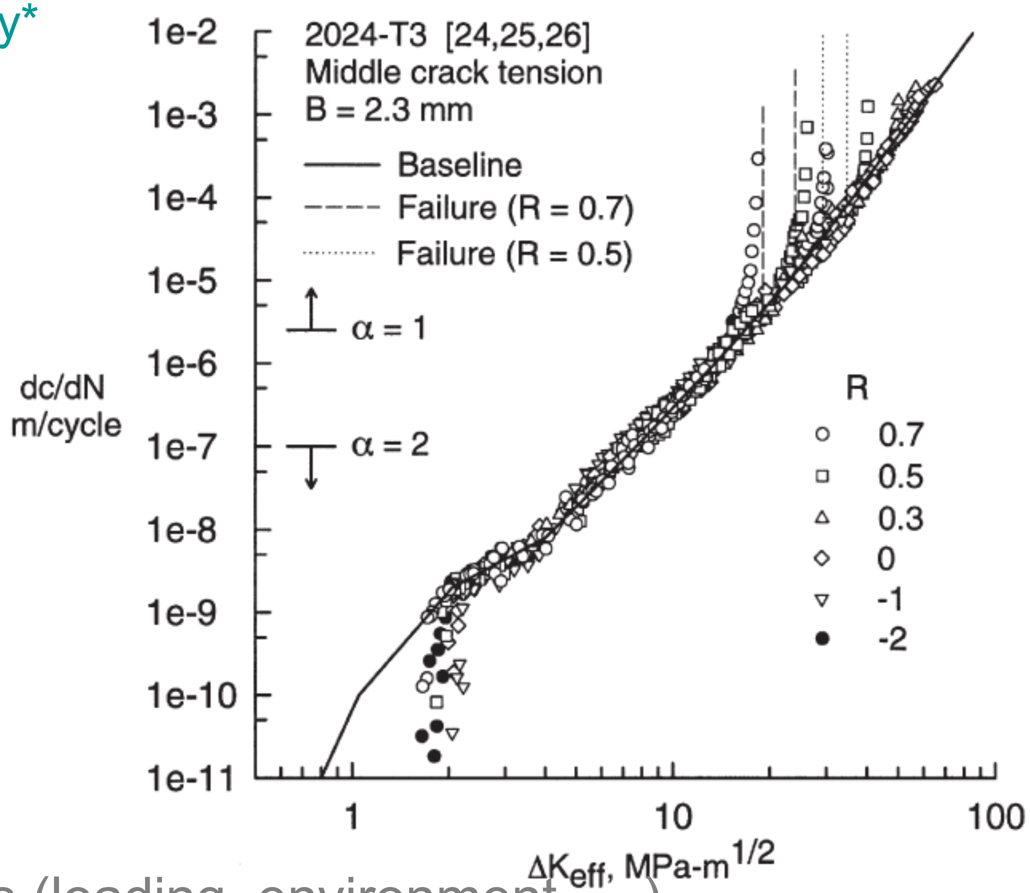
When applying the Paris law, it is important to remember that K depends on the crack size.

Crack propagation in Stage III

In this zone the crack grows rapidly until failure of the structure. The failure is reached as soon as the crack size reaches af such that $K(P_{max}, a_f)$ is equal to K_c.

- Effect of $R=K_{min}/K_{max}$ on crack growth rate (Zone II)
  - Due to crack closure life of structure is improved for low R
    - Example of 2024-T3 aluminum alloy*





  - ΔK_{eff} depends on many parameters (loading, environment, …)
    - Example: model of Elber & Schijve for Al. 2024-T3
      - $\Delta K_{eff} = (0.55 + 0.33\,R + 0.12\,R^2)\;\Delta K$ for $-1<R<0.54$
    - Models can be inaccurate in non-adequate circumstances

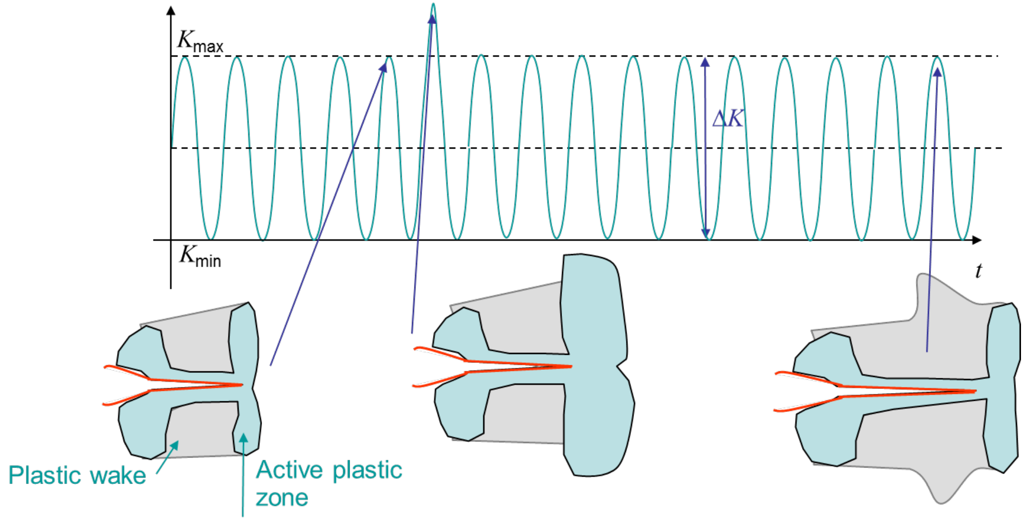
Typical material parameters for fatigue analyzes

– Parameters in Paris law

Material	ΔK_{th} [MPa · m^{½}]	m [-]	C [m (MPa · m$^{1/2}$)$^{-m}$]
Mild steel	3.2-6.6	3.3	$0.24 \cdot 10^{-11}$
Structural steel	2.0-5.0	3.85-4.2	$0.07\text{-}0.11 \cdot 10^{-11}$
Structural steel is sea water	1.0-1.5	3.3	$1.6 \cdot 10^{-11}$
Aluminum	1.0-2.0	2.9	$4.56 \cdot 10^{-11}$
Aluminum alloy	1.0-2.0	2.6-2.9	$3\text{-}19 \cdot 10^{-11}$
Copper	1.8-2.8	3.9	$0.34 \cdot 10^{-11}$
Titanium alloy (6Al-4V, R=0.1)	2.0-3.0	3.22	$1 \cdot 10^{-11}$

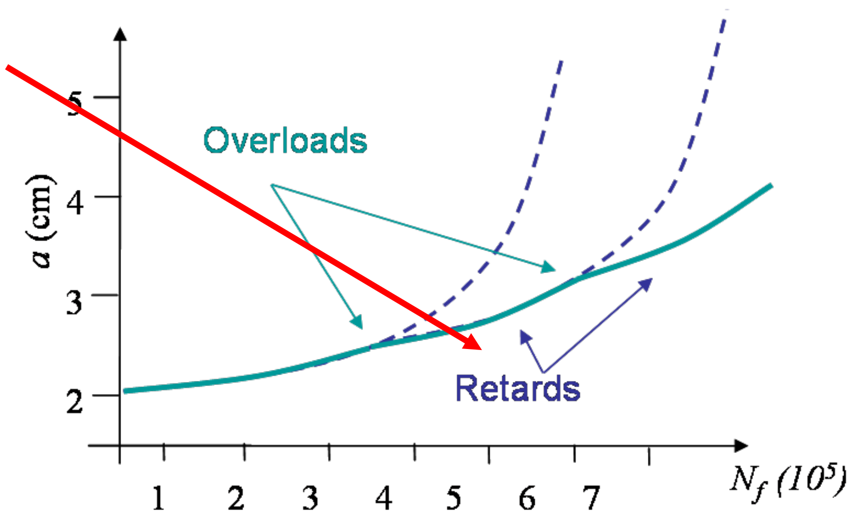
Overload effect (PICC = Plasticity Induced Crack Closure)

During a structure operation, the loading is never as regular as depicted. What happens if there is a few (or a moderate) number of overloads?



Before the overload, the crack propagates with its plastic wake; During the overloading, the active plastic zone is higher than for the other cycles; The plastic wake is temporarily increased (Phase 1) for the coming cycles until the active plastic zone at the crack tip passes the plastic zone created by the overload (Phase 2);

During Phase 1, ΔK_{eff} is reduced due to the PICC and the crack propagates slower: there exists a retard effect in the crack propagation rate. Once the crack tip has passed the modified wake in Phase 2, ΔK_{eff} is as expected and the initial propagation rate is recovered. However, too frequent overloads are damaging as they actually correspond to increasing K_{max}



The total number of cycles for the crack to reach 2 a_f can be obtained by integrating the fatigue crack propagation law

$$N_f = \int_{a0}^{af} \frac{da}{C(\Delta K)^m}$$

Using equation (2.31) we obtain

$$N_f = \int_{a0}^{af} \frac{da}{C\left[Y(a)\Delta\sigma\sqrt{\pi a}\right]^m}$$

Assuming that the function $Y(a)$ is equal to its initial value $Y(a_0)$ so that

$$\Delta K = \Delta K_0 \sqrt{\frac{a}{a_0}} \quad \text{where} \quad \Delta K_0 = Y(a_0)\Delta\sigma\sqrt{\pi a_0}$$

thereafter

$$N_f = \begin{cases} \frac{2a_0}{(m-2)C(\Delta K_0)^m}\left[1 - \left(\frac{a_0}{a_f}\right)^{m/2-1}\right] & \text{for } m \neq 2 \\ \frac{a_0}{C(\Delta K_0)^2} \ln \frac{a_f}{a_0} & \text{for } m = 2 \end{cases}$$

Example 5.14 A large centre-cracked plate containing an initial crack of length $2a_0 = 10$ mm is subjected to a constant amplitude cyclic tensile stress ranging between a minimum value of 100 MPa and a maximum of 200 MPa. Assuming the fatigue crack growth rate is governed by the equation

$$\frac{da}{dN} = 0.42 \times 10^{-11} (\Delta K)^3 \text{ (m/cycle)}$$

1. Calculate the crack growth rate when the crack length has the following values $2a_0 = 10$ mm, 30 mm, 50 mm.
2. Assuming further that the relevant fracture toughness is 60 MPa√m, estimate the number of cycles to failure.

Solution

(1) Determine the critical crack size, a_c,

$$a_c = \frac{1}{\pi}\left(\frac{K_{IC}}{\sigma_{max}}\right)^2 = 28.7 \times 10^{-3} \text{ (m)}$$

This means the total crack length at fast fracture is 57.3 mm.

(2) Crack growth rates:

$2a = 10$ mm $\quad \Delta K = \Delta\sigma\sqrt{\pi a}$ = 12.53 MPa√m

$\frac{da}{dN}$ = 0.42×10^{-11} × $(12.53)^3$ = 8.26×10^{-9} (m/cycle)

$2a = 30$ mm $\quad \Delta K = \Delta\sigma\sqrt{\pi a}$ = 21.7 MPa√m

$\frac{da}{dN}$ = 0.42×10^{-11} × $(21.7)^3$ = 4.29×10^{-8} (m/cycle)

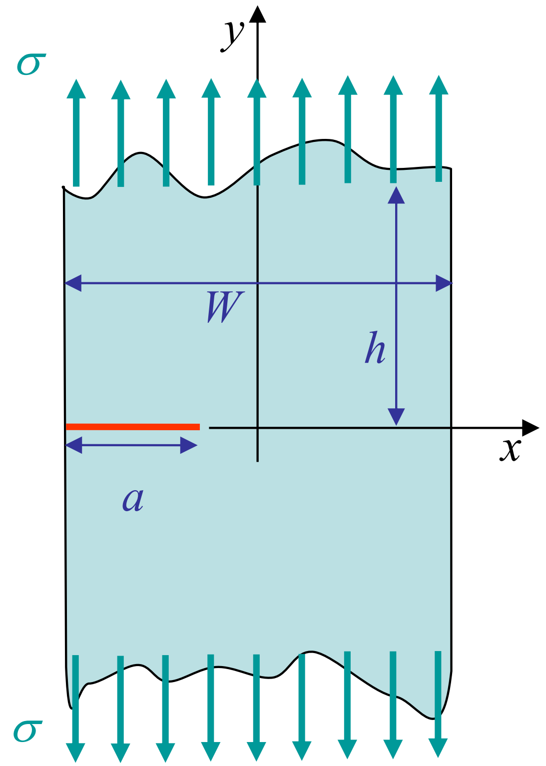
$2a = 50$ mm $\quad \Delta K = \Delta\sigma\sqrt{\pi a}$ = 28 MPa√m

$\frac{da}{dN}$ = 0.42×10^{-11} × $(28)^3$ = 9.24×10^{-8} (m/cycle)

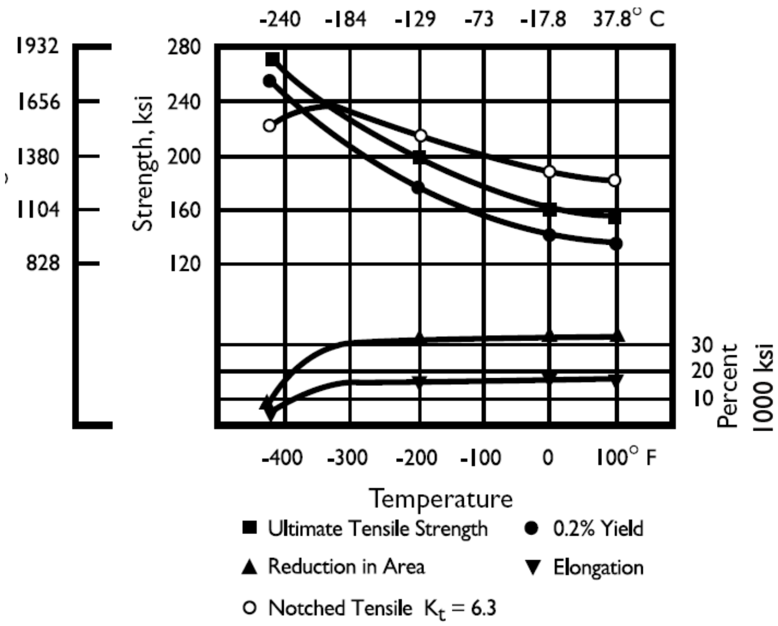
(3) Fatigue life:

$$
\begin{aligned}
N_f &= \int dN = \int_5^{28.7} \frac{da}{0.42 \times 10^{-11} (\Delta K)^3} = \int_5^{28.7} \frac{da}{7.39 \times 10^{-10} a^{3/2}} \\
&= 1.35 \times 10^9 \int_5^{28.7} a^{-3/2} da \\
&= 1.35 \times 10^9 (-2) a^{-1/2} \Big|_5^{28.7} \\
&= 6.76 \times 10^8 \ \text{(cycles)}
\end{aligned}
$$

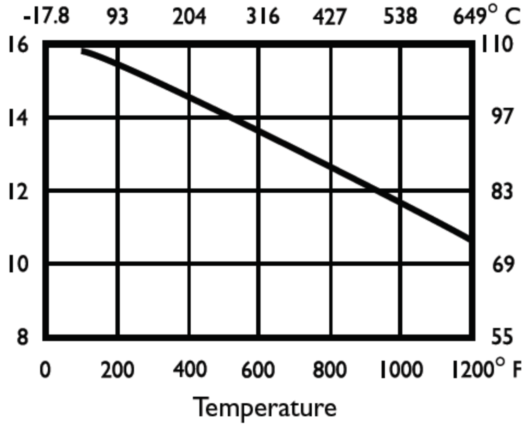
- Edge notch specimen under cyclic loading
  - Assume titanium alloy 6%Al - 4%V
    - See figures below
  - Assume a remains << W and << h
  - Initial crack size a = 1.5 cm
  - Cyclic loading between
    - Minimum value: 8 MPA
    - Maximum value: 80 MPa
  - What is the life of the structure?



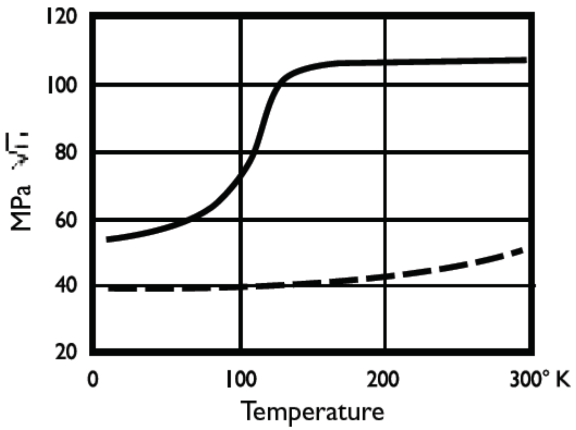
Tensile Properties vs. Temperature



Modulus of Elasticity



Fracture Toughness vs Temperature



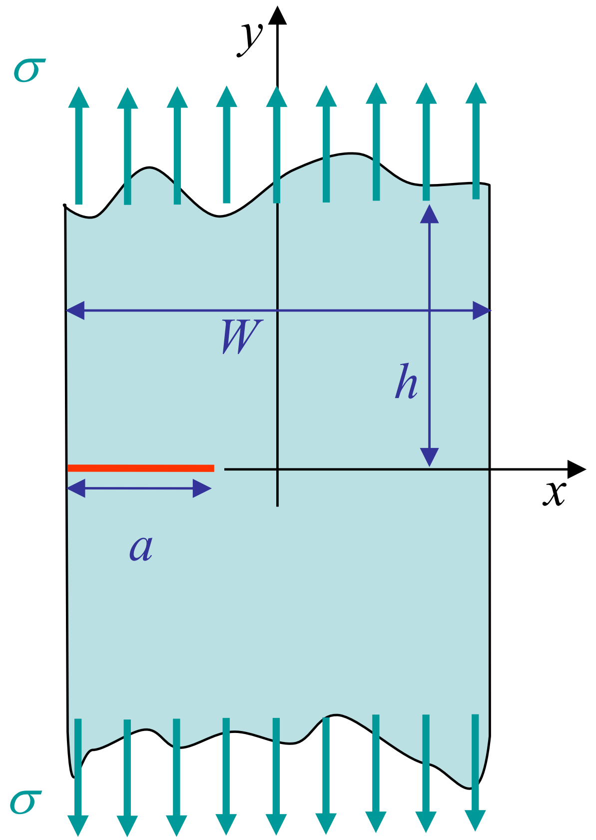
- Edge notch specimen under cyclic loading
  - Material properties at room temperature
    - Yield: 830 MPa
    - Toughness: 55 $\text{MPa} \cdot \text{m}^{1/2}$
  - SIF if a remains < 2% of W
    - $K_I = 1.122\, \sigma\, (\pi\, a)^{1/2}$
  - Plane strain & elastic fracture?
    - Yes if specimen thick enough

$$t > 2.5 \left(\frac{K_C}{\boldsymbol{\sigma}_p^0} \right)^2 = 2.5 \left(\frac{55}{830} \right)^2 \text{m} = 1.1\text{cm}$$

    - Crack is large enough

$$a > 2.5 \left(\frac{K_C}{\boldsymbol{\sigma}_p^0} \right)^2 = 2.5 \left(\frac{55}{830} \right)^2 \text{m} = 1.1\text{cm}$$

    - Moreover the applied stress << the yield stress



- Edge notch specimen under cyclic loading (2)
  - Is the crack critical?

$$K_{I\,\min,0} = 1.122\sigma_{\min}\sqrt{\pi a_0} = 1.95\ \ \mathrm{MPa}\cdot\sqrt{\mathrm{m}} < K_C$$
$$K_{I\,\max,0} = 1.122\sigma_{\max}\sqrt{\pi a_0} = 19.5\ \ \mathrm{MPa}\cdot\sqrt{\mathrm{m}} < K_C$$

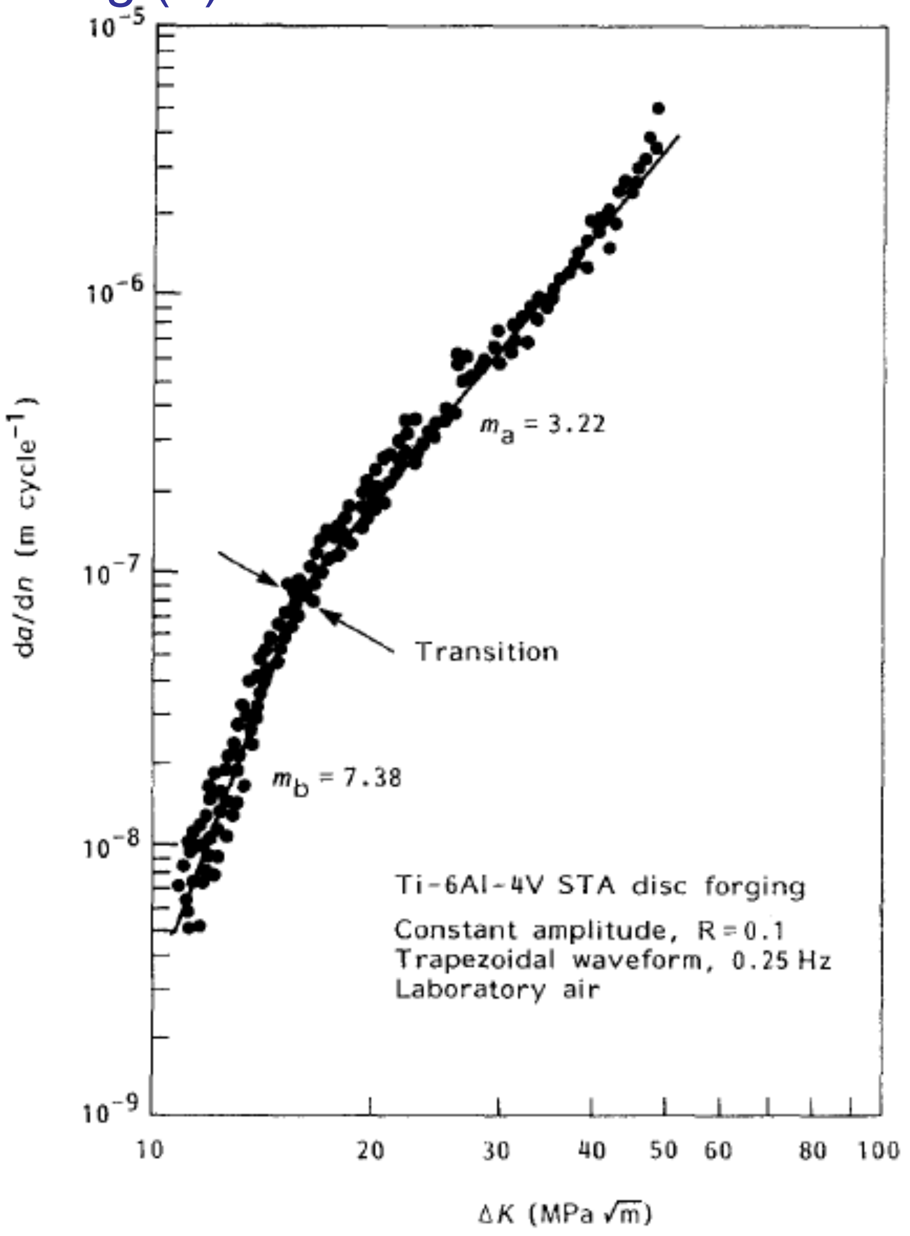
    - The crack will not lead to static failure
  - Fatigue?

$$\Delta K_0 = 17.5\ \mathrm{MPa}\cdot\sqrt{\mathrm{m}} > \Delta K_{\mathrm{th}}$$

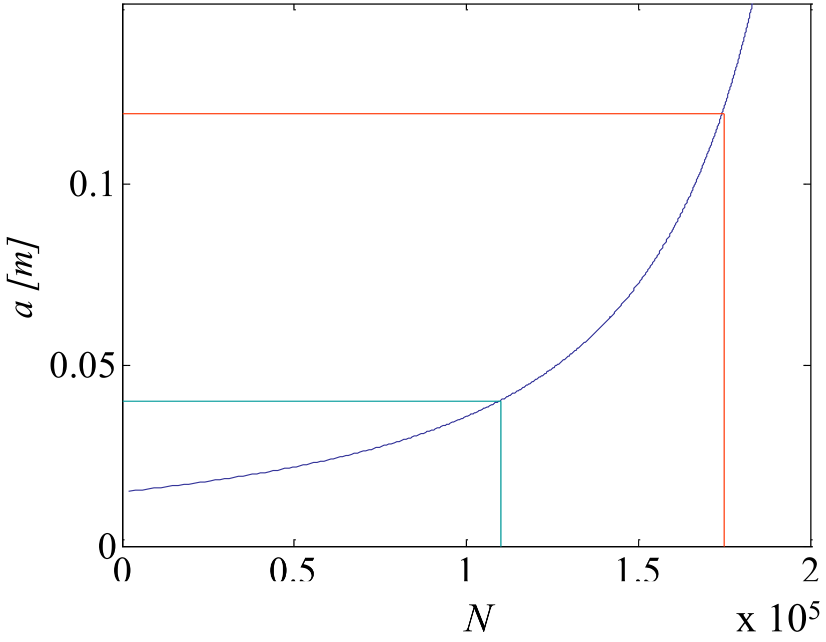
    - As we are above the threshold there will be fatigue
    - $R = 0.1$, so crack experiences closing effect
  - What is the critical crack length leading to static failure?

$$a_f = \frac{1}{\pi}\left(\frac{K_C}{1.122\sigma_{\max}}\right)^2 = \frac{1}{\pi}\left(\frac{55}{1.122\,80}\right)^2\ \mathrm{m} = 11.95\,\mathrm{cm}$$

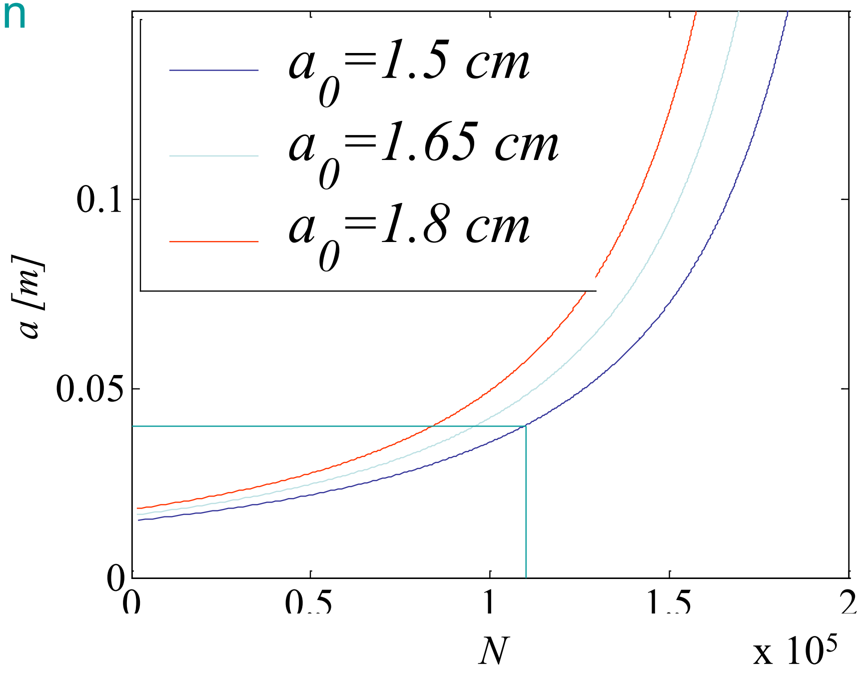
- Edge notch specimen under cyclic loading (3)
  - Parameters in Paris' law
    - There is a phase transition around $\Delta K = 17\ \mathrm{MPa}\cdot\mathrm{m}^{1/2}$ but we are above
    - Paris' coefficients:
      - $C = 10^{-11}\ \mathrm{m}\ (\mathrm{MPa}\cdot\mathrm{m}^{1/2})^{-m}$
      - $m = 3.22$



- Edge notch specimen under cyclic loading (4)
  - Paris' law $$\int_{a_0}^{a} \frac{da'}{C\left(\sigma_{\max}-\sigma_{\min}\right)^m 1.122^m \pi^{\frac{m}{2}} \left(a'\right)^{\frac{m}{2}}} = \int_0^N dN'$$
    - Can be integrated explicitly, and, for $m \neq 2$, it yields
$$N_f = \frac{1}{C\left(\sigma_{\max}-\sigma_{\min}\right)^m 1.122^m \pi^{\frac{m}{2}}} \left[\frac{2}{2-m}\left(a'\right)^{\frac{2-m}{2}}\right]_{a_0}^{a}$$
    - So the number of cycles in terms of the crack size is
$$N_f = \frac{-1}{10^{-11}\, 72^{3.22}\, 1.122^{3.22}\, \pi^{1.61}\, 0.61} \left[(a)^{-0.61} - (a_0)^{-0.61}\right]$$
  - Critical size is reached after 1.74 10^5 cycles, but this value cannot be used as
    - Paris' law is not valid is zone III
    - After 1.5 10^5 cycles the crack is growing too quickly for allowing inspections
    - 1.1 10^5 is a conservative time life

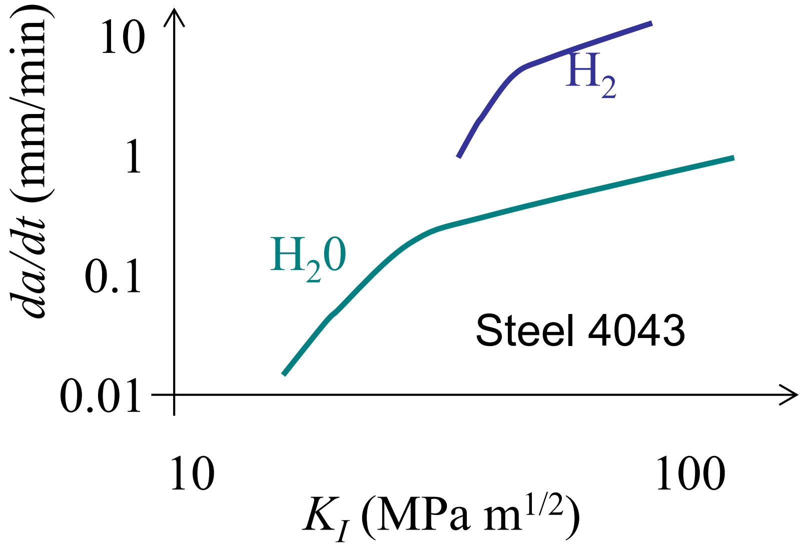


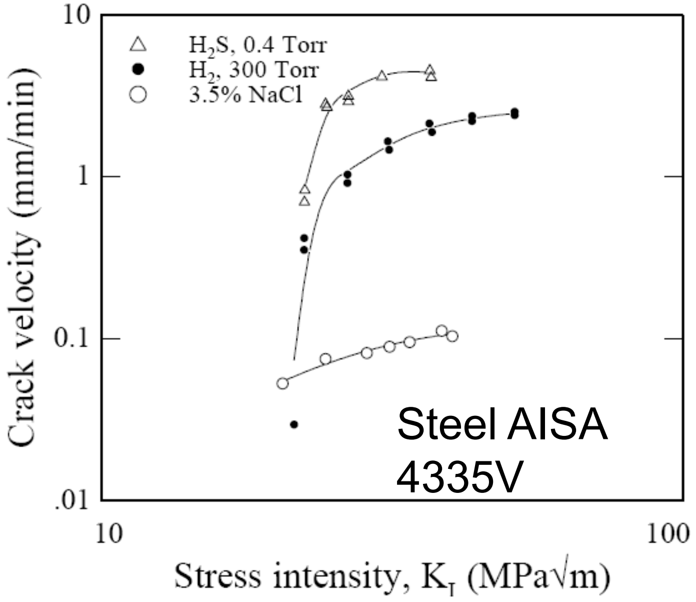
- Edge notch specimen under cyclic loading (5)
  - Some remarks $N_f = \frac{-1}{10^{-11}\ 72^{3.22}\ 1.122^{3.22}\ \pi^{1.61}\ 0.61}\left[(a)^{-0.61} - (a_0)^{-0.61}\right]$
    - If there is an error of 10% in the estimation of the initial crack length size
      - Life of the structure is reduced by 15%
      - Using a 1.5 10^5-cycle life prediction would actually lead to a crack size located in zone III
    - We have assumed $a < 0.02\ W$, which corresponds to a six-meter wide specimen. In practice
      - Crack size cannot be considered infinitely small
      - SIF must then be evaluated using
        - » Either FEM simulations
        - » Or SIF handbooks
      - Paris' law
        - » Cannot be integrated in a closed form
        - » Integration has to be performed numerically



Stress corrosion cracking

- A crack can grow due to the combination of stress and chemical attack
  - This is not only fatigue but also for static stress with $K<K_C$
  - It happens in particular environments
    - Salt water
    - Hydrogen
    - Chlorides
    - …
  - Mainly for metals
    - Steel in salt water, chloride, hydrogen
    - Aluminum alloys in salt water





Design methods

When an engineer has to design a structure submitted to cyclic loading, there exist different design approaches, appeared chronlogically in time with the improvment of fracture mechanics theory

Infinite life design. In this case we always ensure that the stress amplitude σ_a is lower than the endurance limit σ_e so that the life is "infinite". **In practice this design is never used as it is economically inefficient.**

Safe life design. For this design we make sure that no cracks appear before a given number of cycles (the structure life) is reached. **At the end of the expected life the component is changed even if no failure has occurred**. This method clearly puts the emphasis on cracks prevention: a crack free structure is assumed.

The number of cycles is determined by the the total life approach. **Nowadays this method is used for rotating structures vibrating with the flow as turbine blades.** Indeed because of the very high number of cycles to be sustained by these structures, once a crack has formed the remaining life time is very short in hours. However a particular attention should be paid to the quality of the components as the structure is assumed to be defect-free.

Fail Safe design. The philosophy of this design is to consider failure of a component as possible. **However the design is such that there remains enough integrity to operate the structure safely.** This means that in such a design the crack path has to be determined and crack arresters have to be added to the structure. To avoid too many critical cracks at once, it is necessary to regularly check the structure.

The difference with the approaches described before **is that one focuses more on crack arrest than on crack initiation.** An example of such a design is the Boeing 737. In 1988 the Aloha Airlines flight 243 suffered from a production defect: **two fuselage plates had not been glued, allowing sea water to induce corrosion, resulting in a loading of the rivets due to the increase of volume between the plates (due to corrosion). This led to fatigue of the rivets, which failed. But due to the design, the crack followed the defined path, and the plane could still be operated.**

Damage tolerant design This design is more recent and is used for recent aircraft structures. **This method assumes that cracks are present from the beginning of service.** To be able to predict the life time of the structure it is necessary to be able to characterize the significance of the existing cracks.

To do so one needs to determine the initial crack sizes through non-destructive inspections. Once the critical crack sizes are known the Paris-Erdogan law is used to estimate the crack growth rate during service and thus the life of the structure. Conservative inspection intervals, like every so many years or after a certain amount of flight hours, are thus scheduled to validate or correct the predictions. During these inspections one verifies the crack growth, and predicts the end of life a_f. If this end of life is too close in the future, the part is either replaced or an implemented repair-rehabilitation strategy is planned.