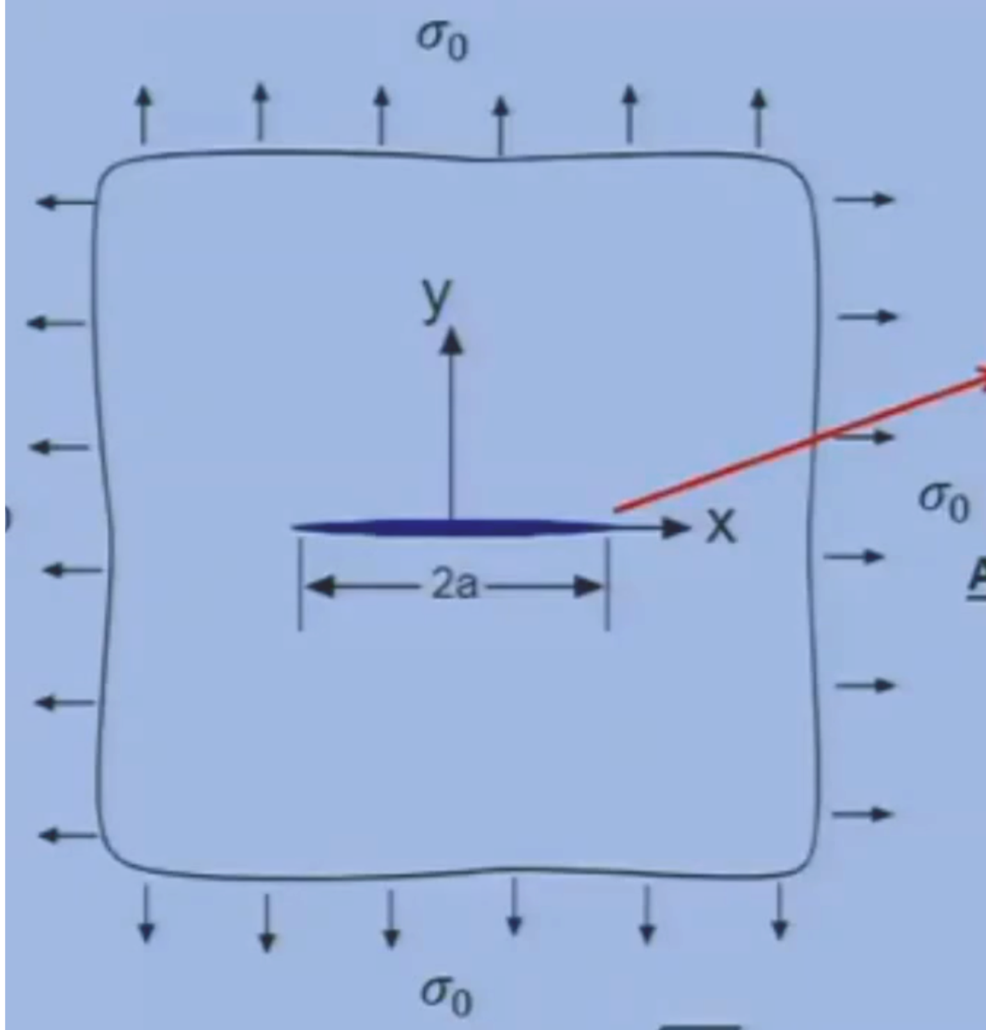
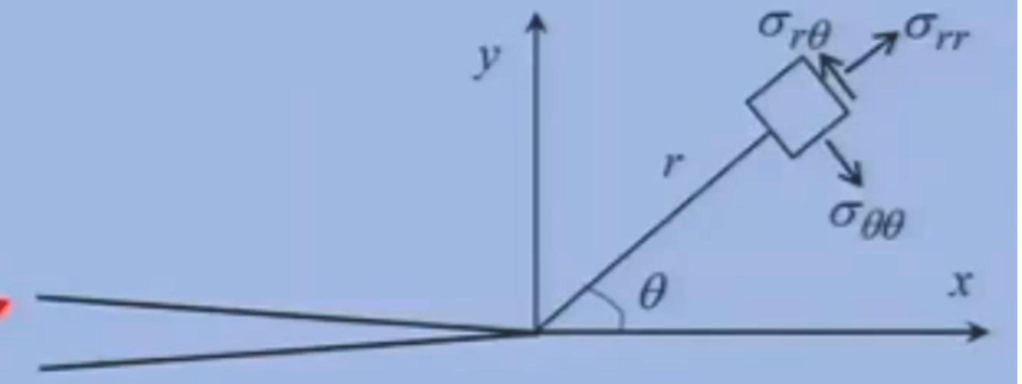
Crack propagation
and
crack path

Stress field around a crack tip





$$\theta = 0: \quad \sigma_{yy} = \frac{\sigma_0\sqrt{\pi a}}{\sqrt{2\pi r_1}}$$

Approx. Stress field at the crack tip (Analytical):

$$\sigma_{xx} = \frac{\sigma_0\sqrt{a}}{\sqrt{2r_1}}\cos\frac{1}{2}\theta_1\left(1-\sin\frac{1}{2}\theta_1\sin\frac{3}{2}\theta_1\right)$$

$$\sigma_{yy} = \frac{\sigma_0\sqrt{a}}{\sqrt{2r_1}}\cos\frac{1}{2}\theta_1\left(1+\sin\frac{1}{2}\theta_1\sin\frac{3}{2}\theta_1\right)$$

$$\sigma_{xy} = \frac{\sigma_0\sqrt{a}}{\sqrt{2r_1}}\sin\frac{1}{2}\theta_1\cos\frac{1}{2}\theta_1\cos\frac{3}{2}\theta_1$$

Stress intensity factor

G. Irwin (1957) find that stresses around a crack in terms of a scaling factor:

stress intensity factor $\boldsymbol{K}$, defined as:

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi x}}$$

From the analytical solution:

$\theta = 0$:
(Crack line)

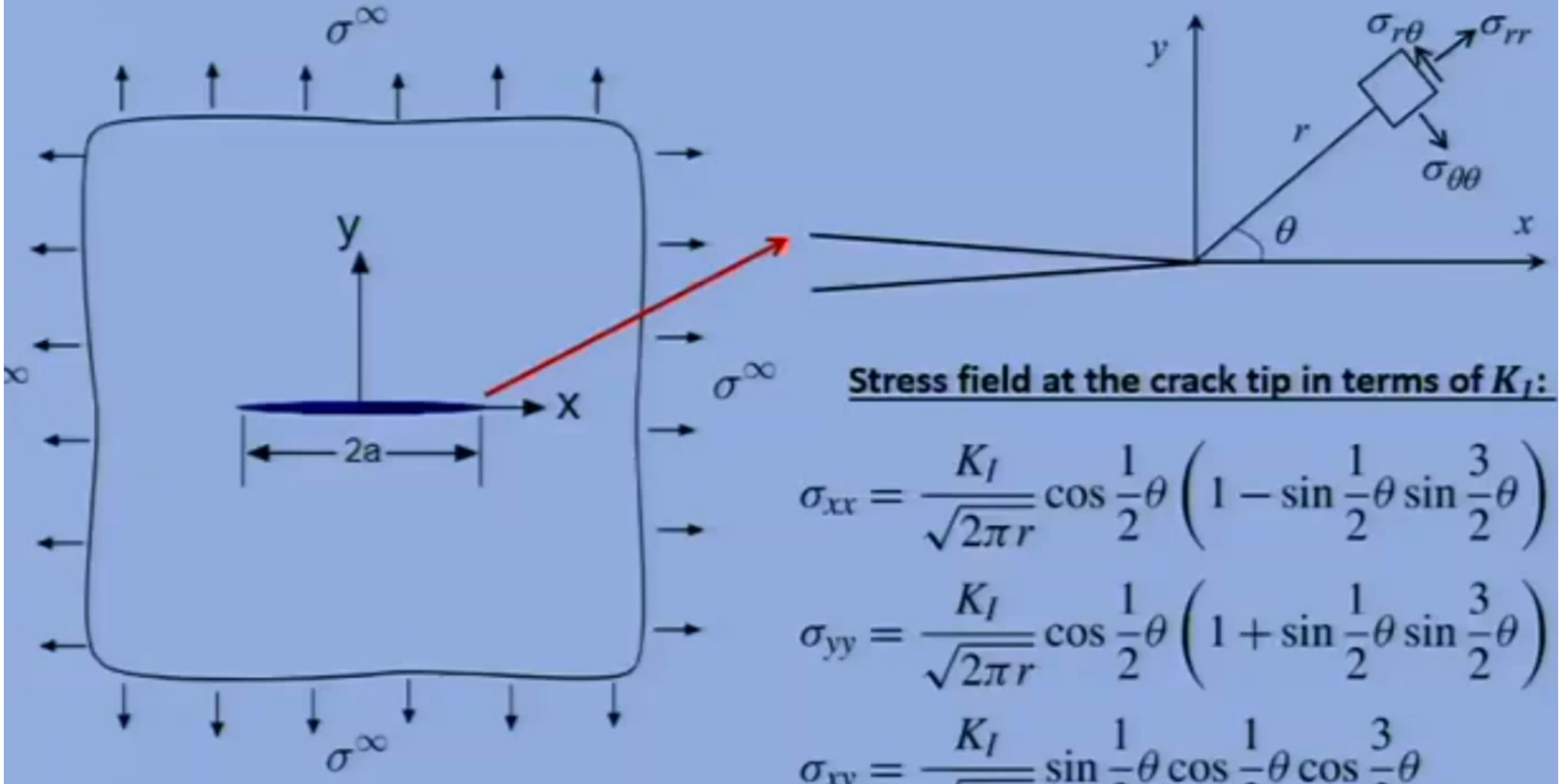
$$\sigma_{yy} = \frac{\sigma_0 \sqrt{\pi a}}{\sqrt{2\pi r_1}}$$

Here $x = r_1$, we then obtain:

$$K_I = \sigma_0 \sqrt{\pi a}$$

This relates the $\boldsymbol{K_I}$ with remote stress $\boldsymbol{\sigma_0}$ and crack length $\boldsymbol{a}$.

Stress field around a crack tip



Stress field at the crack tip in terms of K_I:

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{1}{2}\theta \left(1 - \sin\frac{1}{2}\theta \sin\frac{3}{2}\theta\right)$$

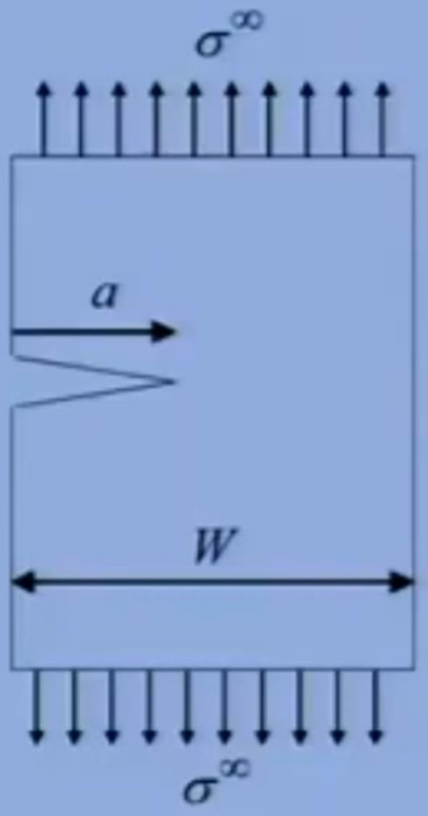
$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{1}{2}\theta \left(1 + \sin\frac{1}{2}\theta \sin\frac{3}{2}\theta\right)$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{1}{2}\theta \cos\frac{1}{2}\theta \cos\frac{3}{2}\theta$$

K_I depends on boundary conditions

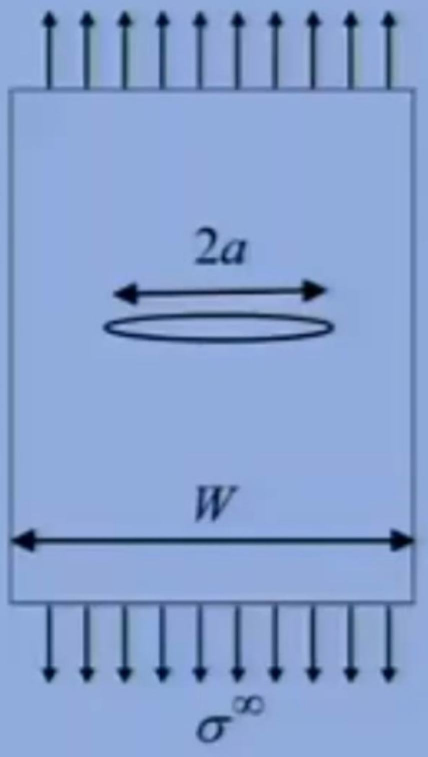
$$K_I = Y\sigma_0\sqrt{\pi a}$$

Y: geometrical correction factor, $Y(\frac{a}{W})$ calibrated by finite element !



$W >> a$

$$K_I = 1.122\sigma^\infty\sqrt{\pi a}$$



$$K_I = Y\sigma^\infty\sqrt{\pi a}$$

$$Y \approx \left(\cos\frac{\pi a}{W}\right)^{-1/2}$$

Fracture toughness

Fracture toughness characterizes the resistance of a material to fracture.

When the stress intensity factor reaches a critical value, crack starts to grow,

$$K_{Ic} = Y\sigma_c\sqrt{\pi a}$$

Remote stress

Fracture toughness is the critical stress intensity factor, as a **material constant.** It is independent on the loading conditions and crack length. We measure it from experiments.

Questions you may ask

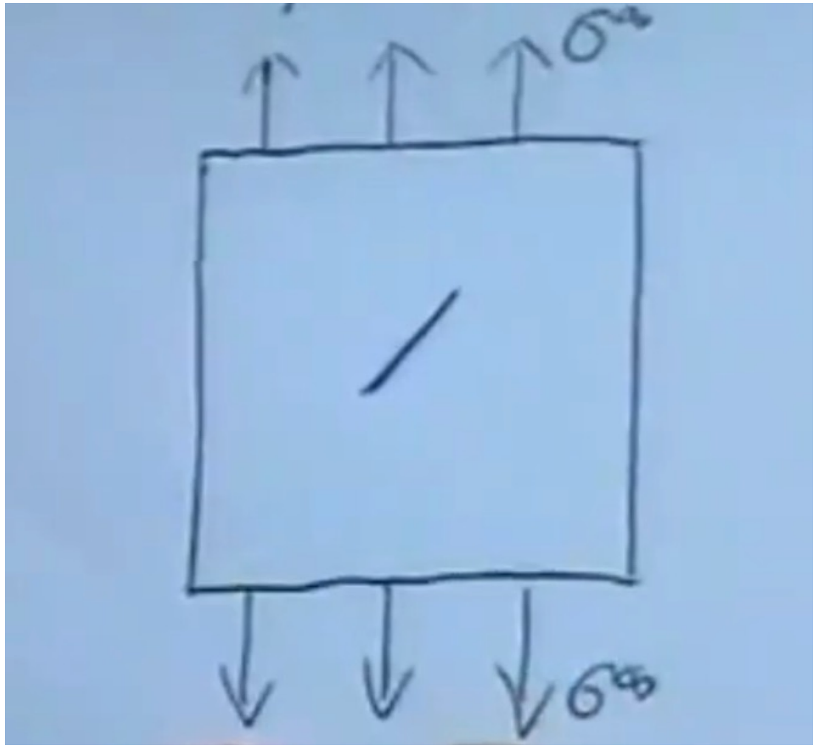
$$K_{Ic} = Y\sigma_c\sqrt{\pi a}$$

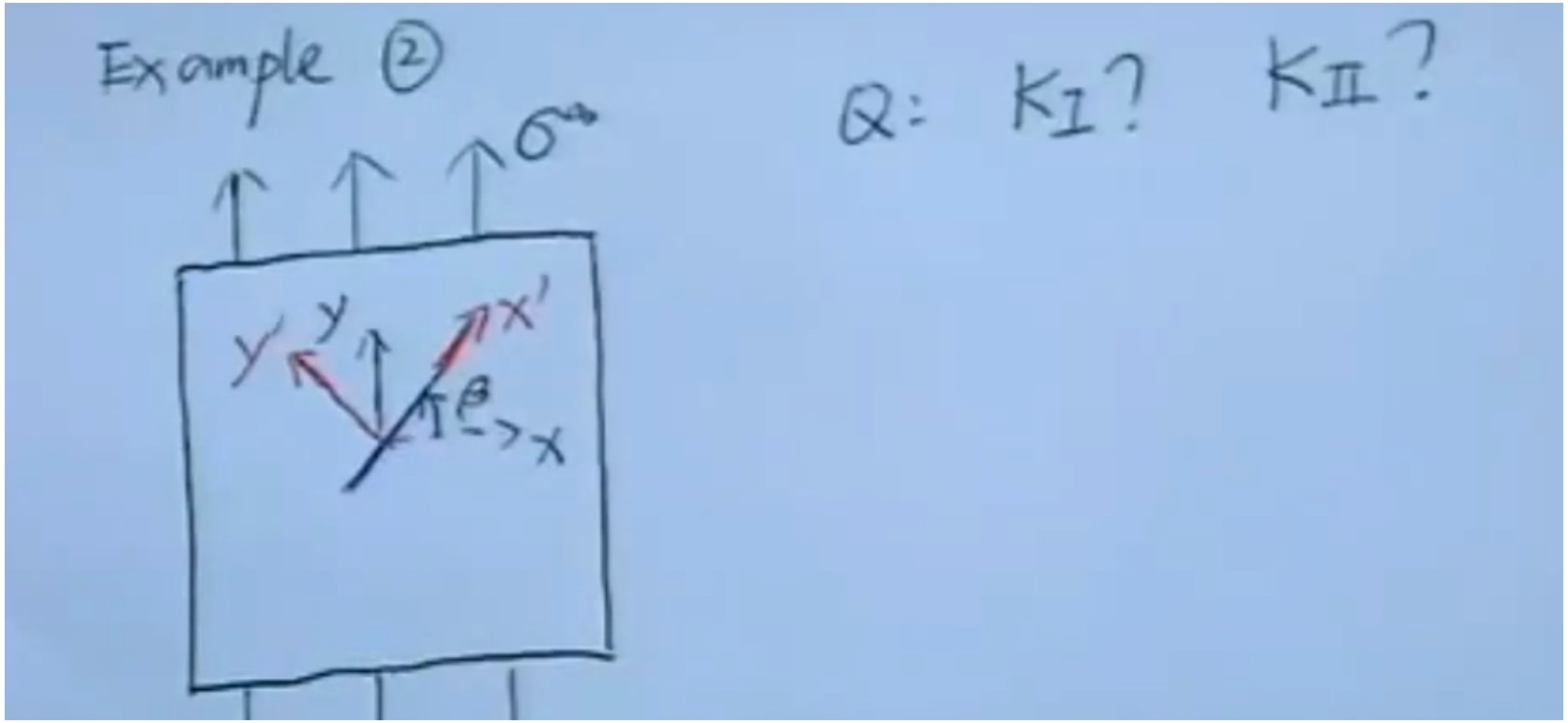
- Why we often use $2a$ to define the crack length?
  A: We only use it in the centre-notched panel, because crack will grow in both directions.

- Why we use fracture toughness instead of stress to predict the crack growth?
  A: Fracture toughness considering both stress and crack length, it represents the stress intensity around the crack tip

- How to measure the fracture toughness ?
  A: Various methods using notched specimen, measure the remote stress and crack growth. Compact tension, three-point bending, etc.

But what about an inclined crack?





Q: K_I? K_{II}?

Stress transformation:

$c = \cos\beta \qquad s = \sin\beta$

$$\begin{Bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

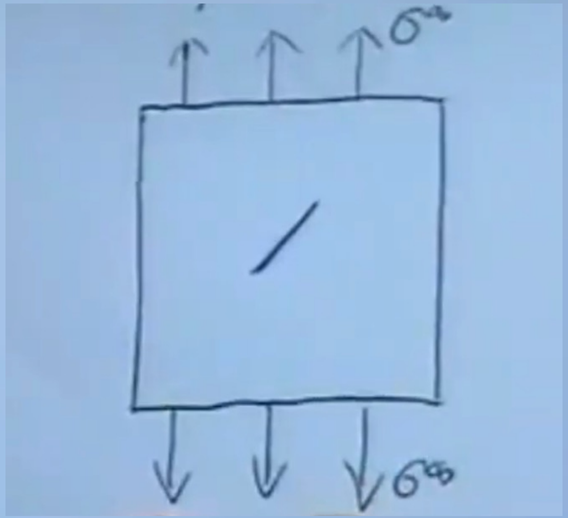
$$\begin{cases} \sigma_{x'} = \sigma_x \cos^2\beta + \sigma_y \sin^2\beta + 2\tau_{xy} \sin\beta\cos\beta \\ \sigma_{y'} = \sigma_x \sin^2\beta + \sigma_y \cos^2\beta - 2\tau_{xy} \sin\beta\cos\beta \\ \tau_{x'y'} = (\sigma_y - \sigma_x)\sin\beta\cos\beta + \tau_{xy}(\cos^2\beta - \sin^2\beta) \end{cases} \quad [1]$$

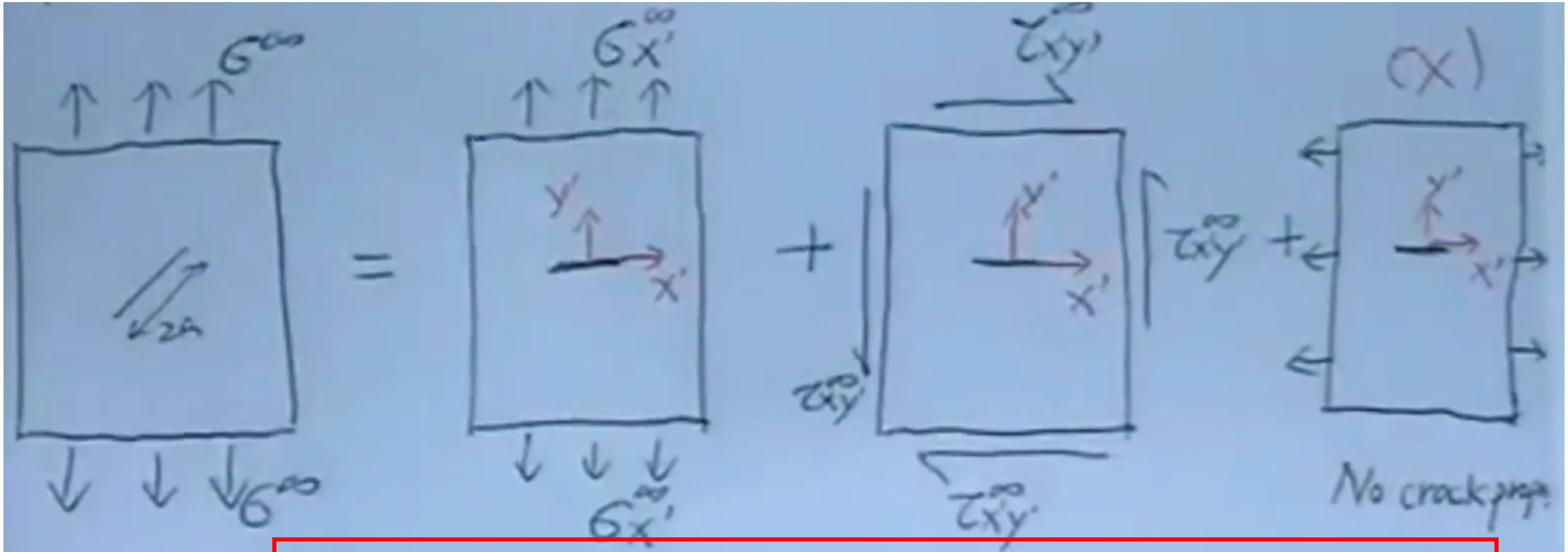
Loading condition: $\sigma_x = 0 \quad \sigma_y = \sigma^{\infty} \quad \tau_{xy} = 0$

Caso particolare

sub. into [1]

$$\begin{cases} \sigma_{x'} = \sigma^{\infty} \sin^2\beta \\ \sigma_{y'} = \sigma^{\infty} \cos^2\beta \\ \tau_{x'y'} = \sigma^{\infty} \sin\beta\cos\beta \end{cases} \quad [2]$$



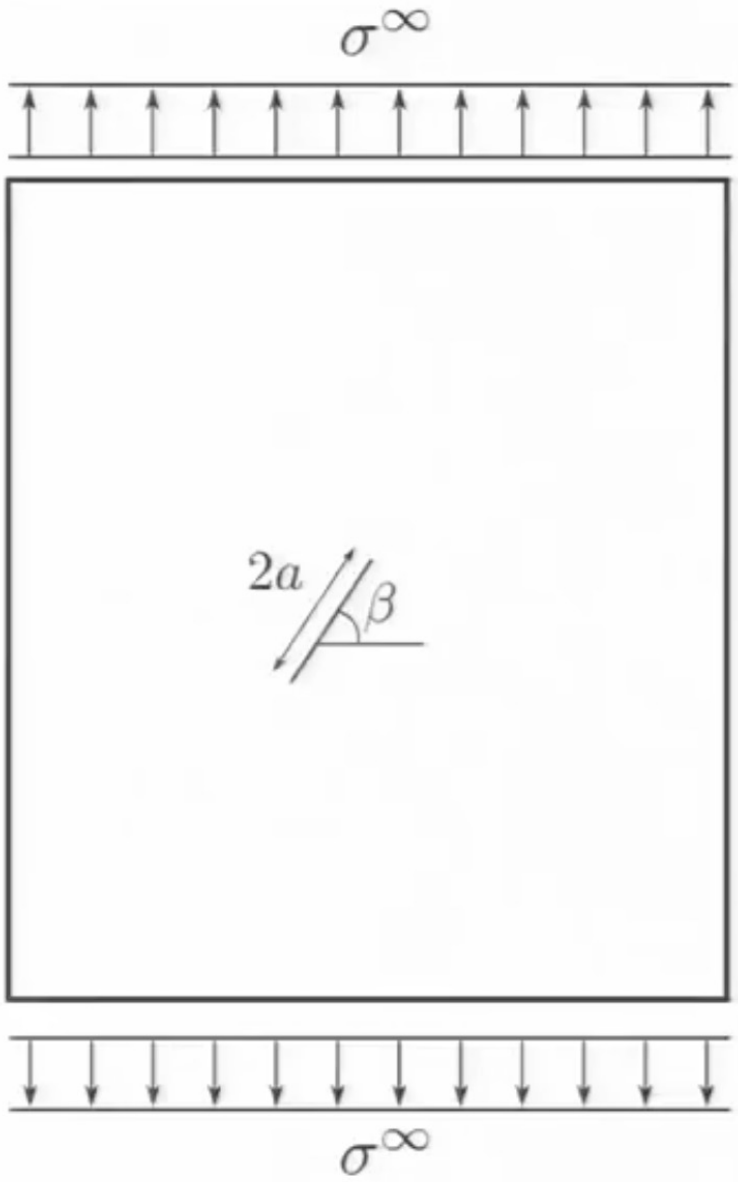


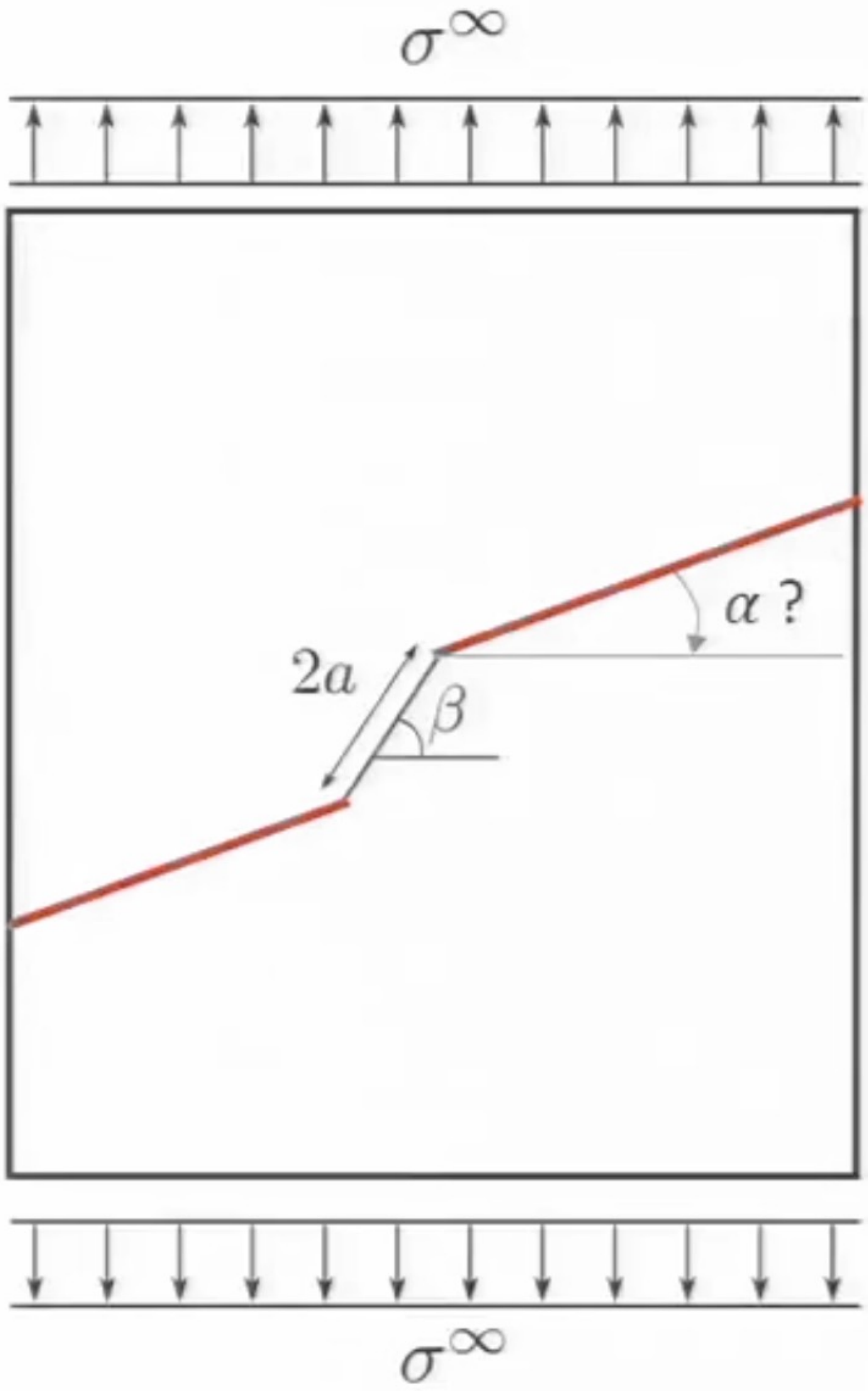
Answer:

$$K_I = Y\,\sigma_{x'}^{\infty}\sqrt{\pi a} = Y\,\sigma^{\infty}\sin^2\beta\sqrt{\pi a}$$

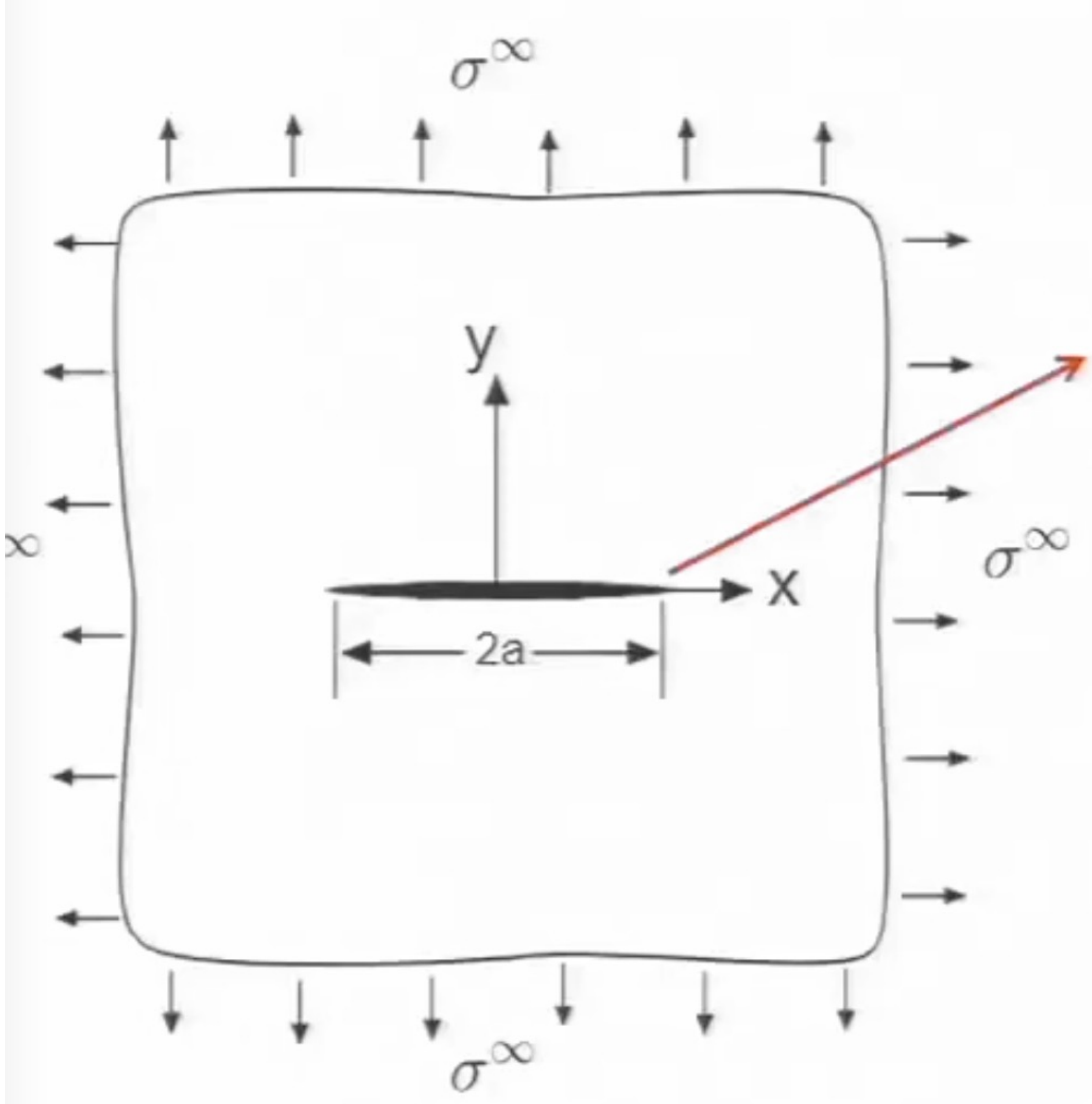
$$K_{II} = Y\,\tau_{xy'}^{\infty}\sqrt{\pi a} = Y\,\sigma^{\infty}\sin\beta\cos\beta\sqrt{\pi a}$$

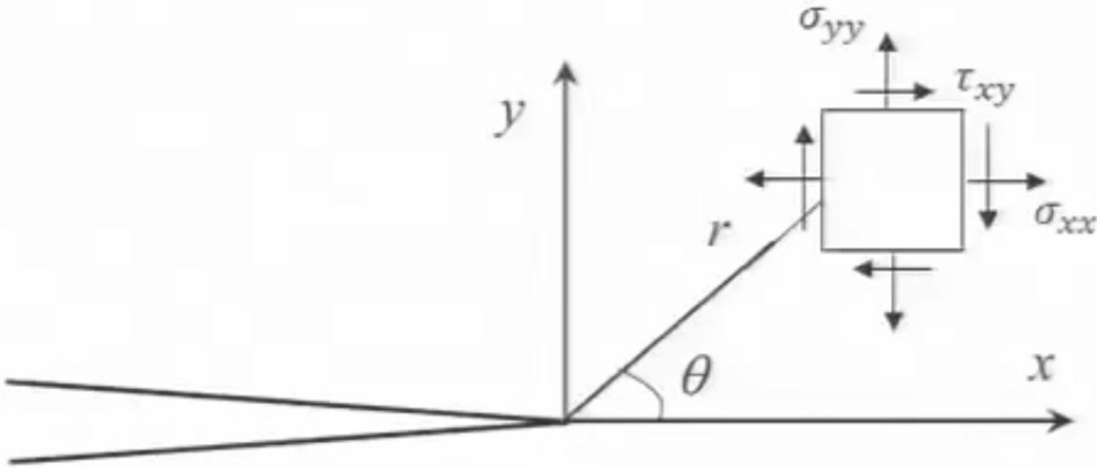
How to predict the crack path?





- To predict the crack path, we first need to know the **stress field** near the crack tip.
- Since this is a mixed-mode problem, let's first look at the stress field for **each mode**.
- Then we **combine** the stress field together to analysis.



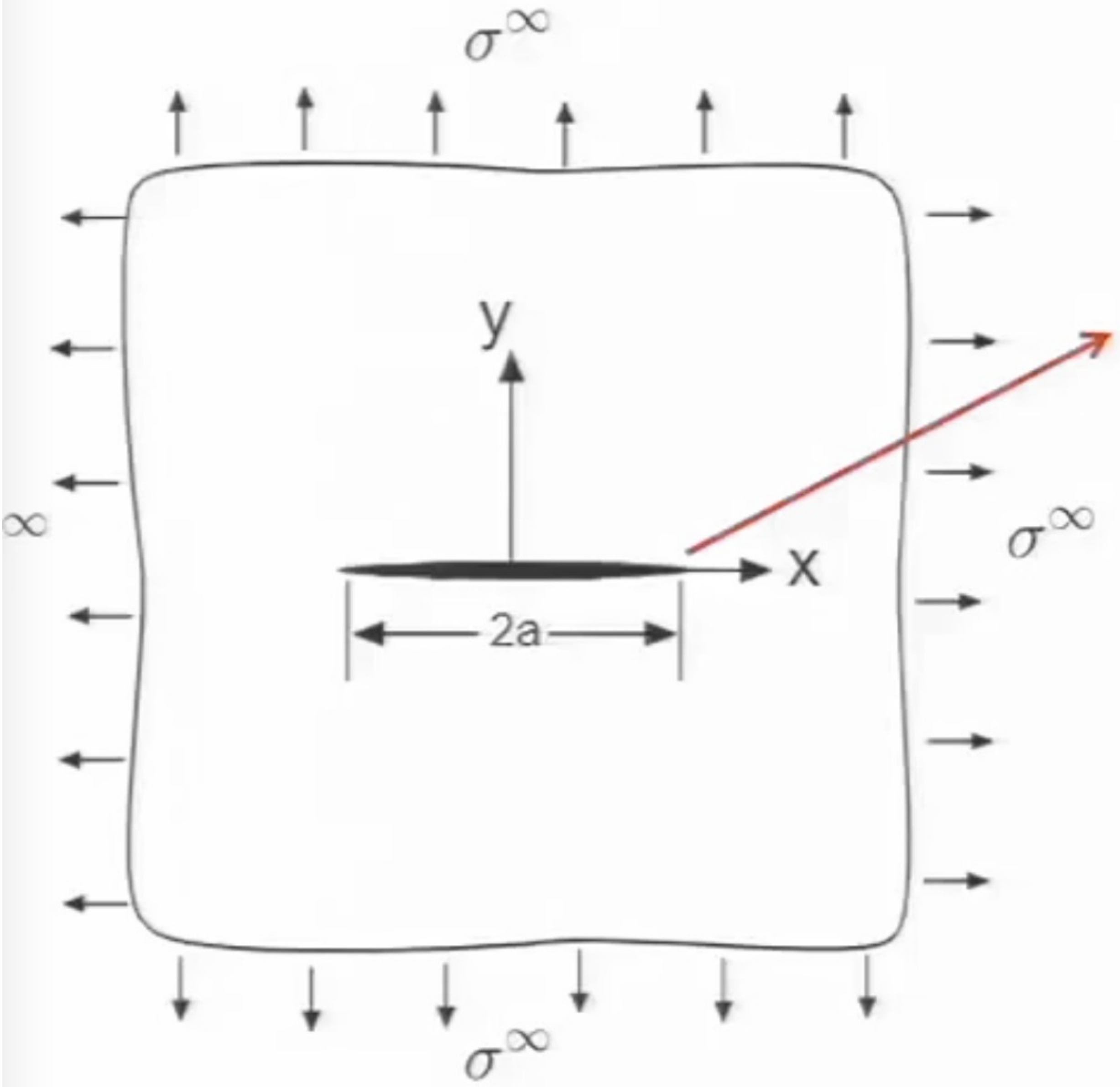


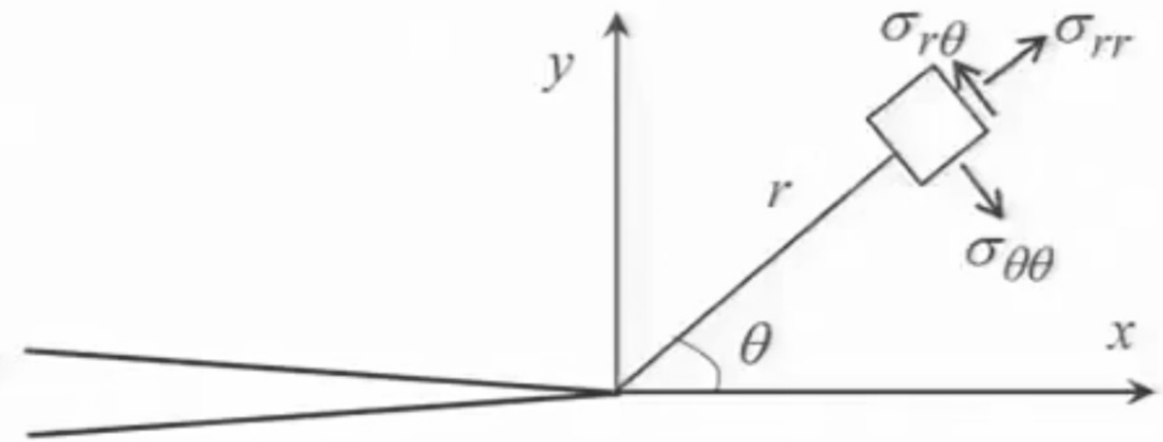
Stress field at the crack tip in terms of K_I:

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{1}{2}\theta\left(1 - \sin\frac{1}{2}\theta\sin\frac{3}{2}\theta\right)$$

$$\sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{1}{2}\theta\left(1 + \sin\frac{1}{2}\theta\sin\frac{3}{2}\theta\right)$$

$$\sigma_{xy} = \frac{K_I}{\sqrt{2\pi r}} \sin\frac{1}{2}\theta\cos\frac{1}{2}\theta\cos\frac{3}{2}\theta$$





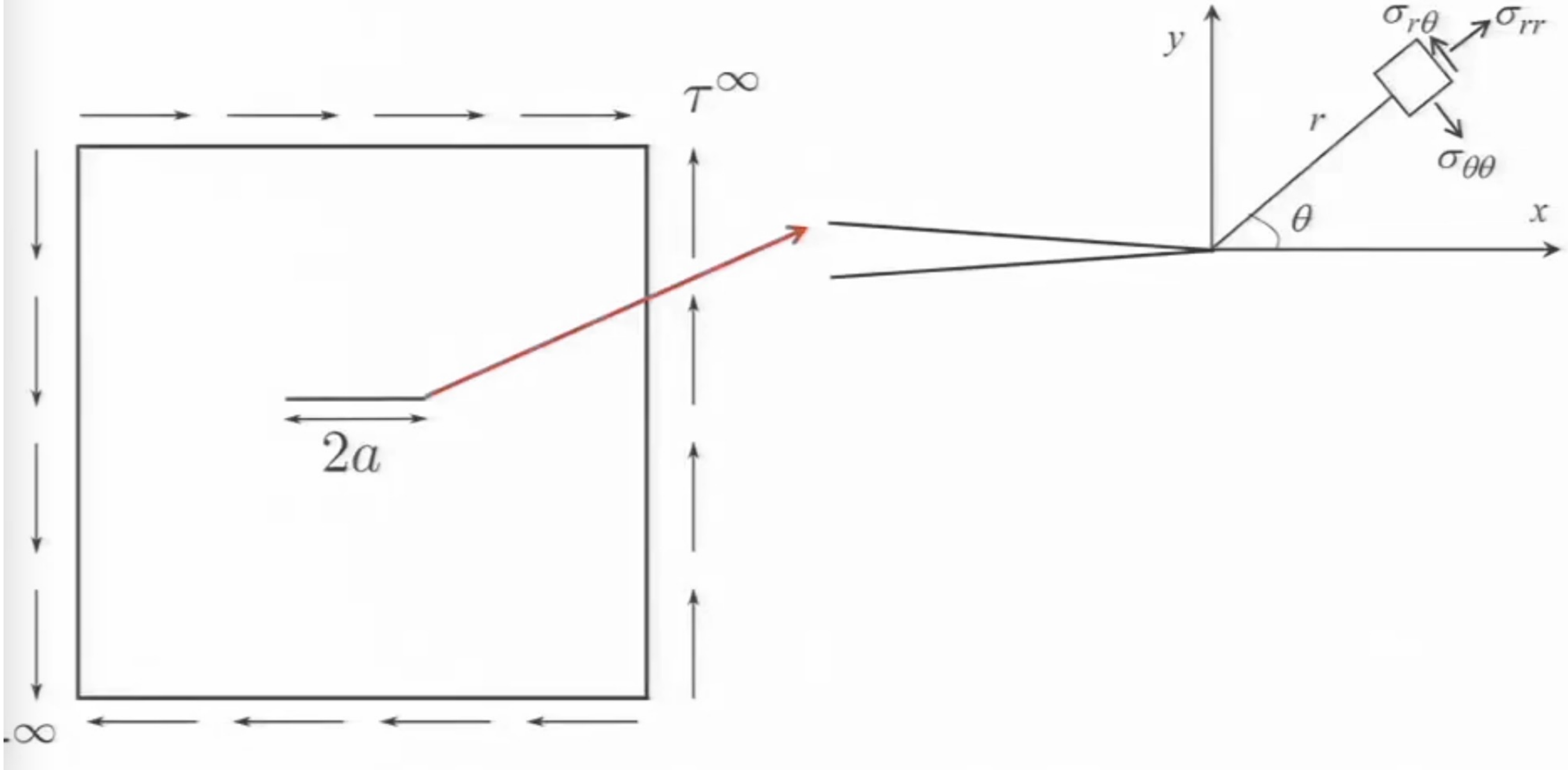
Stress field at the crack tip in terms of K_I:

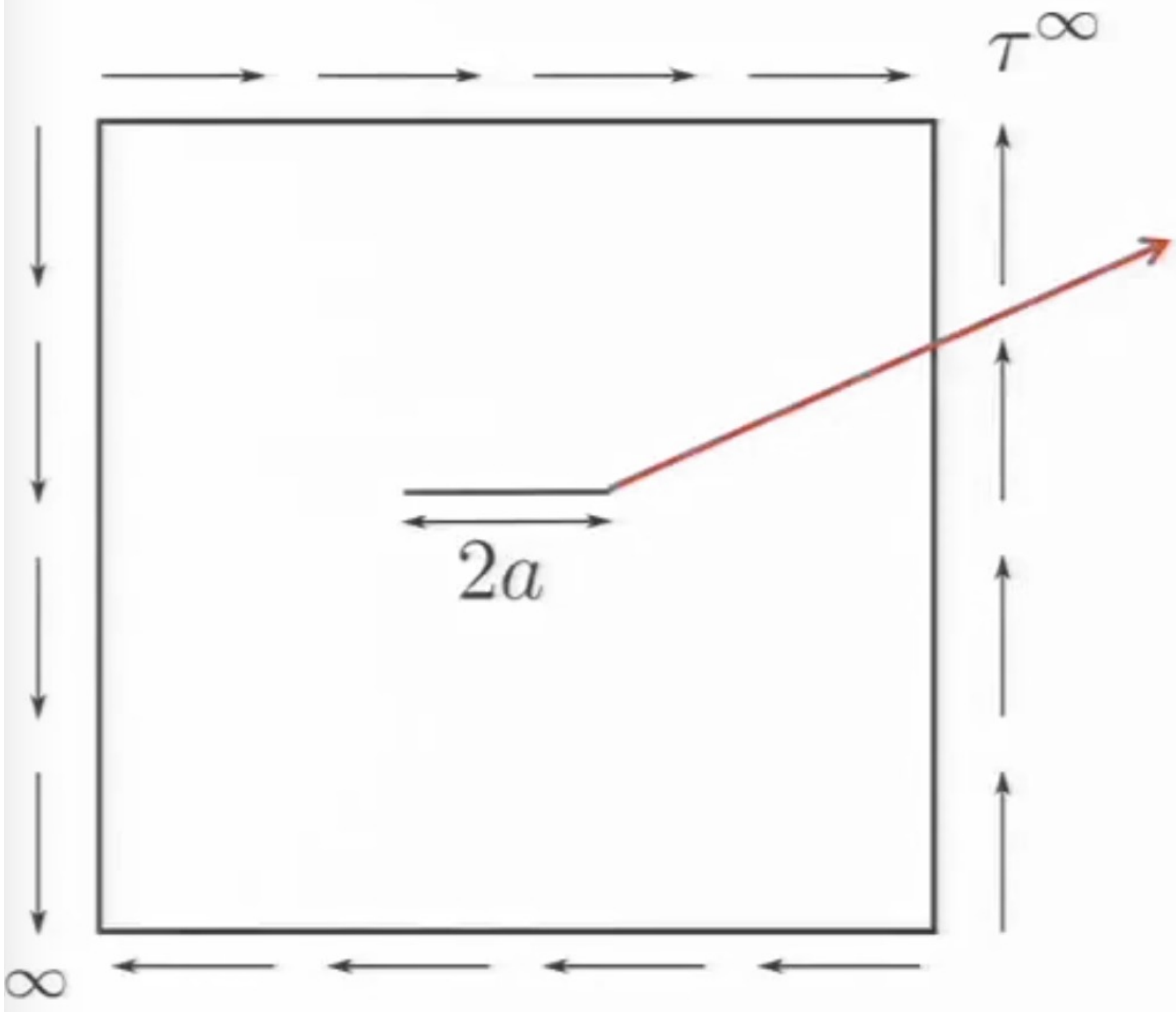
$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{5}{4}\cos\left(\frac{\theta}{2}\right) - \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right]$$

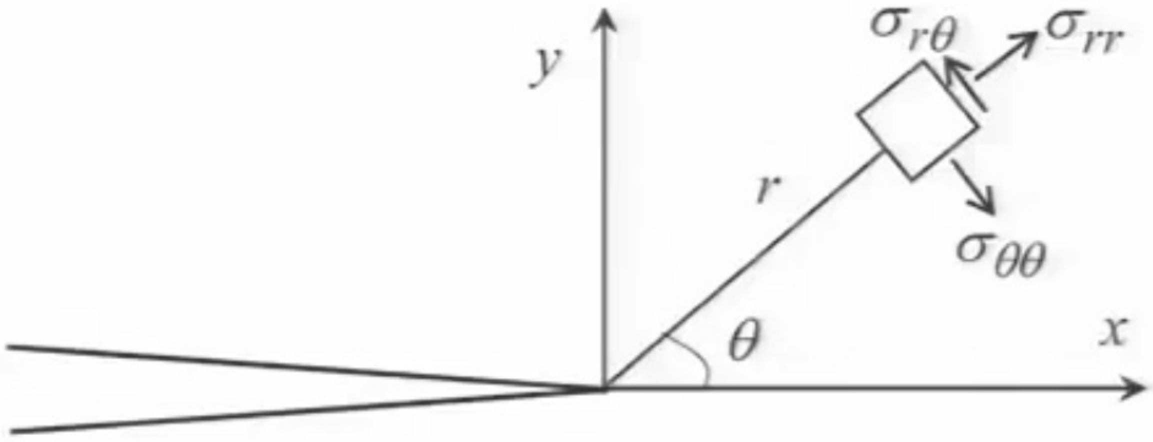
$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{3}{4}\cos\left(\frac{\theta}{2}\right) + \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin\left(\frac{\theta}{2}\right) + \frac{1}{4}\sin\left(\frac{3\theta}{2}\right)\right]$$

Mode II stress field- Polar coordinate







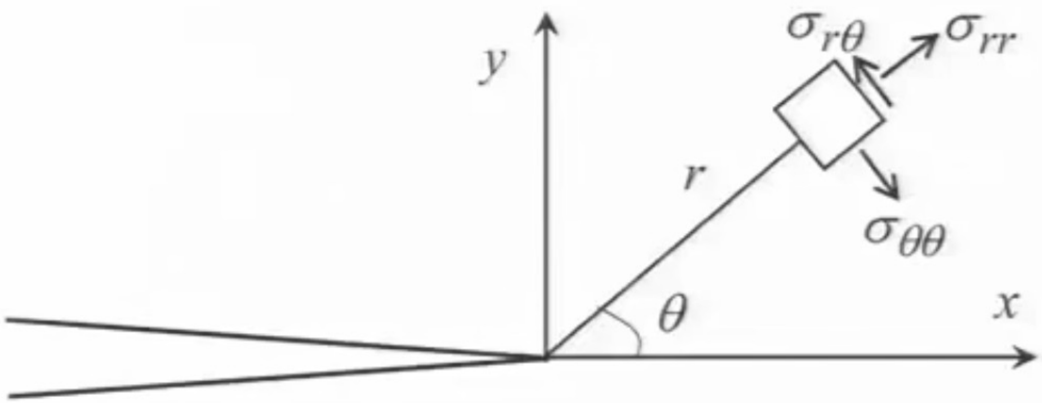
Stress field at the crack tip in terms of K_{II}:

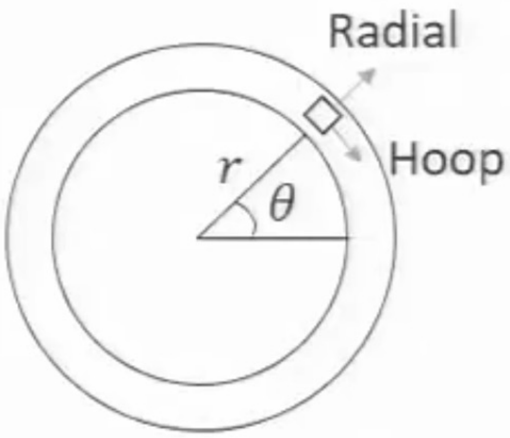
$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{5}{4}\sin\left(\frac{\theta}{2}\right)+\frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{\theta\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{3}{4}\sin\left(\frac{\theta}{2}\right)-\frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\cos\left(\frac{\theta}{2}\right)+\frac{3}{4}\cos\left(\frac{3\theta}{2}\right)\right]$$

Maximum hoop stress criterion





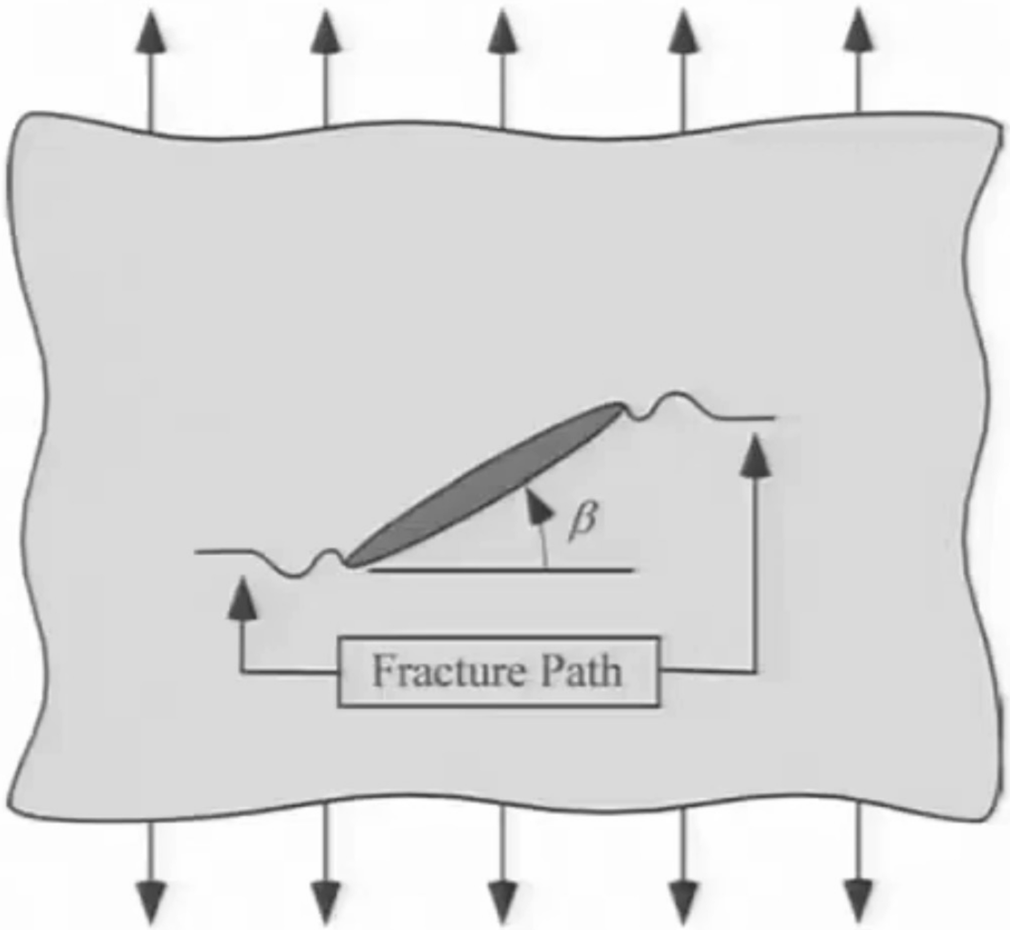
- Crack begins to extend from the tip in the direction along with maximum hoop stress.
- Maximum tensile stress along the hoop direction, no shear stress (principal stress).

Max hoop stress criterion:

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0 \qquad \frac{\partial^2 \sigma_{\theta\theta}}{\partial^2 \theta} < 0$$

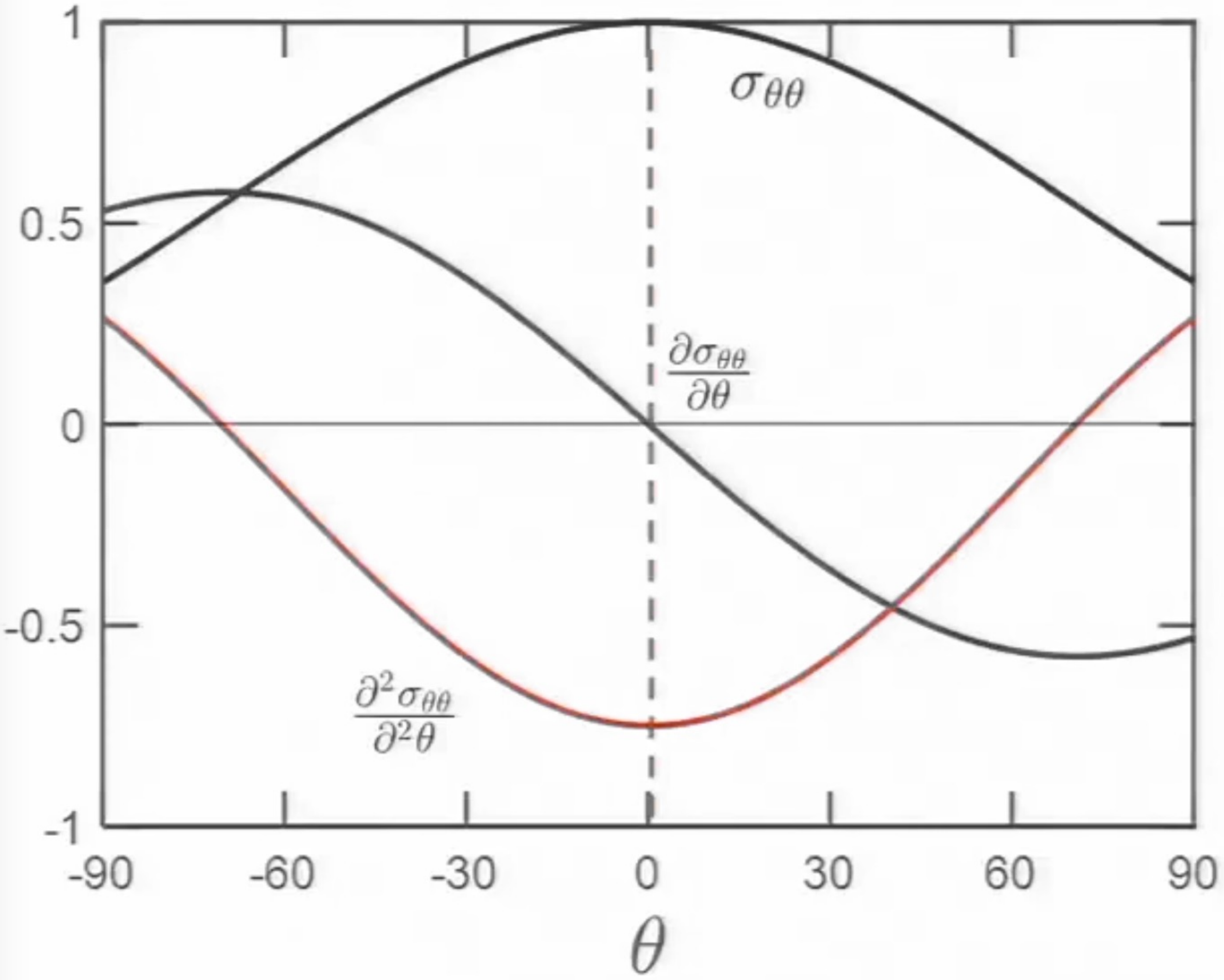
Or shear stress $\tau_{r\theta} = 0$

Physical meaning



- The crack tends to propagate normal to the applied stress, in **pure Mode I loading.**
- No shear stress, think of principal stress.

Physical meaning from math



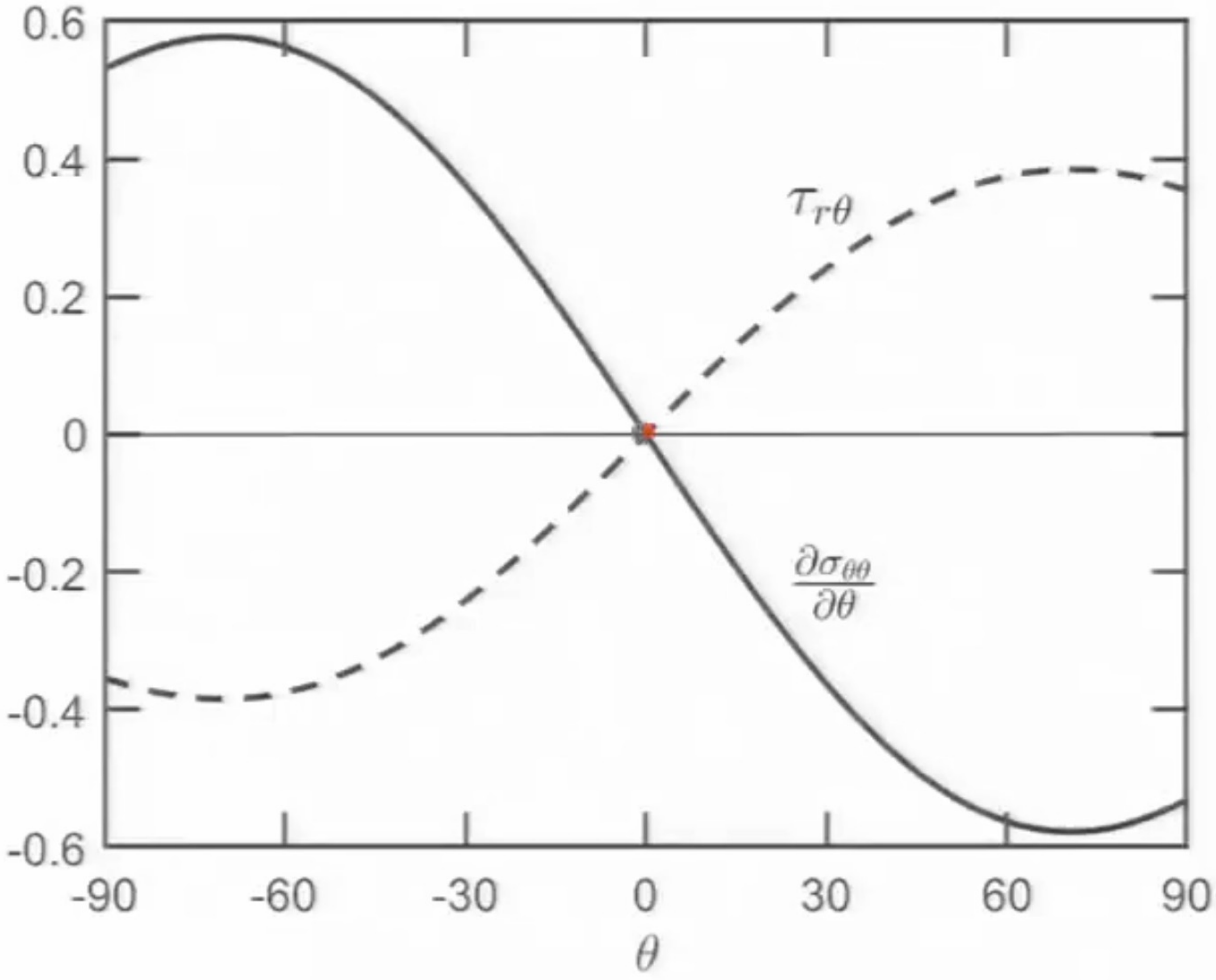
$$\frac{\partial\sigma_{\theta\theta}}{\partial\theta} = 0$$

To find the max/min stress in hoop direction

$$\frac{\partial^2\sigma_{\theta\theta}}{\partial^2\theta} < 0$$

To make sure this the maxima!

Physical meaning from math



$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0$$

is equivalent to

$$\tau_{r\theta} = 0$$

Maximum hoop stress criterion

Stress field at the crack tip in terms of K_I:

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{5}{4}\cos\left(\frac{\theta}{2}\right) - \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{3}{4}\cos\left(\frac{\theta}{2}\right) + \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin\left(\frac{\theta}{2}\right) + \frac{1}{4}\sin\left(\frac{3\theta}{2}\right)\right]$$

Stress field at the crack tip in terms of K_{II}:

$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{5}{4}\sin\left(\frac{\theta}{2}\right) + \frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{\theta\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{3}{4}\sin\left(\frac{\theta}{2}\right) - \frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\cos\left(\frac{\theta}{2}\right) + \frac{3}{4}\cos\left(\frac{3\theta}{2}\right)\right]$$

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0 \qquad \exists \qquad \tau_{r\theta} = 0$$

Mode I:

$$(\cos\theta)' = -\sin\theta \qquad (\sin\theta)' = \cos\theta$$

$$f(\alpha(x))' = \alpha'(x) f'(\alpha(x))$$

$$-\frac{3}{4}\left(\frac{1}{2}\right)\sin\frac{\theta}{2} - \frac{1}{4}\frac{3}{2}\sin\frac{3\theta}{2}$$

$$-\frac{3}{4}\cdot\left(\frac{1}{2}\right)\sin\frac{\theta}{2} - \frac{1}{4}\,\frac{3}{2}\sin\frac{3\theta}{2}$$

$$= -\frac{3}{8}\sin\frac{\theta}{2} - \frac{3}{8}\sin\frac{3\theta}{2}$$

$$= -\frac{3}{8}\left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right) = 0$$

$$\tau_{r\theta} = \frac{1}{4}\left[\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right] = 0$$

Stress field at the crack tip in terms of K_I:

$$\sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{5}{4}\cos\left(\frac{\theta}{2}\right) - \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{3}{4}\cos\left(\frac{\theta}{2}\right) + \frac{1}{4}\cos\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin\left(\frac{\theta}{2}\right) + \frac{1}{4}\sin\left(\frac{3\theta}{2}\right)\right]$$

Stress field at the crack tip in terms of K_{II}:

$$\sigma_{rr} = \frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{5}{4}\sin\left(\frac{\theta}{2}\right) + \frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\sigma_{\theta\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left[-\frac{3}{4}\sin\left(\frac{\theta}{2}\right) - \frac{3}{4}\sin\left(\frac{3\theta}{2}\right)\right]$$

$$\tau_{r\theta} = \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\cos\left(\frac{\theta}{2}\right) + \frac{3}{4}\cos\left(\frac{3\theta}{2}\right)\right]$$

Consider crack is under **mode I (tensile)** and **mode II (shear)** loading:

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right] + \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right]$$

Maximum hoop stress criterion

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}] + \frac{K_{II}}{\sqrt{2\pi r}}[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}] = 0$$

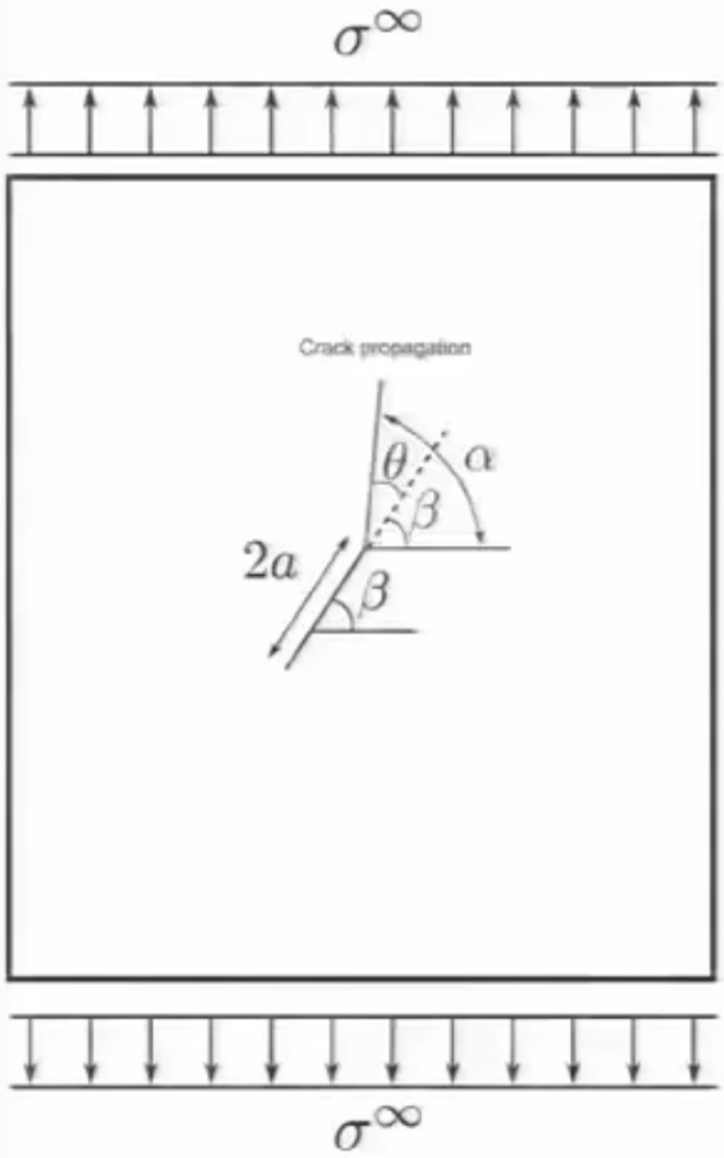
$$\frac{K_I}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}}\cos\frac{\theta}{2}[1 - 3\sin^2\frac{\theta}{2}] = 0$$

$$\boxed{K_I \sin\theta + 3K_{II}\cos\theta - K_{II} = 0}$$

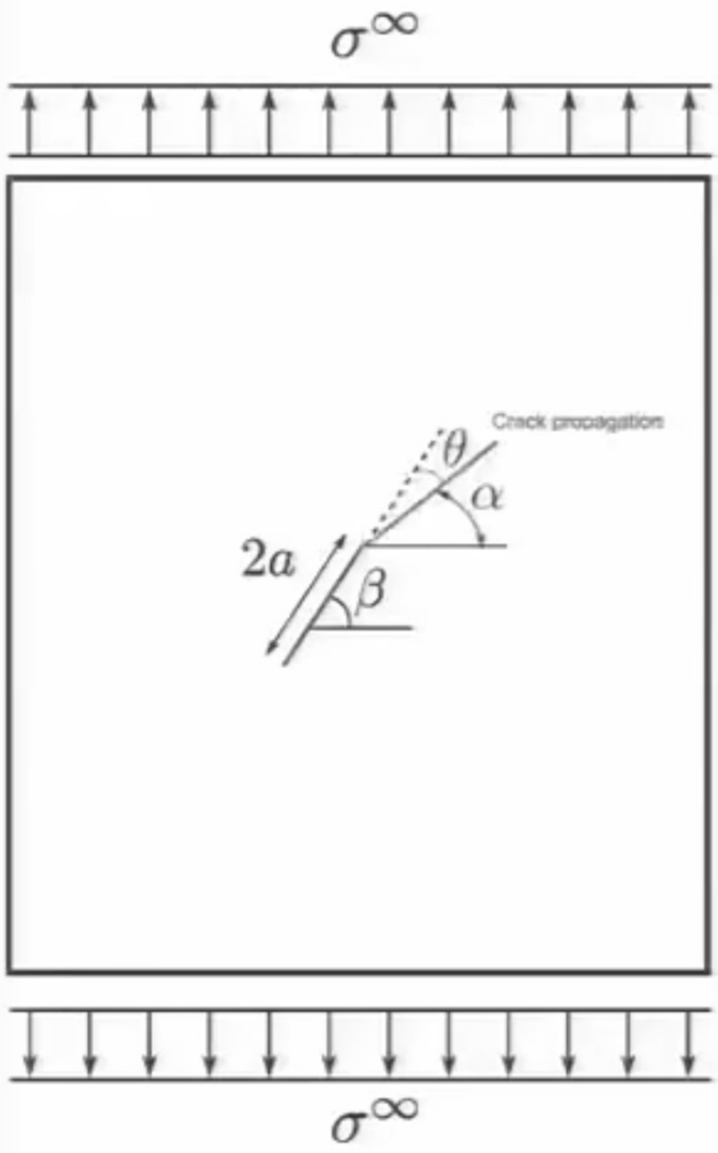
$$2\tan^2\frac{\theta}{2} - \frac{K_I}{K_{II}}\tan\frac{\theta}{2} - 1 = 0$$

$$(\tan\frac{\theta}{2})_{1,2} = \frac{1}{4}\frac{K_I}{K_{II}} \pm \sqrt{(\frac{K_I}{4K_{II}})^2 + \frac{1}{2}} \qquad (\theta)_{1,2} = 2\tan^{-1}(\frac{1}{4}\frac{K_I}{K_{II}} \pm \sqrt{(\frac{K_I}{4K_{II}})^2 + \frac{1}{2}})$$

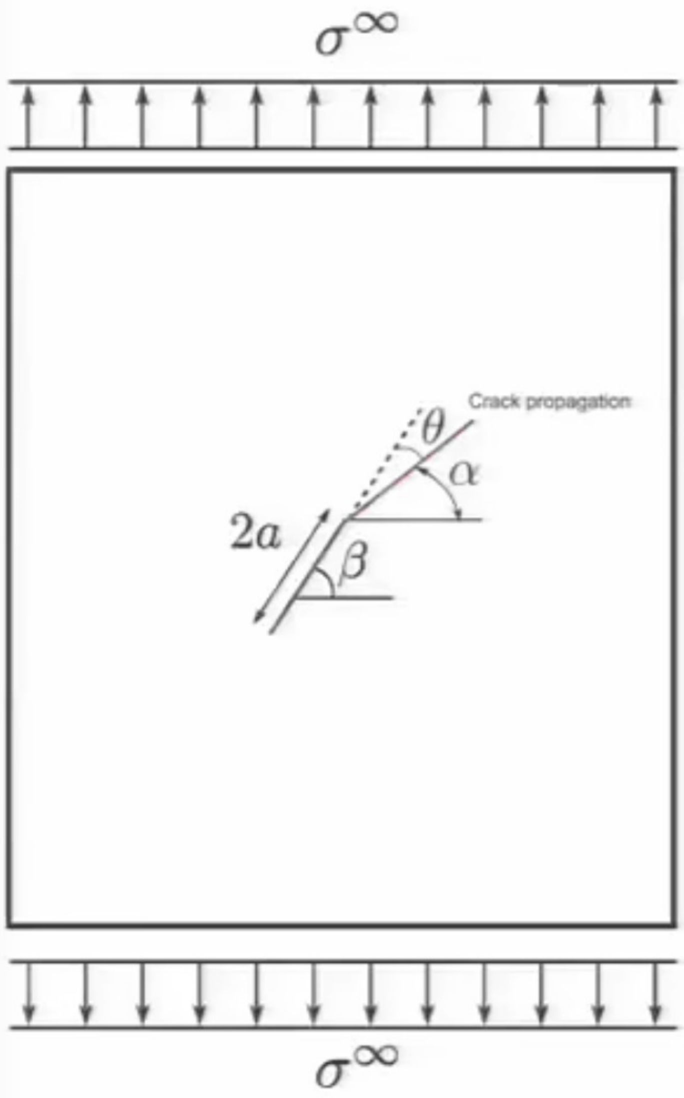
For uniaxial loading



For uniaxial loading



For uniaxial loading



$$\sigma_{x'} = \cos^2\theta\,\sigma_x + \sin^2\theta\,\sigma_y + 2\cos\theta\sin\theta\,\tau_{xy}$$

$$\sigma_{y'} = \sin^2\theta\,\sigma_x + \cos^2\theta\,\sigma_y - 2\cos\theta\sin\theta\,\tau_{xy}$$

$$\tau_{x'y'} = \sin\theta\cos\theta(\sigma_y - \sigma_x) + (\cos^2\theta - \sin^2\theta)\tau_{xy}$$

Since $\sigma_x = 0, \sigma_y = \sigma^{\infty}, \tau_{xy} = 0$

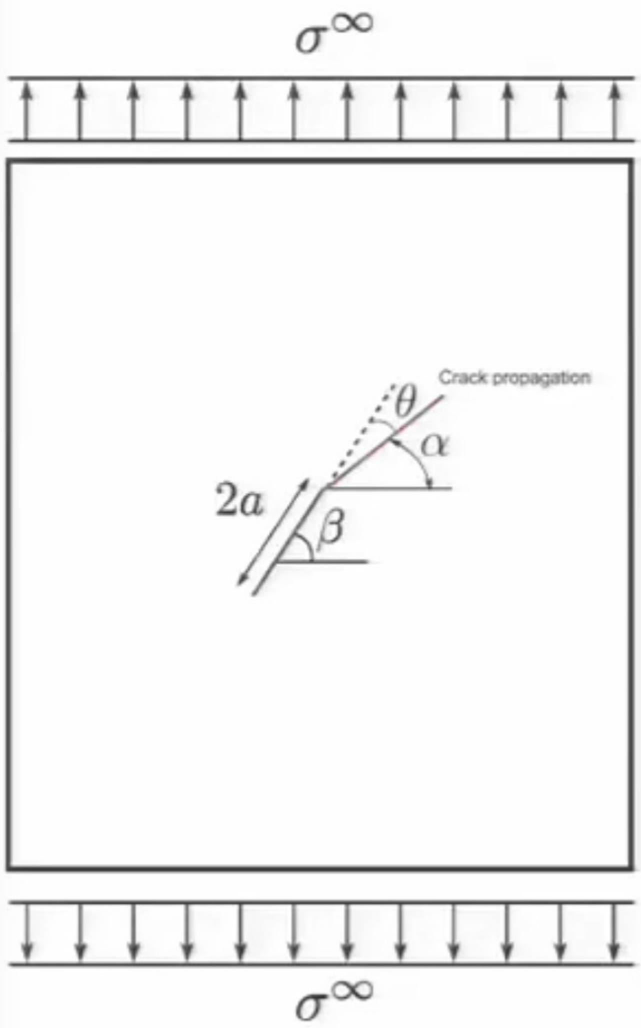
$$K_I = Y_I \sigma_{y'} \sqrt{\pi a}$$

$$K_{II} = Y_{II} \tau_{x'y'} \sqrt{\pi a}$$

$$\boxed{K_I = \sigma^{\infty} \cos^2 \beta \sqrt{\pi a}}$$

$$\boxed{K_{II} = \sigma^{\infty} \sin\beta \cos\beta \sqrt{\pi a}}$$

For uniaxial loading



$$\alpha = \beta + \theta$$

$$K_I = \sigma^\infty \cos^2 \beta \sqrt{\pi a}$$

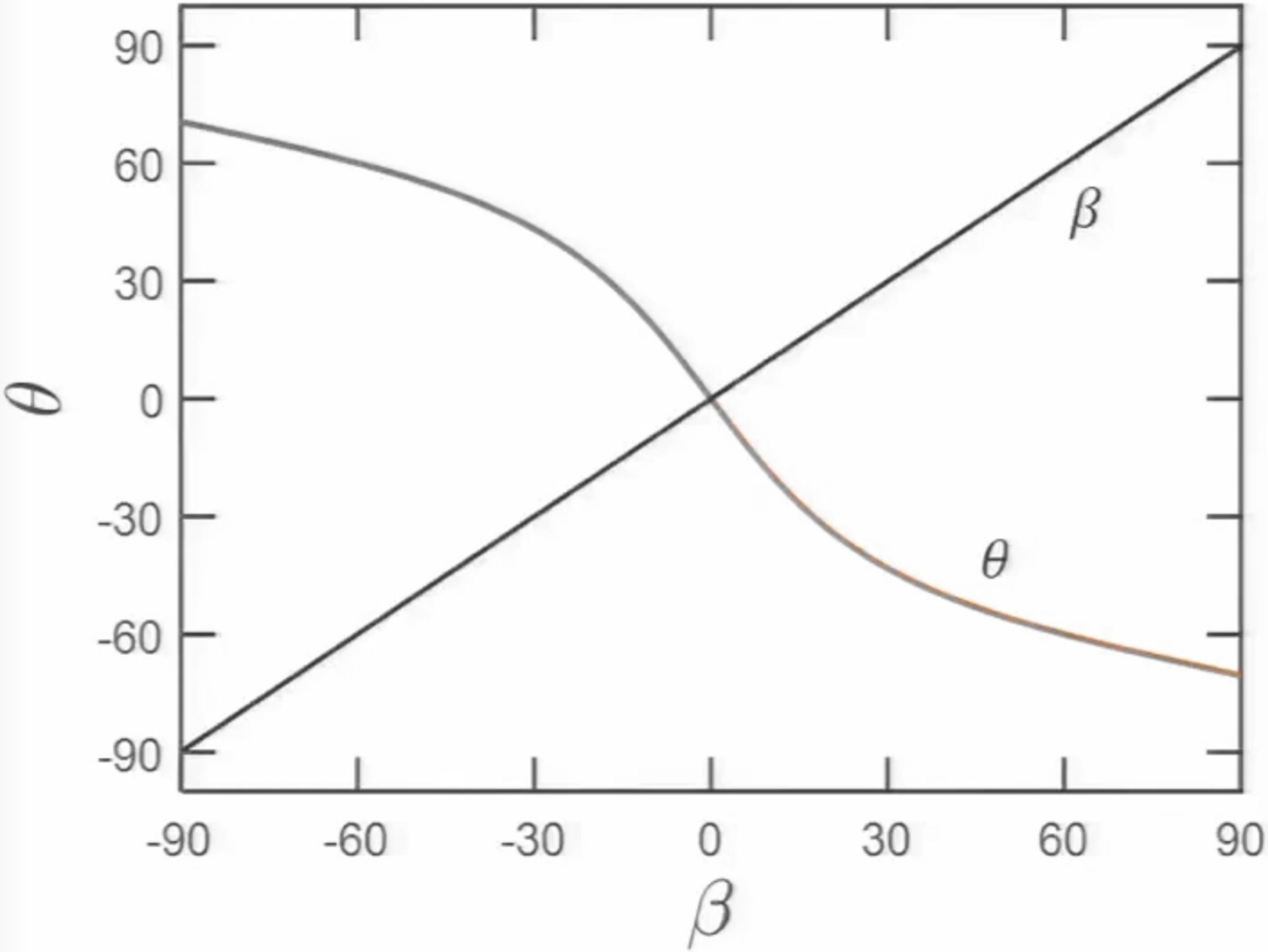
$$K_{II} = \sigma^\infty \sin \beta \cos \beta \sqrt{\pi a}$$

$$\frac{K_I}{K_{II}} = \frac{\cos^2 \beta}{\sin \beta \cos \beta} = \frac{1}{\tan \beta}$$

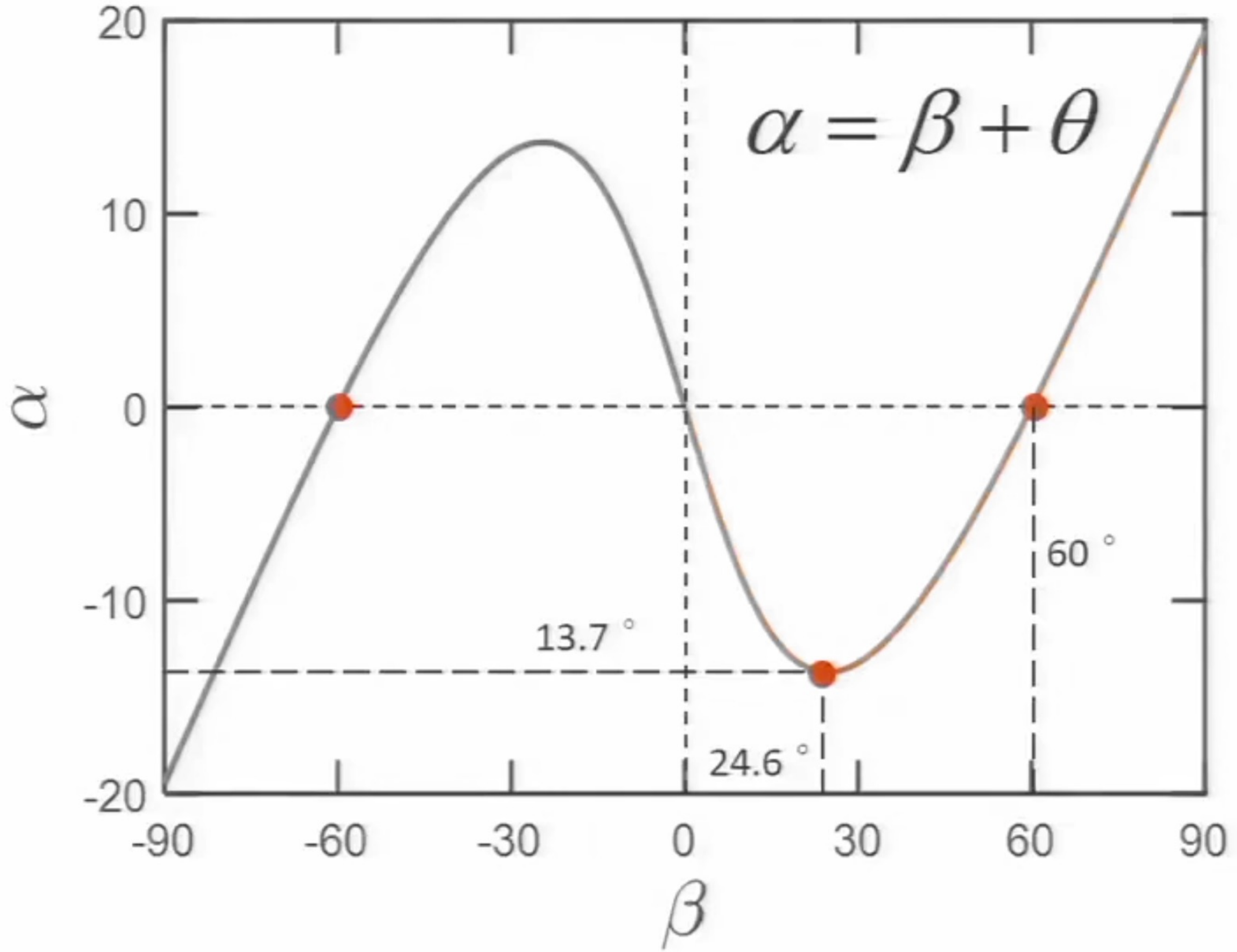
$$\theta = 2 \tan^{-1}\left(\frac{1}{4}\frac{K_I}{K_{II}} \pm \sqrt{\left(\frac{K_I}{4K_{II}}\right)^2 + \frac{1}{2}}\right)$$

$$\alpha = \beta + 2 \tan^{-1}\left(\frac{1}{4 \tan \beta} \pm \sqrt{\left(\frac{1}{4 \tan \beta}\right)^2 + \frac{1}{2}}\right)$$

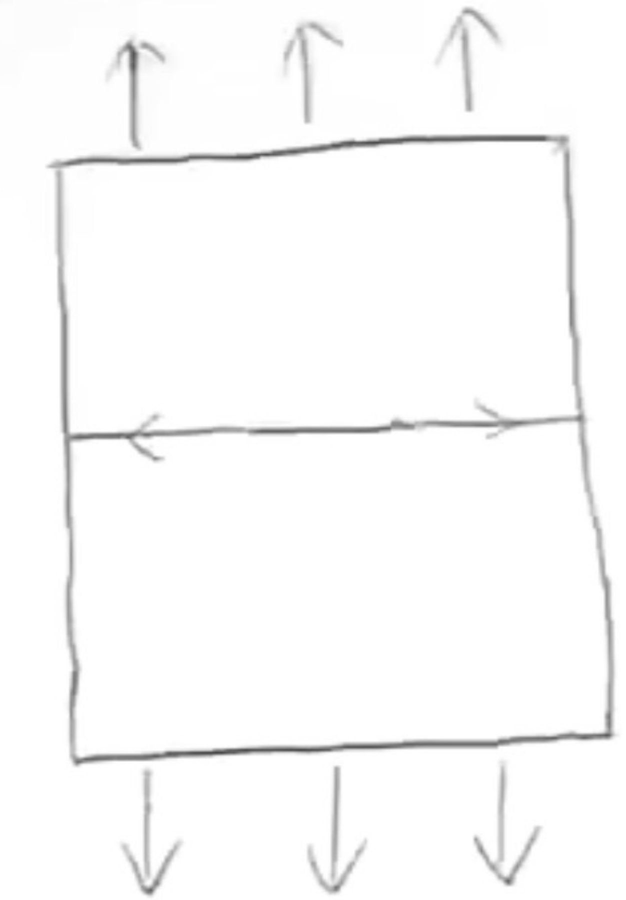
How does θ change with β?



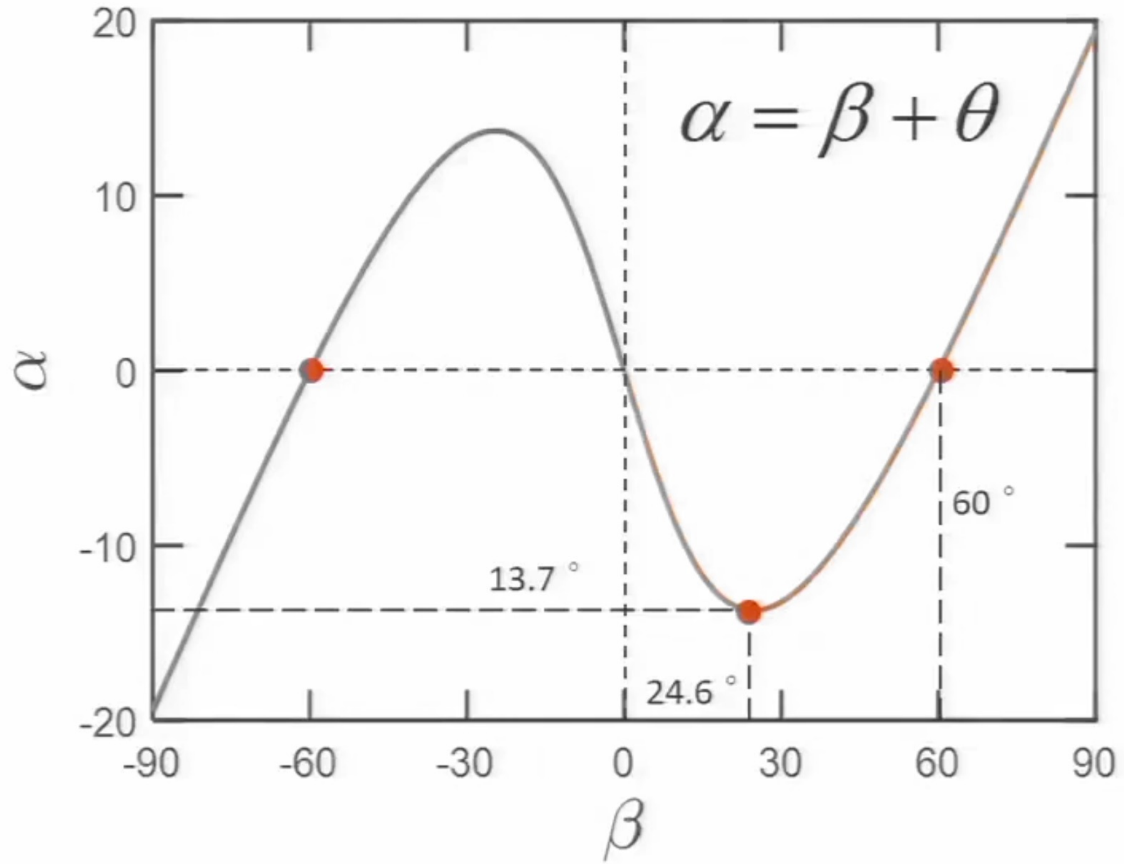
The relation between $\alpha - \beta$



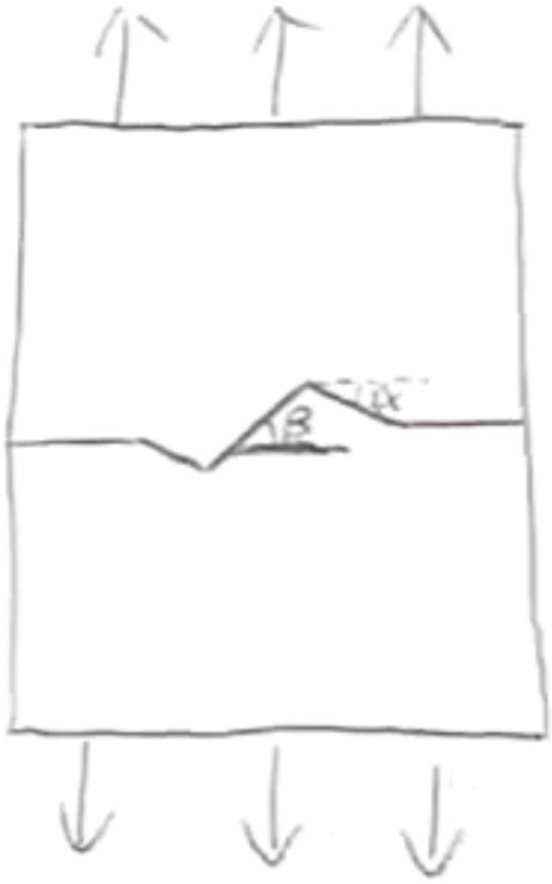
Let set $\beta=0^\circ$



The relation between $\alpha-\beta$



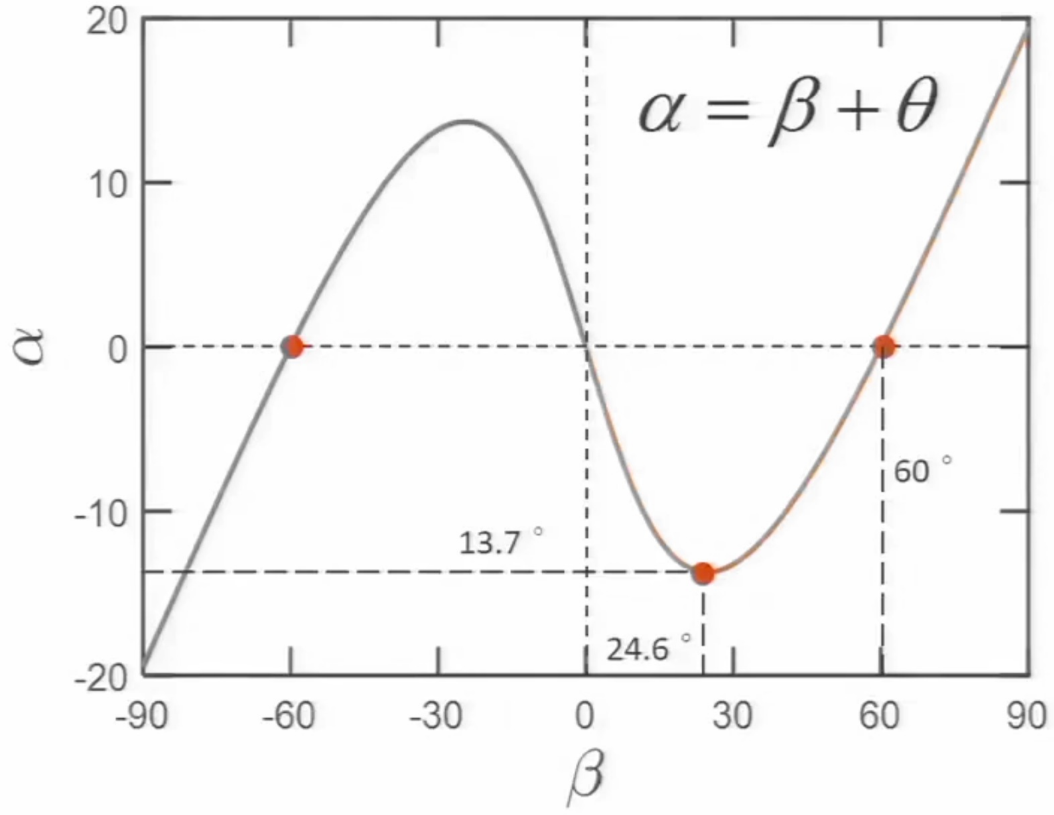
$\beta = 24.6^\circ$

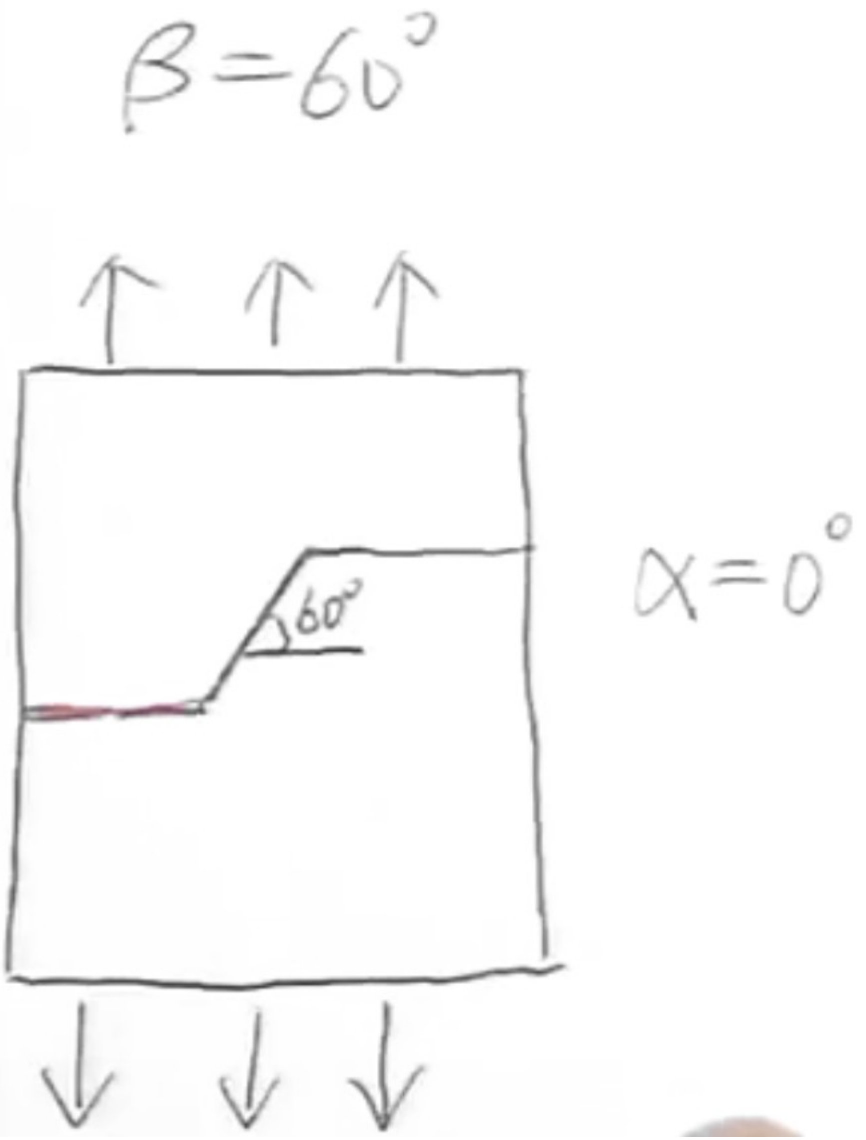


$\alpha = -13.7^\circ$

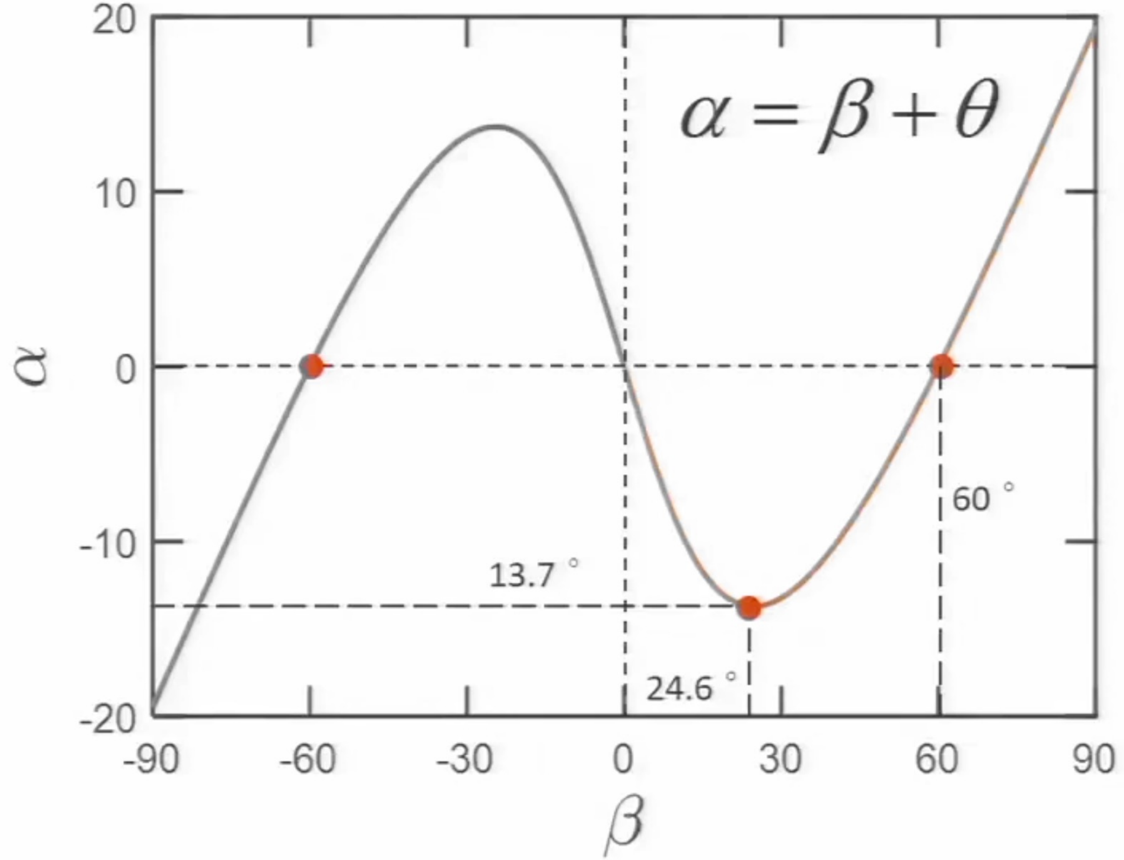
Crack kinking

The relation between $\alpha - \beta$





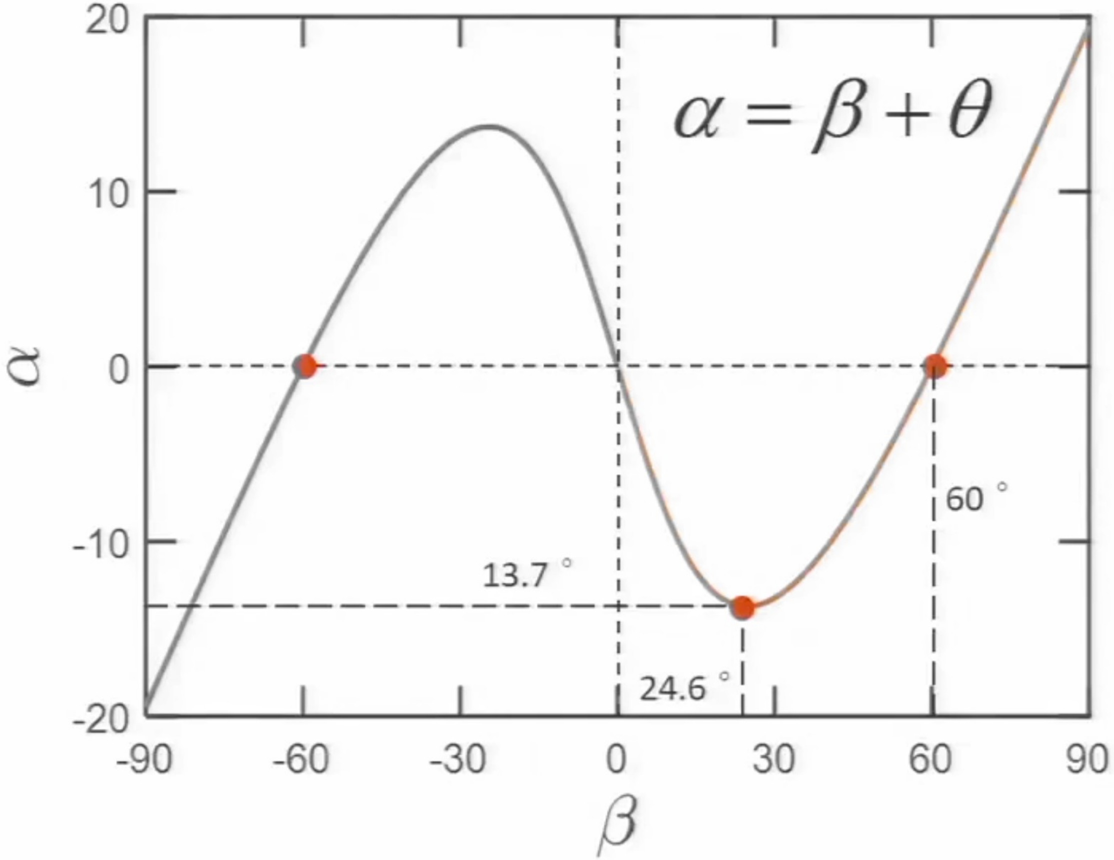
The relation between $\alpha - \beta$



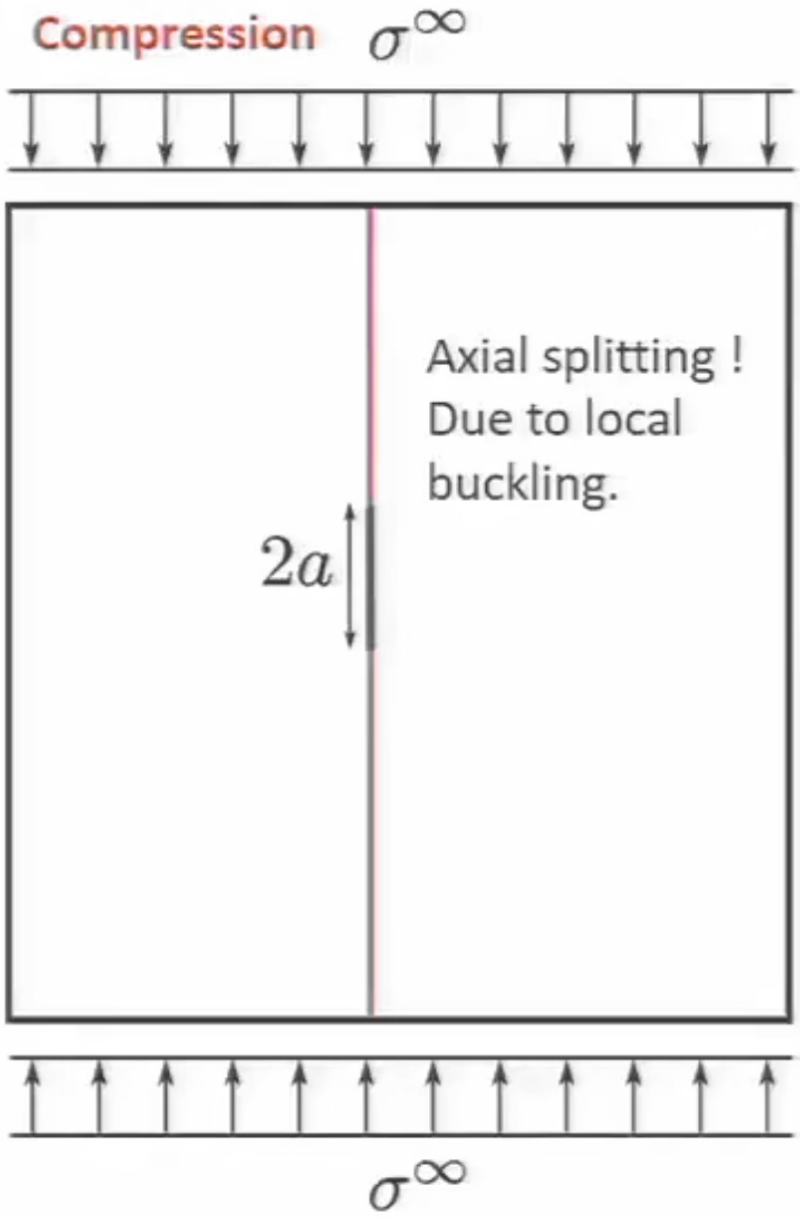
$60^\circ < \beta < 90^\circ \quad \beta = 80^\circ$

$\alpha = 12^\circ$

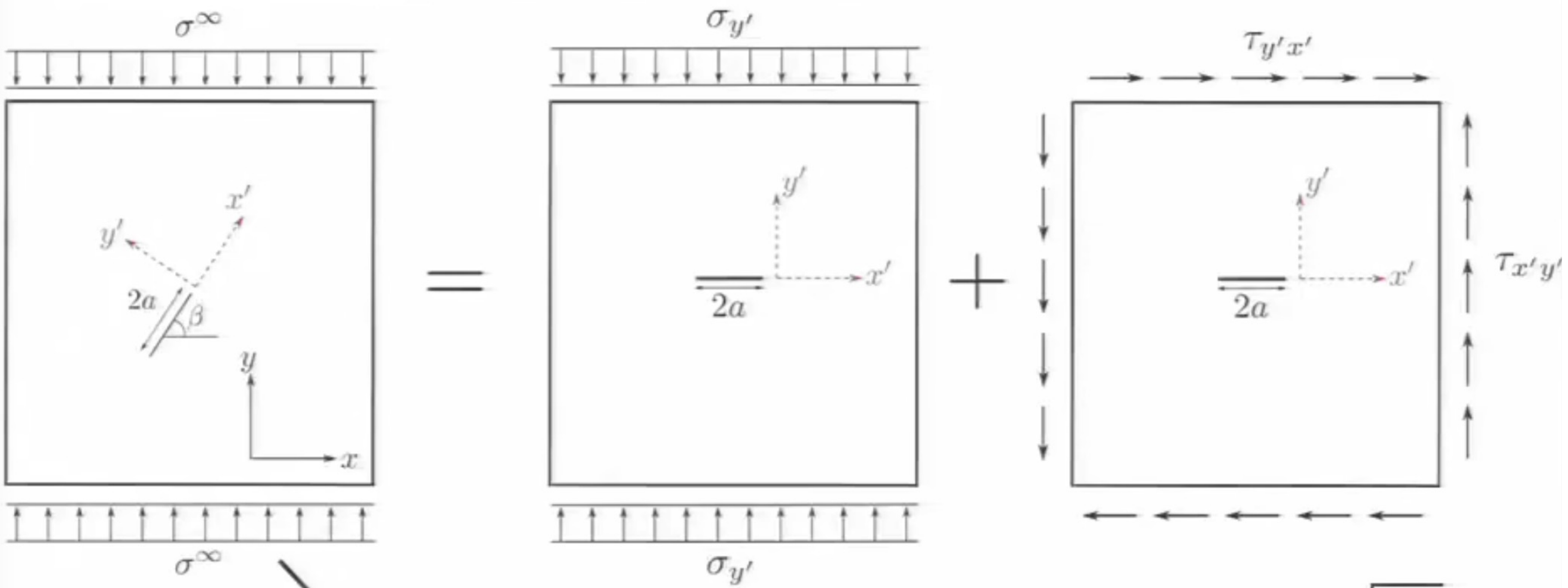
The relation between $\alpha - \beta$



How about these ones?



Uniaxial compression on an inclined crack

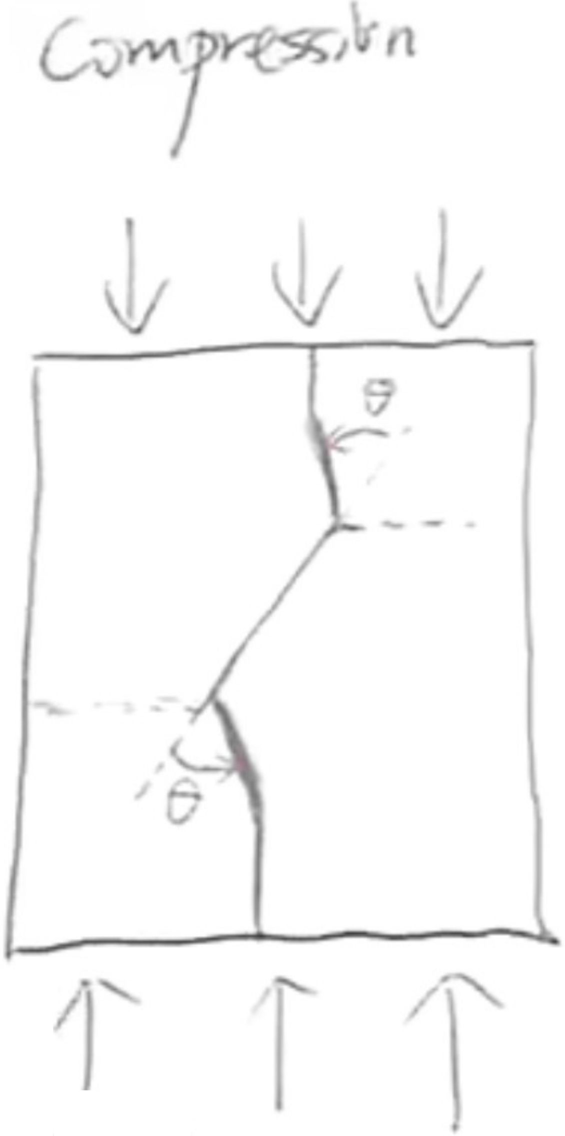


$$K_I = 0 \qquad K_{II} = Y_{II}\tau_{x'y'}\sqrt{\pi a}$$

0

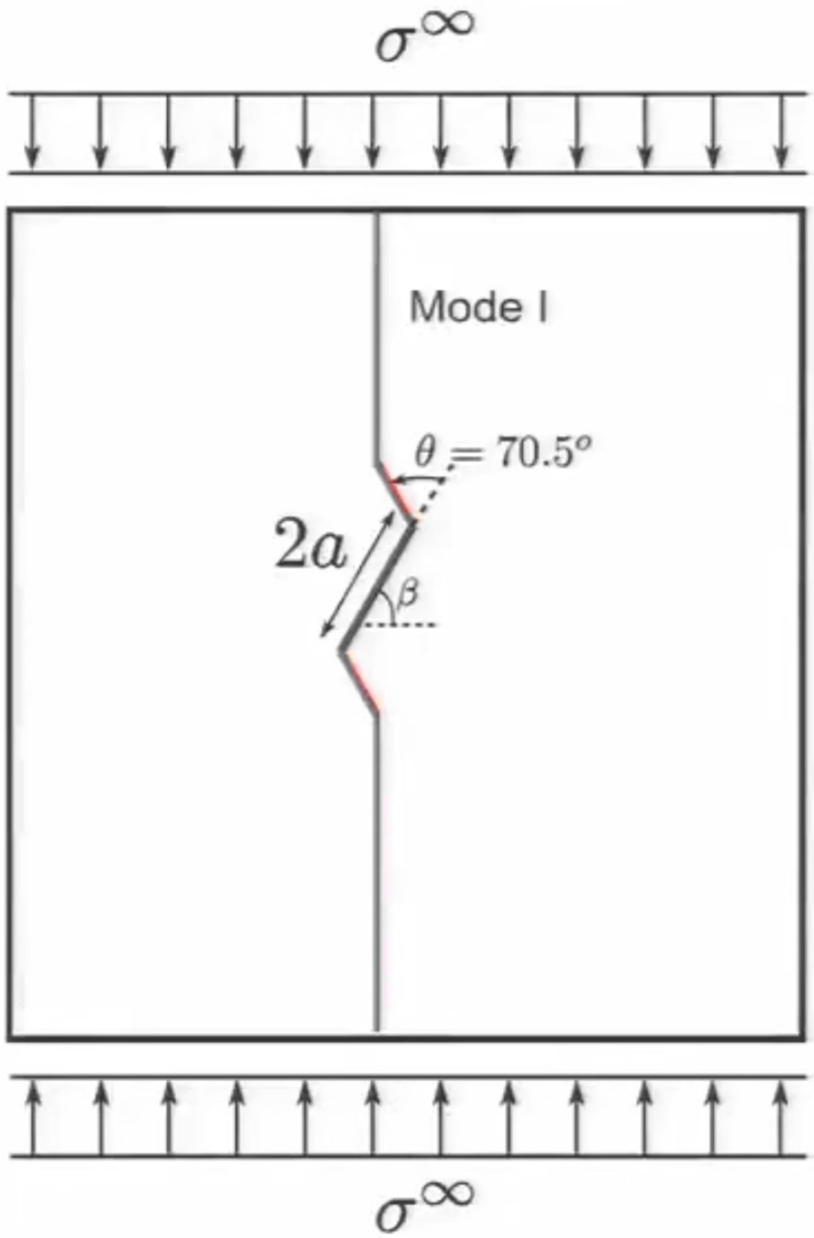
$$K_I \sin\theta + 3K_{II}\cos\theta - K_{II} = 0$$

$$3K_{II}\cos\theta - K_{II} = 0 \qquad \cos\theta = \frac{1}{3} \qquad \theta = 70.5^\circ$$

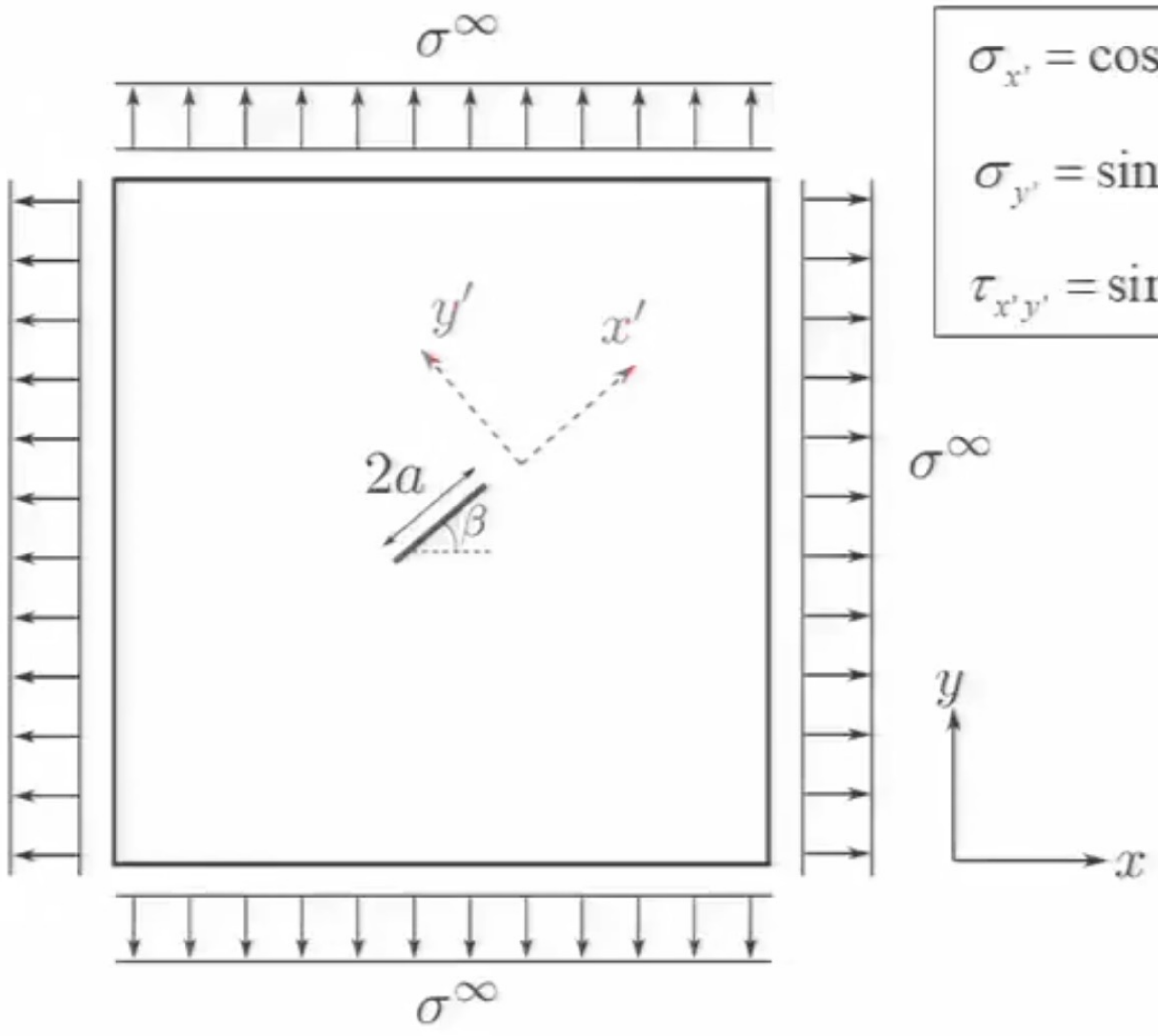


$$\alpha = \beta + \theta$$
$$= 45^\circ + 72.5^\circ$$
$$= 114.5^\circ > 90^\circ$$

Crack path for compression



Biaxial loading



$$\sigma_{x'} = \cos^2\theta\,\sigma_x + \sin^2\theta\,\sigma_y + 2\cos\theta\sin\theta\,\tau_{xy}$$

$$\sigma_{y'} = \sin^2\theta\,\sigma_x + \cos^2\theta\,\sigma_y - 2\cos\theta\sin\theta\,\tau_{xy}$$

$$\tau_{x'y'} = \sin\theta\cos\theta(\sigma_y - \sigma_x) + (\cos^2\theta - \sin^2\theta)\tau_{xy}$$

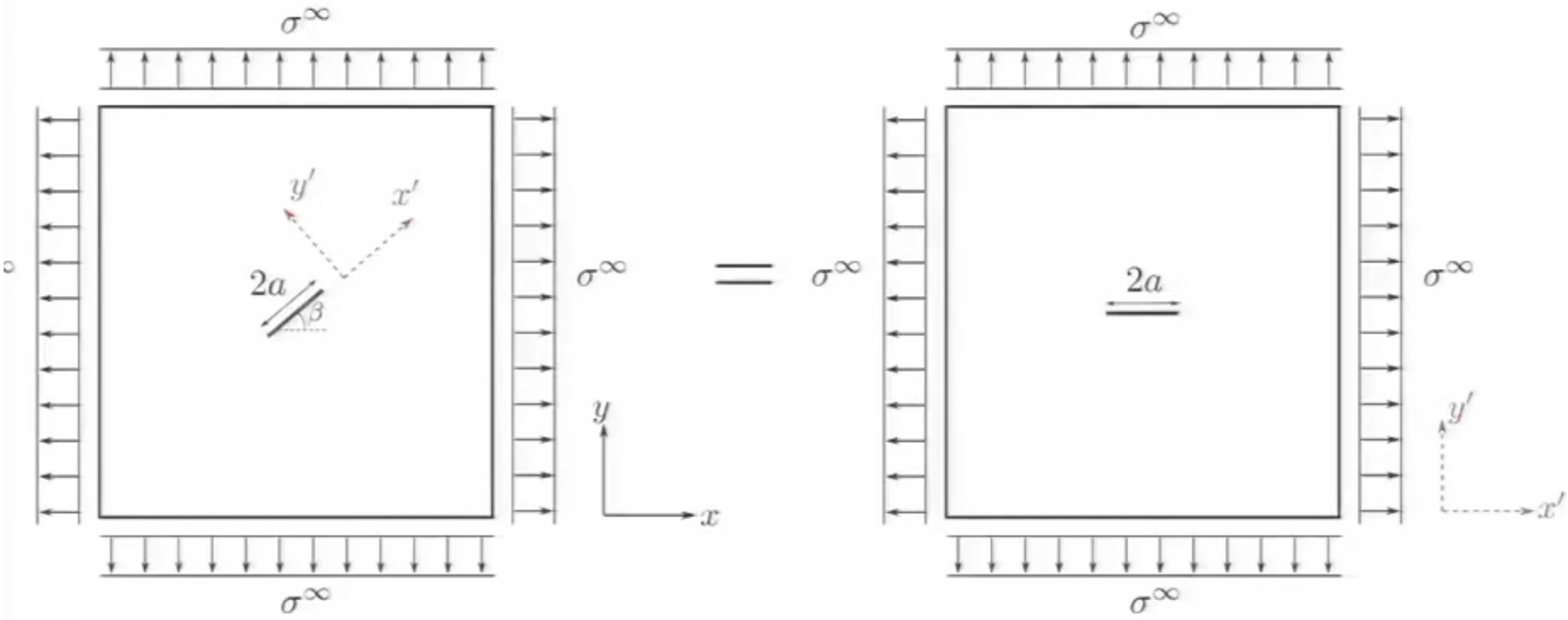
Since $\sigma_x = \sigma^\infty, \sigma_y = \sigma^\infty, \tau_{xy} = 0$

$$\sigma_{x'} = \cos^2\theta\,\sigma^\infty + \sin^2\theta\,\sigma^\infty = \sigma^\infty$$

$$\sigma_{y'} = \sin^2\theta\,\sigma^\infty + \cos^2\theta\,\sigma^\infty = \sigma^\infty$$

$$\tau_{x'y'} = (\sigma^\infty - \sigma^\infty)\sin\beta\cos\beta + 0 = 0$$

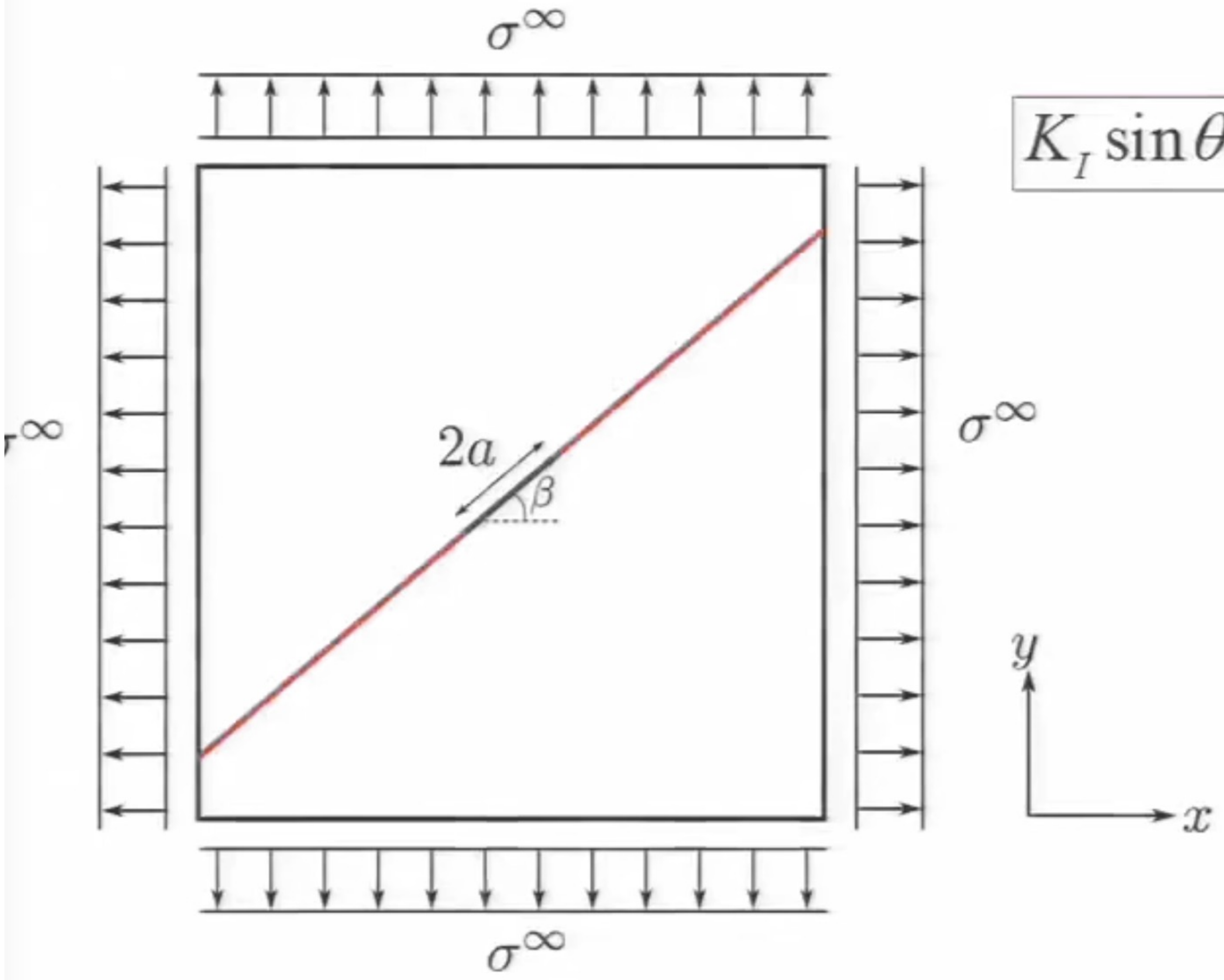
Biaxial loading



$$K_I = Y_I \sigma^{\infty} \sqrt{\pi a}$$

$$K_{II} = 0$$

Crack direction



$$\boxed{K_I \sin\theta + 3K_{II}\cos\theta - K_{II} = 0}$$

$$K_{II} = 0$$

$$K_I \sin\theta = 0$$

$$\sin\theta = 0$$

$$\theta = 0^\circ$$

$$\alpha = \beta + \theta = \beta$$

Appendix-derivation of θ

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}] + \frac{K_{II}}{\sqrt{2\pi r}}[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}] = 0 \quad \textbf{(1)}$$

$$\sin\frac{3\theta}{2} = \sin(\theta + \frac{\theta}{2}) = \sin\theta\cos\frac{\theta}{2} + \cos\theta\sin\frac{\theta}{2}$$

$$\cos\frac{3\theta}{2} = \cos(\theta + \frac{\theta}{2}) = \cos\theta\cos\frac{\theta}{2} - \sin\theta\sin\frac{\theta}{2}$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$$

***Trigonometric function**

Appendix-derivation of θ

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}\left[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right] + \frac{K_{II}}{\sqrt{2\pi r}}\left[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}\right] = 0 \quad \textbf{(1)}$$

$$\begin{aligned}\left[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}\right] &= \frac{1}{4}\left[\sin\frac{\theta}{2} + \sin\theta\cos\frac{\theta}{2} + \cos\theta\sin\frac{\theta}{2}\right] \\ &= \frac{1}{4}\left[\sin\frac{\theta}{2} + 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2} + \cos\theta\sin\frac{\theta}{2}\right] \\ &= \frac{1}{4}\sin\frac{\theta}{2}\left[1 + 2\cos^2\frac{\theta}{2} + \cos\theta\right] \\ &= \frac{1}{4}\sin\frac{\theta}{2}\left[1 + 2\cos^2\frac{\theta}{2} + 2\cos^2\frac{\theta}{2} - 1\right] \\ &= \sin\frac{\theta}{2}\cos^2\frac{\theta}{2} \qquad \textbf{(2)}\end{aligned}$$

Appendix-derivation of θ

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}] + \frac{K_{II}}{\sqrt{2\pi r}}\boxed{[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}]} = 0 \quad \textbf{(1)}$$

Appendix-derivation of θ

$$\tau_{r\theta} = \frac{K_I}{\sqrt{2\pi r}}[\frac{1}{4}\sin\frac{\theta}{2} + \frac{1}{4}\sin\frac{3\theta}{2}] + \frac{K_{II}}{\sqrt{2\pi r}}[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}] = 0 \quad \textbf{(1)}$$

$$[\frac{1}{4}\cos\frac{\theta}{2} + \frac{3}{4}\cos\frac{3\theta}{2}] = \frac{1}{4}[\cos\frac{\theta}{2} + 3(\cos\theta\cos\frac{\theta}{2} - \sin\theta\sin\frac{\theta}{2})]$$

$$= \frac{1}{4}[\cos\frac{\theta}{2} + 3(\cos\theta\cos\frac{\theta}{2} - 2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2})]$$

$$= \frac{1}{4}\cos\frac{\theta}{2}[1 + 3\cos\theta - 6\sin^2\frac{\theta}{2}]$$

$$= \frac{1}{4}\cos\frac{\theta}{2}[1 + 3(1 - \sin^2\frac{\theta}{2}) - 6\sin^2\frac{\theta}{2}]$$

$$= \frac{1}{4}\cos\frac{\theta}{2}[4 - 12\sin^2\frac{\theta}{2}] = \cos\frac{\theta}{2}(1 - 3\sin^2\frac{\theta}{2}) \quad \textbf{(3)}$$