April 29th.

Ex. 1 Let h be a $\mathcal{C}^1([0,1]\times\mathbb{R})$ function

Define $g:\mathbb{R}\to\mathbb{R}$, $g(x)=\int_0^1 h(s x)\,ds$

Prove that g is a $\mathcal{C}^1$ function

solution

let's prove that g is continuous

consider $(x_n)_n$ and $\bar{x}$ with $x_n \xrightarrow{n} \bar{x}$ $\quad \bar{x}-1\le x_n\le \bar{x}+1$

$$g(x_n)=\int_0^1 \underbrace{h(s,x_n)}_{f_n(s)}\,ds$$

$f_n(s)\longrightarrow f(s) = h(s,\bar{x})$ (1)

and $|f_n(s)| = |h(s,x_n)| \le \max\limits_{\substack{s\in[0,1]\\ x\in[\bar{x}-1,\bar{x}+1]}} |h(s,x)| \le \underline{\underline{\text{cost}}}$ (2)

(1) + (2) $\Rightarrow$ we can use the dominated convergence ten.

$$\lim_n \int_0^1 f_n(s)\,ds = \int_0^1 f(s)\,ds$$

$$\| \qquad\qquad \|$$

$$g(x_n) \qquad\qquad g(\bar{x}) \quad \Rightarrow g \text{ is continuous.}$$

Similarly consider

$$\underbrace{\frac{g(x_n)-g(\bar{x})}{x_n-\bar{x}}}_{\int_0^1 \tilde{f}_n(x)} \longrightarrow \int_0^1 \frac{\partial h}{\partial x}(s,x)\,ds$$

$\int_0^1 \tilde{f}_n(x)$ ← apply the dominited convergence.

Ex. Prove that $id_{\mathbb{R}} \cdot \delta_0' = -\delta_0$

$$\boxed{id_{\mathbb{R}} \cdot PV_{\frac{1}{x}} = T_1}$$

sol. $f \cdot T(\varphi) = T(f \cdot \varphi)$ $\qquad$ $id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$, $x \longmapsto x$

$$x(\delta_0')(\varphi(x)) = \delta_0'(x\varphi(x))$$

$$= -\delta_0(\cdot(x\varphi(x))') = -\delta_0(\varphi(x) + x\varphi'(x))$$

$$= -(\varphi(x) + x\varphi'(x))\Big|_{x=0} = -(\varphi(0) + \underbrace{0 \cdot \varphi'(0)}_{=0})$$

$$= -\varphi(0)$$

$$= -\delta_0(\varphi)$$

$$x(\delta_0')(\varphi) = -\delta_0(\varphi)$$

$$\boxed{x\,\delta_0' = -\delta_0}$$

$$\boxed{x \cdot PV_{\frac{1}{x}} = 1}$$

$$\boxed{\text{for functions} \quad x \cdot \frac{1}{x} = 1}$$

$$x \cdot PV_{\frac{1}{x}}(\varphi) = PV_{\frac{1}{x}}(x \cdot \varphi(x)) = \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{x\varphi(x)}{x}\,dx$$

$$= \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \varphi(x)$$

(pass to the limit under the sign of $\int$)

$$= \int_{\mathbb{R}} \varphi(x)\,dx = \int_{\mathbb{R}} 1 \cdot \varphi(x)\,dx$$

$$= T_1(\varphi)$$

$$x \cdot PV_{\frac{1}{x}}(\varphi) = T_1(\varphi)$$

$$\boxed{x \cdot PV_{\frac{1}{x}} = T_1}$$

Ex. Let $f \in \mathcal{C}^\infty(\mathbb{R})$

Prove that there exist $\alpha \in \mathbb{R}$ and $g \in \mathcal{C}^\infty(\mathbb{R})$

s.t. $$f \cdot PV_{\frac{1}{x}} = \alpha PV_{\frac{1}{x}} + T_g$$

sol.

$$f \cdot PV_{\frac{1}{x}}(\varphi) = PV_{\frac{1}{x}}(f\varphi)$$

$$= \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{f(x)\varphi(x)}{x} dx$$

$$= \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \left(\underbrace{\frac{f(x) - f(0)}{x}}_{g(x) \text{ from ex. 2.}} \cdot \varphi(x) + \frac{f(0)}{x} \varphi(x) \right) dx$$

$$= \underbrace{\lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} g(x) \cdot \varphi(x)} + \lim_{\varepsilon \to 0^+} \overset{\alpha \in \mathbb{R}}{\overset{\shortparallel}{f(0)}} \cdot \int_{|x| \geq \varepsilon} \frac{\varphi(x)}{x} dx$$

$$= \int_{\mathbb{R}} g(x) \varphi(x) + \alpha \cdot PV_{\frac{1}{x}}(\varphi)$$

$$= T_g(\varphi) + \alpha PV_{\frac{1}{x}}(\varphi)$$

Ex. 5. Find all the distributions $T \in \mathcal{D}'(\mathbb{R})$

s.t. $x \cdot T = 0$ and $x \cdot T = 1 = T_1$

$x \cdot T = 1$

$\parallel$

$T \sim \frac{1}{x}$ Hint $PV\frac{1}{x}$?

sol.

$x \cdot T = 0$

means that, for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$

$x \cdot T(\varphi) = 0$

so for all $\varphi \in \mathcal{D}(\mathbb{R})$, $T(x\varphi(x)) = 0$

I take $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\chi = 1$ in a nbh of 0

I take a test function ψ (without any restriction)

$$\psi(x) = \chi(x)\psi(x) + (1-\chi(x))\psi(x)$$

$$T(\psi) = T(\chi\psi) + T((1-\chi)\psi)$$

$$= T(\chi\psi) + \underbrace{T\Big(x \cdot \underbrace{\frac{(1-\chi(x))\psi(x)}{x}}\Big)}_{=0}$$

it is a $\mathcal{C}_0^\infty$ function because $x \longmapsto \frac{1-\chi(x)}{x}$ is $\mathcal{C}^\infty$

$\forall \psi \in \mathcal{C}_0^\infty(\mathbb{R})$

$$T(\psi) = T(\chi\psi)$$

$$= T(\chi(x)(\psi(x)-\psi(0))) + \underbrace{T(\chi(x)\psi(0))}$$

$$= \underbrace{T\Big(\underbrace{\chi(x)\Big(\frac{\psi(x)-\psi(0)}{x}\Big)}_{\in\mathcal{D}(\mathbb{R})} \cdot x\Big)}_{=0} \qquad \psi(0) \cdot \underbrace{T(\chi)}_{\text{constant}}$$

fixed

conclusion $\forall \psi \in \mathcal{D}(\mathbb{R})$, $T(\psi) = \psi(0) \cdot \text{cst}$,

in principle depends on χ

not true

if I swap the χ with $\tilde{\chi}$ the constant is the same,

so $T = c \cdot \delta_0$

$$\boxed{x \cdot T = 0 \Rightarrow \mathbf{T} = c \cdot \delta_0 \quad \text{with } c \in \mathbb{R}}$$

$xT = 1$?

we have seen that $x \cdot PV\frac{1}{x} = 1$

so let T s.t. $x \cdot T = 1$

we know $x \cdot PV\frac{1}{x} = 1$ so

$$\boxed{x \cdot (T - PV\tfrac{1}{x}) = 0}$$ so that

$$T - PV\tfrac{1}{x} = c\,\delta_0$$

conclusion if $x \cdot T = 1$ then $\boxed{T = PV\frac{1}{x} + c\,\delta_0}$

Ex. find all the distributions T in $\mathcal{D}'(\mathbb{R})$ s.t.

$$x\,T' + T = 0$$

sol. for form o ?

$$\underbrace{x \cdot u'(x) + u(x)}_{(x\,u(x))'} = 0$$

$(x\,u(x))' = 0$

$x\,u(x) = \text{const}$

$u(x) = \frac{c}{x}$

Hint. $T \sim c \cdot PV_{\frac{1}{x}}$

$$x T' + T = 0$$
$$\Downarrow$$
$$x T'(\varphi) + T(\varphi) = 0$$
$$\Downarrow$$
$$T'(x\varphi(x)) + T(\varphi(x)) = 0$$
$$T\left(-(x\varphi(x))'\right) + T(\varphi(x)) = 0$$
$$T\left(-\varphi(x) + x\varphi'(x)\right) + T(\varphi(x))$$
$$-T(\varphi(x)) + T(x\varphi'(x)) + T(\varphi(x)) = 0$$

if $xT' + T = 0$ then $\boxed{T(x\varphi'(x)) = 0 \quad \forall \varphi \in \mathcal{D}(\mathbb{R})}$

take $\psi \in \mathcal{D}(\mathbb{R})$

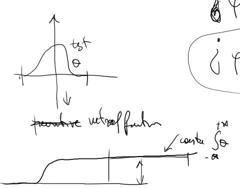
take $\theta \in \mathcal{D}(\mathbb{R})$ s.t. $\int_{-\infty}^{+\infty} \theta(t)\,dt = 1$

fixed

so consider $\varphi(x) = \int_{-\infty}^{x} \psi(t)\,dt - \int_{0}^{x} \theta(t)\,dt \cdot \int_{-\infty}^{+\infty} \psi(t)\,dt$

¿ $\varphi \in \mathcal{E}^{\infty}$? yes (integral function of $\mathcal{E}^{\infty}_{0}$ functions)

¿ $\varphi \in \mathcal{E}^{\infty}_{0}$? yes



$$\varphi'(x) = \psi(x) - \left(\int_{-\infty}^{+\infty} \psi(t)\,dt\right) \cdot \theta(x)$$

const

$$T(x\varphi'(x)) = T(x\psi(x)) - T\left(x \cdot \int_{-\infty}^{+\infty} \psi(t)\,dt \cdot \theta(x)\right)$$

if $T(x\varphi'(x)) = 0$

then, for all ψ, $T(x\psi(x)) - \underbrace{\int_{-\infty}^{+\infty} \psi(t)\,dt}_{T_1(\psi)} \cdot \underbrace{T(x\theta(x))}_{\text{fixed constant}} = 0$

so that $T(x\psi(x)) - c\,T_1(\psi) = 0$

so that $T(x\psi(x)) = c\,T_1$

so $\boxed{x \cdot T = c\,T_1}$

as before

$$xT = T_1$$
$$\Downarrow$$
$$T = PV_{\frac{1}{x}} + c\,\delta_0$$

$$xT = c\,T_1 \Rightarrow \boxed{T = c_1 PV_{\frac{1}{x}} + c_2 \delta_0}$$

$\boxed{c_1, c_2 \in \mathbb{R}}$

verify that $c_1 PV_{\frac{1}{x}} + c_2\delta_0 = T$ satisfies $xT' + T = 0$.

Fourier Transform of functions

def. let $f \in L^1(\mathbb{R}^d)$ let $\xi \in \mathbb{R}^d$

define $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i x \cdot \xi} f(x)\,dx$ $\quad x\cdot\xi = x_1\xi_1 + \dots + x_n\xi_n$

$\forall \xi \in \mathbb{R}^d$ $\hat{f}(\xi)$ is well defined (because $x \longmapsto e^{-i x\cdot\xi} f(x)$ is L^1 for all ξ)

$\hat{f} = \mathcal{F}(f)$ the Fourier transform of f

Th. • $\hat{f} \in L^\infty(\mathbb{R}^d) \cap \mathcal{C}(\mathbb{R}^d)$

• $\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$

• (Riemann-Lebesgue th) $\lim_{|\xi| \to +\infty} \hat{f}(\xi) = 0$

proof.

$\forall \xi \in \mathbb{R}$ $|\hat{f}(\xi)| = \left|\int_{\mathbb{R}^d} e^{-i x\cdot\xi} f(x)\,dx\right| \leq \int_{\mathbb{R}^d} |f(x)|\,dx = \|f\|_{L^1}$

$\|\hat{f}\|_\infty \leq \|f\|_{L^1}$

For the continuity of $\hat{f}$, consider $\xi_n \xrightarrow{n} \bar{\xi}$

then $\hat{f}(\xi_n) \to \hat{f}(\bar{\xi})$ through dominated convergence th.

R-L.

Consider first $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$

$\hat{\varphi}(\xi) = \int e^{-i x\cdot\xi} \varphi(x)\,dx$ $\quad D_{x_j} = -i \frac{\partial}{\partial x_j}$

Then
$$\xi_j \hat{\varphi}(\xi) = \int_{\mathbb{R}^d} \underbrace{\xi_j e^{-i\xi\cdot x}}_{-D_{x_j}(e^{-i\xi\cdot x})} \varphi(x)\,dx$$
$$= \int_{\mathbb{R}^d} \underbrace{-D_{x_j}\left(e^{-i\xi x}\varphi(x)\right)} + e^{-i\xi\cdot x} D_{x_j}\varphi(x)\,dx$$
$$= \int e^{-i x\cdot\xi} D_{x_j}\varphi(x)\,dx = \widehat{D_{x_j}\varphi}(\xi)$$

using several times

$$(1+|\xi|^2)\hat{\varphi}(\xi) = \widehat{(1-\Delta)\varphi}(\xi)$$ Laplacian

but $(1-\Delta)\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d) \subset L^1$

$\widehat{(1-\Delta)\varphi} \in L^\infty$

so $(1+|\xi|^2)\hat{\varphi}(\xi) \in L^\infty$ $\Rightarrow$ $\hat{\varphi}(\xi) \leq \frac{C}{1+|\xi|^2}$

$\Rightarrow \lim_{|\xi| \to +\infty} \hat{\varphi}(\xi) = 0$

let $f \in L^1(\mathbb{R}^d)$

fix $\varepsilon > 0$ then $\underbrace{\exists \bar{\varphi} \in \mathcal{C}_0^\infty(\mathbb{R}^d) : \|f - \bar{\varphi}\|_{L^1} < \varepsilon/2}_{\text{density of } \mathcal{C}_0^\infty(\mathbb{R}^d) \text{ in } L^1(\mathbb{R}^d)}$

we know that $\lim_{|\xi| \to +\infty} \hat{\bar{\varphi}}(\xi) = 0$

$\exists R > 0 : |\xi| \geq R$ then $|\hat{\bar{\varphi}}(\xi)| < \varepsilon/2$

so $\forall |\xi| \geq R$ $\quad |\hat{f}(\xi)| \leq \underbrace{|\hat{f}(\xi) - \hat{\bar{\varphi}}(\xi)|}_{\substack{\|\hat{f} - \hat{\bar{\varphi}}\|_{L^\infty} \\ \leq \|f - \bar{\varphi}\|_{L^1} \\ \leq \varepsilon/2}} + \underbrace{|\hat{\bar{\varphi}}(\xi)|}_{\leq \varepsilon/2} \leq \varepsilon$

ok

Relation between $\hat{f}$ and ∂_{x_j} and multiplication by ξ_j

Th Let $f \in L^1(\mathbb{R}^d)$

Let f diff. w.r.t. x_j and let $\partial_{x_j} f \in L^1(\mathbb{R}^d)$

then

$$\widehat{D_{x_j} f} = \xi_j \hat{f}(\xi)$$

Th, let $f \in L^1(\mathbb{R}^d)$ and

let $x \mapsto x_j f(x) \in L^1(\mathbb{R}^d)$

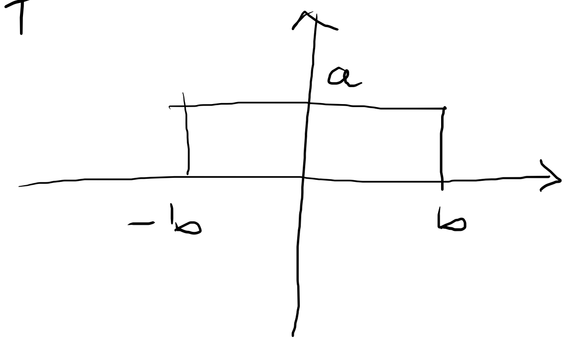
then $\hat{f}$ is diff. w.r.t. ξ_j

and $$D_{\xi_j}\left(\hat{f}(\xi)\right) = -\left(\widehat{x_j f}\right)(\xi)$$

proof (try to use the dominated convergence th).

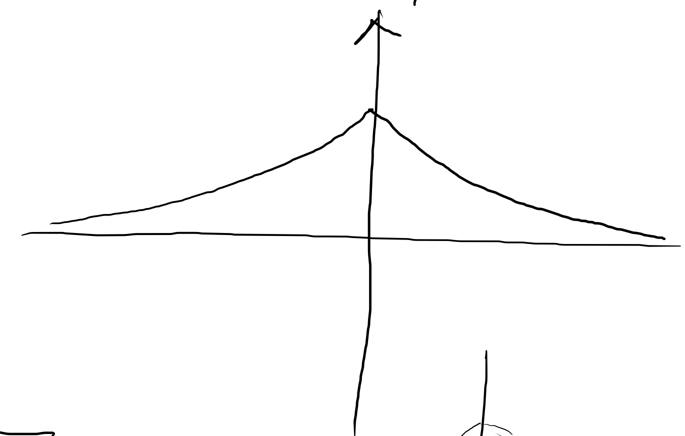
Ex. compute the Fourier transform of

$$f(x) = \begin{cases} a & \text{if } -b \le x \le b \\ 0 & \text{if } |x| > b \end{cases}$$

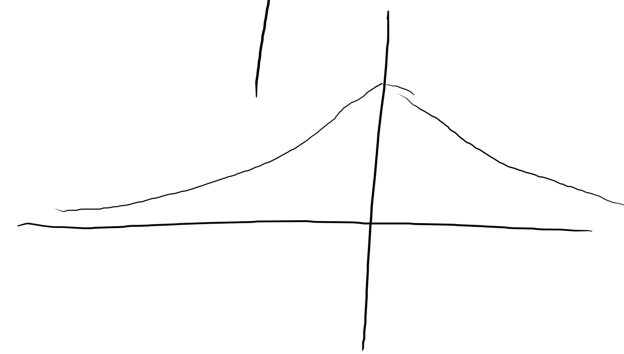


$a > 0$

$$f(x) = \begin{cases} e^{-ax} & \text{if } x \ge 0 \\ e^{ax} & \text{if } x \le 0 \end{cases}$$



$$\boxed{f(x) = e^{-ax^2} \qquad a > 0}$$



$$\hat{f}(\xi) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{|\xi|^2}{4a}}$$

Ex. let f be a $\mathcal{C}^\infty(\mathbb{R})$

let $g(x) = \begin{cases} \dfrac{f(x) - f(0)}{x} & \text{if } x \neq 0 \\ f'(0) & \text{if } x = 0 \end{cases}$

Prove that g is $\mathcal{C}^\infty(\mathbb{R})$

sol.

$$f(x) = f(0) + \int_0^x f'(t)\,dt \qquad t = sx \quad dt = x\,ds$$

$$= f(0) + \int_0^1 x f'(sx)\,ds$$

$$\frac{f(x) - f(0)}{x} = \int_0^1 f'(sx)\,ds \qquad x \neq 0$$

for $x = 0$ $\qquad \int_0^1 f'(0)\,ds = f'(0)$

we obtain that $\qquad g(x) = \int_0^1 \underbrace{f'(sx)}_{h(s,x)}\,ds \quad \forall x \in \mathbb{R}$

$f' \in \mathcal{C}^\infty$

$h \in \mathcal{C}^\infty [0,1] \times \mathbb{R}$

Ex 1 $\Rightarrow$ $g \in \mathcal{C}^1$

$$g'(x) = \int_0^1 s f''(sx)\,ds$$

again g' is $\mathcal{C}^1$

and so on

$$g^{(n)}(x) = \int_0^1 s^n f^{(n+1)}(sx)\,ds$$