Ex. consider $f(x,t) = \frac{H(t)}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}}$ ← fundamental solutions to the heat equation

$$= \begin{cases} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}} & \text{for } t>0 \\ 0 & \text{for } t \le 0 \end{cases}$$

prove that $(\partial_t - \partial_x^2)\, T_f = \delta_0$

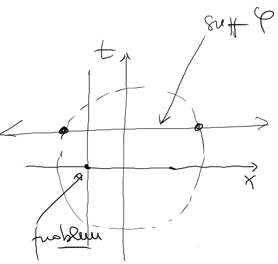
rem. if $\underbrace{P(D)}\, T = \delta_0$ T is the fundamental solution to $P(D)$

diff. polynomial $P(D) = \sum_{|\alpha| \le m} a_\alpha D^\alpha$

if $P(D)T = \delta_0$ then $T * f$ is such that $P(D)(T*f) = f$

1) check that $f \in L^1_{loc}(\mathbb{R}^2)$ $\quad f(x,t) = \frac{H(t)}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}}$

2) compute $(\partial_t - \partial_x^2)\, T_f(\varphi) =$ $\quad \varphi \in C_0^\infty(\mathbb{R}^2)$



$$= T_f\big((-\partial_t - \partial_x^2)\varphi\big)$$

$$= -\int_0^{+\infty}\int_{-\infty}^{+\infty} f(x,t)\,(\partial_t + \partial_x^2)\,\varphi \; dx\, dt$$

$t \quad x$

$$= \lim_{\tau \to 0^+} -\int_\tau^{+\infty}\int_{-\infty}^{+\infty} f(x,t)\,(\partial_t + \partial_x^2)\,\varphi \; dx\,dt$$

here some cancellations →

$$= \lim_{\tau \to 0^+} \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{\pi \tau}}\, e^{-\frac{x^2}{4\tau}}\, \varphi(\tau, x)\, dx$$

use dominated convergence.

Fourier transf. of L^1 functions

Ex. $f(x) = e^{-ax^2}$ with $a>0$

compute $\hat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\cdot\xi - ax^2}\,dx$

$\| \; u(\xi)$

$$u'(\xi) = \int_{-\infty}^{+\infty} -ix\, e^{-ix\xi - ax^2}\,dx$$

$$= \frac{+i}{2a}\int_{-\infty}^{+\infty} e^{-ix\xi}\underbrace{(-2ax)\cdot e^{-ax^2}}_{(e^{-ax^2})'}\,dx$$

$(e^{-ax^2})'$ w.r.t. x

$$= \frac{i}{2a}\int_{-\infty}^{+\infty} \left(e^{-ix\xi - ax^2}\right)' + i\xi\, e^{-ix\xi}\, e^{-ax^2}\,dx$$

then $\int_{-\infty}^{+\infty}\left(e^{-ix\xi - ax^2}\right)'dx = \lim_{\tau\to+\infty}\int_{-\tau}^{+\tau} e^{-ix\xi - ax^2}\,dx$

$$= \lim_{\tau\to+\infty}\left(e^{-ix\xi - ax^2}\Big|_{-\tau}^{+\tau}\right) = 0$$

$$= \frac{i}{2a}\cdot i\xi \underbrace{\int_{-\infty}^{+\infty} e^{-ix\xi - ax^2}\,dx}_{\hat{f}(\xi)}$$

$$= -\frac{1}{2a}\,\xi\, u(\xi)$$

finally $u(\xi)$ satisfies

$$u'(\xi) = \frac{1}{2a}\,\xi\, u(\xi)$$

if $\boxed{u = \hat{f}}$ then $\Longleftarrow$

i.e.

$$\begin{cases} 2a\,u'(\xi) + \xi\, u(\xi) = 0 \\ u(0) = \hat{f}(0) = \int_{-\infty}^{+\infty} e^{-i0\cdot x} f(x)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}} \end{cases}$$

$\hat{f}(\xi)$ is the unique solution of the Cauchy problem

$$\begin{cases} 2a\,u' + \xi u = 0 \\ u(0) = \sqrt{\frac{\pi}{a}} \end{cases} \qquad \left(u(\xi)\cdot e^{\frac{|\xi|^2}{4a}}\right)' = 0$$

$$\boxed{\hat{f}(\xi) = \sqrt{\frac{\pi}{a}}\, e^{-\frac{\xi^2}{4a}}}$$

and if $f(x) = e^{-a|x|^2}$, $x \in \mathbb{R}^d$

$$\boxed{\hat{f}(\xi) = \left(\sqrt{\frac{\pi}{a}}\right)^{n} e^{-\frac{|\xi|^2}{4a}}}$$

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$

def. Let $f \in \mathcal{C}^\infty(\mathbb{R}^d)$. $f \in \mathcal{S}(\mathbb{R}^d)$

Schwartz space or space of rapidly decreasing functions

if $\forall \alpha, \beta \in \mathbb{N}^n$

$$\sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta f(x)| < +\infty$$

e.g. I want that

$|f| \in L^\infty$ $(\alpha = 0 = \beta)$

$|\partial^\beta f| \in L^\infty$ $(\alpha = 0, \beta \dots)$

$|x^\alpha \partial^\beta f| \in L^\infty$

$\Rightarrow$ $|\partial^\beta f| \leq \frac{C}{|x|^\alpha}$ $x \neq 0$

on $\mathcal{S}(\mathbb{R}^d)$ I put the topology from the countable family of seminorms

$$p_{\alpha,\beta}(f) = \sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta f(x)|$$

$\mathcal{S}(\mathbb{R}^d)$ is a Fréchet space.

rem. the topology is same if you consider

$$r_k(f) = \sup_{x \in \mathbb{R}^d} \left((1+|x|)^k \sum_{|\alpha| \leq k} |D^\alpha f(x)| \right)$$

rem. $$\mathcal{C}_0^\infty(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{C}^\infty(\mathbb{R}^d)$$

It can be proved that

i) $(\varphi_n)_n$ in $\mathcal{D}(\mathbb{R}^d)$ s.t. $\varphi_n \to 0$ in the sense of $\mathcal{D}$

then $\varphi_n \to 0$ w.r. to the top. of $\mathcal{S}(\mathbb{R}^d)$

(ex. prove that $p_{\alpha\beta}(\varphi_n) \to 0$ $\forall \alpha, \beta$)

ii) f $(f_n)_n$ in sequence in $\mathcal{S}(\mathbb{R}^d)$

s.t. $f_n \to 0$ in the top. of $\mathcal{S}(\mathbb{R}^d)$

then $f_n \to 0$ in the top. of $\mathcal{E}(\mathbb{R}^d)$

$$\mathcal{D}(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{E}(\mathbb{R}^d)$$

continuous immersion

iii) $\mathcal{D}(\mathbb{R}^d)$ is dense in $\mathcal{S}(\mathbb{R}^d)$

Hint. take $f \in \mathcal{S}(\mathbb{R}^d)$ and take

$(\chi_n)_n$ in $\mathcal{D}(\mathbb{R}^d)$, $\chi_n = \begin{cases} 1 & \text{in } \overline{B(0,n)} \\ 0 & \text{outside } B(0,n+1) \end{cases}$

consider $f_n = f \cdot \chi_n$ prove that $p_{\alpha,\beta}(f - f\chi_n) \xrightarrow{n} 0$

$p_{\alpha,\beta}((1-\chi_n)f)$

iv) $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{E}(\mathbb{R}^d)$

conclusion

$$\mathcal{D}(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{E}(\mathbb{R}^d)$$

immersion: continuous with dense immage

$\Downarrow$

$$\mathcal{E}'(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d) \subseteq \mathcal{D}'(\mathbb{R}^d)$$

distribution with compact support

Temperate (or tempered) distributions

distribution

Ex. if $f \in \mathcal{S}(\mathbb{R}^d)$ then $f \in L^p(\mathbb{R}^d)$ for all p.

in fact $f \in L^\infty(\mathbb{R}^d)$ choose $\alpha = \beta = 0$ in def.

$$f(x)(1+|x|)^{d+1} \in L^\infty$$

$$\Downarrow$$

$$|f(x)| \le \underbrace{\frac{C}{(1+|x|)^{d+1}}}_{\in L^1}$$

so $f \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$

Ex. take $f \in L^p(\mathbb{R}^d)$ then $T_f \in \mathcal{S}'(\mathbb{R}^d)$

Fourier inversion formula

Th. Let $f \in \mathcal{S}(\mathbb{R}^d)$. Then $\hat{f} \in \mathcal{S}(\mathbb{R}^d)$

and
$$\widehat{D^\alpha f}(\xi) = \xi^\alpha \hat{f}(\xi)$$
$$\widehat{x^\alpha f}(\xi) = (-1)^{|\alpha|} D^\alpha \hat{f}(\xi)$$

proof (exercise)

we know that: $f \in L^1$ and $f' \in L^1$

then $\widehat{f'}(\xi) = (-i)\xi \hat{f}(\xi)$

we know that $f \in L^1$ and $x f(x) \in L^1$

then $\hat{f}$ is diff. and $\hat{f}' = i \widehat{x f(x)}$

$(\pm ?)$

applying recursively this formula we obtain the thesis

remark that we can prove also that

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^d_x) \longrightarrow \mathcal{S}(\mathbb{R}^d_\xi)$$
$$f \longmapsto \hat{f}$$

is continuous

$\forall \alpha, \beta \in \mathbb{N}^d$ $\exists$ $\tilde{\alpha}_1 \tilde{\beta}_1, \tilde{\alpha}_2, \tilde{\beta}_2, \dots \tilde{\alpha}_k \tilde{\beta}_k$

$\exists C > 0$

s.t.
$$p_{\alpha\beta}(\hat{f}) \leq C \sum_{j=1}^{k} p_{\tilde{\alpha}_j \tilde{\beta}_j}(f)$$

$\|\xi^\alpha D^\beta \hat{f}\|_{L^\infty}$

$\|\hat{f}\|_{L^\infty} \leq \|f\|_{L^1}$

recall: we have $\mathcal{F} : \mathcal{S} \longrightarrow \mathcal{S}$

$f \longmapsto \mathcal{F}(f) = \hat{f}$

linear and continuous

¿ is it invertible? ¿ the inverse it is continuous? — surjective and injective? — yes!!

Fourier inversion formula

Theorem let $f \in \mathcal{S}(\mathbb{R}^d)$. Then

$$\forall x \in \mathbb{R}^d, \quad f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^d_\xi} e^{i x\cdot\xi} \hat{f}(\xi)\, d\xi$$

proof. idea: consider $\int_{\mathbb{R}^d_\xi} e^{i x\cdot\xi} \hat{f}(\xi)\, d\xi$

$$= \int_{\mathbb{R}^d_\xi} e^{i x\cdot\xi} \left(\int_{\mathbb{R}^d_y} e^{-i y\cdot\xi} f(y)\, dy \right) d\xi$$

try to exchange the order of integration

difficulty:

$$= \dots \iint \left(e^{-i(y-x)\cdot\xi} f(y) \right) dy\, d\xi$$

$(y,\xi) \longmapsto e^{-i(y-x)\xi} f(y)$ is not in $L^1(\mathbb{R}^d_y \times \mathbb{R}^d_\xi)$

We overcome in this way: let $g \in \mathcal{S}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d_\xi} e^{i x\cdot\xi} g(\xi) \hat{f}(\xi)\, d\xi = \int_{\mathbb{R}^d_\xi} e^{i x\cdot\xi} g(\xi) \cdot \int_{\mathbb{R}^d_y} e^{-i y\cdot\xi} f(y)\, dy\, d\xi$$

$$= \int_{\mathbb{R}^d_\xi} \int_{\mathbb{R}^d_y} e^{-i(y-x)\cdot\xi} g(\xi) f(y)\, dy\, d\xi$$

x fixed: $(y,\xi) \longmapsto e^{-i(y-x)\cdot\xi} g(\xi) f(y)$ is in $L^1(\mathbb{R}^d_y \times \mathbb{R}^d_\xi)$

$$= \int_{\mathbb{R}^d_y} f(y) \left(\int_{\mathbb{R}^d_\xi} e^{-i(y-x)\cdot\xi} g(\xi)\, d\xi \right) dy$$

$$= \int_{\mathbb{R}^d_y} f(y)\, \hat{g}(y-x)\, dy$$

for $f, g \in \mathcal{S}$

$$\int_{\mathbb{R}^d_\xi} e^{i x\cdot\xi} g(\xi) \hat{f}(\xi)\, d\xi = \int_{\mathbb{R}^d_y} f(y)\, \hat{g}(y-x)\, dy \overset{\substack{y-x=z \\ y=x+z}}{=} \int_{\mathbb{R}^d_z} f(x+z)\, \hat{g}(z)\, dz$$

I take $g(\xi) = e^{-|\xi|^2/k}$ — $\hat{g}(z) = (\sqrt{k\pi})^n e^{-k|z|^2/4}$

$$\underbrace{\int_{\mathbb{R}^d_\xi} e^{-i x\cdot\xi} e^{-|\xi|^2/k} \cdot \hat{f}(\xi)\, d\xi}_{} = \underbrace{\int_{\mathbb{R}^d_z} f(x+z) (\sqrt{k\pi})^n e^{-k|z|^2/4}\, dz}_{\sqrt{k}\, z = y}$$

$k \to +\infty$ (dominated convergence)

$$\int_{\mathbb{R}^d_\xi} e^{-i x\cdot\xi} \hat{f}(\xi)\, d\xi \;=\; (2\pi)^n \cdot f(x)$$

OK

$(\sqrt{k})^n dz = dy$

$$\int f\left(x + \frac{y}{\sqrt{k}}\right) (\pi)^{n/2} e^{-|y|^2/4}\, dy$$

$\downarrow k \to +\infty$

$$f(x) \int_{\mathbb{R}^d_y} (\pi)^{n/2} e^{-|y|^2/4}\, dy$$

$$\int_{\mathbb{R}^d} e^{-|y|^2/4}\, dy = 2^n (\sqrt{\pi})^n$$

conclusion $f\in\mathcal{S}(\mathbb{R}^n)$

$$f(x)=\frac{1}{(2\pi)^n}\int e^{ix\cdot\xi}\hat f(\xi)\,d\xi$$

Fourier transform of tempered distributions

def. let $S\in\mathcal{S}'(\mathbb{R}^d)$

then $\hat S(f)=S(\hat f)$

(rem. $f\longmapsto \hat S(f)$, $f\in\mathcal{S}(\mathbb{R}^d)$, $\hat S(f)\in\mathbb{C}$ is a composition

$$f\xrightarrow{\mathcal{F}}\hat f\xrightarrow{S}S(\hat f)$$

both are linear and continuous

Ex. $\delta_0\in\mathcal{E}'(\mathbb{R}^d)\subset\mathcal{S}'(\mathbb{R}^d)$

compute $\hat\delta_0$

$$\hat\delta_0(f)=\delta_0(\hat f)=\hat f(0)=\int_{\mathbb{R}^d}e^{-i0\cdot x}f(x)\,dx=\int_{\mathbb{R}^d}f(x)\,dx=T_1(f)$$

$$\boxed{\hat\delta_0=1}$$

compute $\hat 1$

$$\hat 1(f)=\hat T_1(f)=T_1(\hat f)=\int_{\mathbb{R}^d}\hat f(\xi)\,d\xi=(2\pi)^n\underbrace{\frac{1}{(2\pi)^n}\int_{\mathbb{R}^d}e^{i0\cdot\xi}\hat f(\xi)\,d\xi}_{f(0)}$$

by fourier inv. formula

$$=(2\pi)^n\delta_0(f)$$

$$\boxed{\hat T_1=(2\pi)^n\delta_0}$$

Ex. let H Heaviside function T_H the Heaviside dist.

compute $\hat T_H$

$$\boxed{\hat T_H=i\,PV\tfrac{1}{x}+\pi\delta_0}$$

$\widehat{xf(x)}=i\hat f'$

$\hat T_H=\frac{i}{x}+\pi\delta_0$

idea consider $x\cdot\hat T_H$

$$x\hat T_H(f)=\hat T_H(xf(x))=T_H(\widehat{xf(x)})=T_H(i\hat f')=iT_H(\hat f')=(iT_H)'(\hat f)=i(T_H')(\hat f)=i\delta_0(\hat f)=i\hat f(0)=i\int_{-\infty}^{+\infty}f(x)\,dx=iT_1(f)$$

$$x\hat T_H(f)=iT_1(f)$$

$x(-i\hat T_H)=T_1$ so $-i\hat T_H$ is a solution to $xT=1$

$T=PV\frac{1}{x}+c\delta_0$

$$-i\hat T_H=PV\tfrac{1}{x}+c\delta_0$$

$$\hat T_H=i\,PV\tfrac{1}{x}+c\delta_0$$

compute the value of c applying $\hat T_H$ to e^{-x^2}

$c=\pi$