PARENTESI DI POISSON

Cost. del moto : $f(\bar{p},\bar{q},t)$ t.c. se la valutiamo sulle soluzioni $(\bar{p}(t),\bar{q}(t))$ delle eq. di Hamilton (variabile dinamica)

$$\frac{d}{dt} f(\bar{p}(t),\bar{q}(t),t) = 0 .$$

$$\frac{d}{dt} f(\bar{p}(t),\bar{q}(t),t) = \frac{\partial f}{\partial t} + \sum_{\ell=1}^{n}\left(\frac{\partial f}{\partial p_\ell}\dot{p}_\ell + \frac{\partial f}{\partial q_\ell}\dot{q}_\ell\right) =$$

$$= \frac{\partial f}{\partial t} + \sum_{\ell=1}^{n}\left[\frac{\partial f}{\partial p_\ell}\left(-\frac{\partial H}{\partial q_\ell}\right) + \frac{\partial f}{\partial q_\ell}\left(\frac{\partial H}{\partial p_\ell}\right)\right]$$

prendiamo $\bar{p}(t),\bar{q}(t)$ che soddisfano eq. Ham.

$$= \frac{\partial f}{\partial t} + \underbrace{\sum_{\ell=1}^{n}\left(\frac{\partial f}{\partial q_\ell}\frac{\partial H}{\partial p_\ell} - \frac{\partial f}{\partial p_\ell}\frac{\partial H}{\partial q_\ell}\right)}_{\equiv \{f,H\}}$$

PARENTESI DI POISSON

Def. PARENTESI DI POISSON è un'applicazione bilineare

$$f(\bar{p},\bar{q},t),\ g(\bar{p},\bar{q},t) \mapsto \{f,g\}(\bar{p},\bar{q},t) = \sum_{h=1}^{n}\left(\frac{\partial f}{\partial q_\ell}\frac{\partial g}{\partial p_\ell} - \frac{\partial f}{\partial p_\ell}\frac{\partial g}{\partial q_\ell}\right)$$

soddisfano eq. Ham.

$$\frac{df}{dt}(\bar{p}(t),\bar{q}(t),t) = \frac{\partial f}{\partial t}(\bar{p}(t),\bar{q}(t),t) + \{f,H\}(\bar{p}(t),\bar{q}(t),t)$$

f è una cost. del moto (f indip. da t) $\iff \dfrac{\partial f}{\partial t} + \{f,H\} = 0$ $\quad(\{f,H\}=0)$

Formalismo compatto per sistemi Hamiltoniani

Partiamo da un sist. Ham. a 1 grado di lib. $(n=1)$

$\rightarrow$ spazio delle fasi è parametrizzato da due coord p, q.

$\rightarrow$ Eq. Ham.
$$\begin{cases} \dot{p} = -\dfrac{\partial H}{\partial q} \\ \dot{q} = \dfrac{\partial H}{\partial p} \end{cases}$$

Def. $\bar{x} = \begin{pmatrix} p \\ q \end{pmatrix} \rightsquigarrow \nabla_x H = \begin{pmatrix} \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial q} \end{pmatrix}$

Consideriamo la matrice
$$E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

eq. Ham. $\dot{\bar{x}} = \begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} -\partial H/\partial q \\ \partial H/\partial p \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial q} \end{pmatrix} = E\, \bar{\nabla}_x H$

$$\dot{\bar{x}} = E\, \bar{\nabla}_x H \quad \leftarrow \text{ è della forma } \dot{\bar{x}} = \bar{f}(\bar{x}, t)$$

$$\frac{d}{dt} F(\bar{x}(t), t) = \frac{\partial F}{\partial t} + \underset{E\nabla H = \bar{f}}{\mathcal{L}} F = \frac{\partial F}{\partial t} + \sum_{i=1}^{2} f_i \frac{\partial F}{\partial x_i} =$$

$$= \frac{\partial F}{\partial t} - \frac{\partial H}{\partial q}\frac{\partial F}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial F}{\partial q} = \frac{\partial F}{\partial t} + \{F, H\}$$

In generale
$$\{f, g\} = \mathcal{L}_{E\nabla g} f = \sum_i \underbrace{\left(\sum_j E_{ij} \frac{\partial g}{\partial x_j} \right)}_{= (E\bar{\nabla}_g)_i} \frac{\partial f}{\partial x_i} = \bar{\nabla} f \cdot E\, \bar{\nabla} g$$

$$\{g,f\} = \bar{\nabla} g \cdot E\, \bar{\nabla} f = \sum_{ij} \underset{\uparrow \text{ ANTISIM.}}{E_{ij}} \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial x_j} =$$

$$= -\sum_{ij} E_{ji} \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial x_j} = -\sum_{rs} E_{rs} \frac{\partial f}{\partial x_r} \frac{\partial g}{\partial x_s} = -\{f,g\}$$

Prendiamo ora un sist. a n gradi di lib.

$$\bar{x} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \\ q_1 \\ \vdots \\ q_m \end{pmatrix} = \begin{pmatrix} \bar{p} \\ \bar{q} \end{pmatrix} \qquad \bar{\nabla}_x f = \begin{pmatrix} \partial f/\partial p_1 \\ \partial f/\partial p_2 \\ \vdots \\ \partial f/\partial p_m \\ \partial f/\partial q_1 \\ \vdots \\ \partial f/\partial q_m \end{pmatrix} = \begin{pmatrix} \bar{\nabla}_p f \\ \bar{\nabla}_q f \end{pmatrix}$$

$$E = \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_m & 0 \end{pmatrix} = \begin{pmatrix} \overbrace{0 \cdots 0}^{n} & \overbrace{-1 \; 0 \cdots 0}^{n} \\ \vdots \quad \vdots & \ddots \quad -1 \\ 1 \quad 0 & 0 \cdots 0 \\ \ddots \quad 1 & \vdots \quad 0 \end{pmatrix}$$

matrice $2m \times 2m$

Eq. Ham.: $\dot{\bar{x}} = E \bar{\nabla}_x H$

$$\begin{pmatrix} \dot{\bar{p}} \\ \dot{\bar{q}} \end{pmatrix} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \bar{\nabla}_p H \\ \bar{\nabla}_q H \end{pmatrix} = \begin{pmatrix} -\bar{\nabla}_q H \\ \bar{\nabla}_p H \end{pmatrix}$$

$$\dot{p}_h = -\frac{\partial H}{\partial q_h} \qquad \dot{q}_h = \frac{\partial H}{\partial p_h}$$

Parentesi di Poisson:

$$\{f,g\} = \sum_{i,j=1}^{2m} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g}{\partial x_j} = \bar{\nabla}_x f \cdot E \bar{\nabla}_x g$$

$$E_{ij} = \begin{cases} 0 & i,j = 1,\dots,n \\ -\delta_{hk} & i=h \quad j=k+n \quad h,k=1,\dots,n \\ \delta_{hk} & i=h+n \quad j=k \quad h,k=1,\dots,n \\ 0 & i,j = n+1,\dots,2n \end{cases}$$

Proprietà delle Parentesi di Poisson

a) $\{f, g\} = -\{g, f\} \Rightarrow \{f,f\}=0$ ANTISIM.

b) $\{f, \alpha_1 g_1 + \alpha_2 g_2\} = \alpha_1 \{f, g_1\} + \alpha_2 \{f, g_2\}$ BILINEARE

$\alpha_1, \alpha_2 \in \mathbb{R}$

c) $\{f, g_1 \cdot g_2\} = \{f, g_1\} g_2 + g_1 \{f, g_2\}$

d) Identità di Jacobi

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

La Parentesi di Poisson è un'applicazione BILIN. ANTISIM. che soddisfa l'ID. DI JACOBI (ci permette di definire un' ALGEBRA DI LIE)

Dim

a), b) ovvie

$$\{f,g\} = \sum_h \left(\frac{\partial f}{\partial q_h} \frac{\partial g}{\partial p_h} - \frac{\partial f}{\partial p_h} \frac{\partial g}{\partial q_h} \right) = \sum_{i,j} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g}{\partial x_j}$$

c) $$\sum_{i,j} \frac{\partial f}{\partial x_i} E_{ij} \cdot \underbrace{\frac{\partial}{\partial x_j} (g_1 \cdot g_2)}_{= g_1 \frac{\partial g_2}{\partial x_j} + g_2 \frac{\partial g_1}{\partial x_j}} = \left(\sum_{i,j} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g_1}{\partial x_j} \right) g_2 + \left(\sum_{i,j} \frac{\partial f}{\partial x_i} E_{ij} \frac{\partial g_2}{\partial x_j} \right) g_1$$

//

d) $\underbrace{\{f,\{g,h\}\} + \{g,\{h,f\}\}} + \{h,\{f,g\}\} = 0$

Notazione:

$$\partial_i f \equiv \frac{\partial f}{\partial x_i}$$

$$\{f,\{g,h\}\} - \{g,\{f,h\}\} =$$

$$= \sum_{i,j=1}^{2n} \partial_i f\, E_{ij}\, \partial_j \left[\sum_{k,m=1}^{2n} \partial_k g\, E_{km}\, \partial_m h\right] - \sum_{a,b=1}^{2n} \partial_a g\, E_{ab}\, \partial_b \left[\sum_{c,d=1}^{2n} \partial_c f\, E_{cd}\, \partial_d h\right]$$

$$= \sum_{ijkm} \left[\partial_i f\, E_{ij}\, \partial_j \partial_k g\, E_{km}\, \partial_m h + \partial_i f\, E_{ij}\, \partial_k g\, E_{km}\, \partial_j \partial_m h\right]$$

$$- \sum_{abcd} \left[\partial_a g\, E_{ab}\, \partial_b \partial_c f\, E_{cd}\, \partial_d h + \underset{\overset{\uparrow}{k}}{\partial_a} g\, E_{ab}\, \underset{\overset{\uparrow}{i}}{\partial_c} f\, E_{cd}\, \underset{\overset{\uparrow}{m}}{\partial_b} \underset{\overset{\uparrow}{j}}{\partial_d} h\right]$$

$$= \sum_{ijkm} \left[\partial_i f\, E_{ij}\, \partial_j \partial_k g\, E_{km}\, \partial_m h\right] + \sum_{ijkm} \cancel{\partial_i f\, E_{ij}\, \partial_k g\, E_{km}\, \partial_j \partial_m h}$$

$$- \sum_{abcd} \left[\underset{\overset{\uparrow}{j}}{\partial_a} g\, E_{ab}\, \underset{\overset{\uparrow}{i}}{\partial_b} \underset{\overset{\uparrow}{k}}{\partial_c} f\, E_{cd}\, \underset{\overset{\uparrow}{m}}{\partial_d} h\right] - \sum_{ijkm} \cancel{\partial_k g\, E_{km}\, \partial_i f\, E_{ij}\, \underbrace{\partial_m \partial_j h}_{= \partial_j \partial_m h}}$$

$$= \sum_{ijkm} \left[\partial_i f\, E_{ij}\, \partial_j \partial_k g\, E_{km}\, \partial_m h - \partial_j g\, \underbrace{E_{ji}}_{= -E_{ij}}\, \partial_i \partial_k f\, E_{km}\, \partial_m h\right]$$

$$= \sum_{km} \left(\sum_{ij} \left(\partial_i f\, E_{ij}\, \partial_k \partial_j g + \partial_k \partial_i f\, E_{ij}\, \partial_j g\right)\right) E_{km}\, \partial_m h$$

$$= \sum_{km} \partial_k \left(\sum_{ij} \partial_i f\, E_{ij}\, \partial_j g\right) E_{km}\, \partial_m h = \{\{f,g\},h\} =$$

$$= -\{h,\{f,g\}\} \quad //$$

Parentesi di Poisson FONDAMENTALI

Scegliamo come f e g le funzioni $\bar{p} \mapsto p_h \leftarrow p_h$

$\bar{q} \mapsto q_h \leftarrow q_h$

$$\{p_h, p_k\} = \sum_{\ell} \left(\underset{\nearrow 0}{\frac{\partial p_h}{\partial q_\ell}} \underset{\uparrow \delta_{k\ell}}{\frac{\partial p_k}{\partial p_\ell}} - \frac{\partial p_h}{\partial p_\ell} \underset{\uparrow 0}{\frac{\partial p_k}{\partial q_\ell}} \right) = 0 \qquad \forall h,k = 1,\dots,n$$

$$\{q_h, q_k\} = 0 \qquad \forall h,k = 1,\dots,n$$

$$\{q_h, p_k\} = \sum_{\ell} \left(\underset{\nearrow \delta_{h\ell}}{\frac{\partial q_h}{\partial q_\ell}} \underset{\uparrow \delta_{k\ell}}{\frac{\partial p_k}{\partial p_\ell}} - \underset{\overset{\shortparallel}{0}}{\frac{\partial q_h}{\partial p_\ell}} \underset{\overset{\shortparallel}{0}}{\frac{\partial p_k}{\partial q_\ell}} \right) = \sum_{\ell} \delta_{h\ell}\, \delta_{k\ell} = \delta_{hk}$$

$$\{x_i, x_j\} = \sum_{r,s=1}^{2n} \underset{\nearrow \delta_{ir}}{\frac{\partial x_i}{\partial x_r}} E_{rs} \underset{\uparrow \delta_{js}}{\frac{\partial x_j}{\partial x_s}} = E_{ij}$$

Osservazione :

$$\{x_s, g(\bar{x})\} = \sum_{ij} \overset{\delta_{is}}{\overset{\shortparallel}{\frac{\partial x_s}{\partial x_i}}} E_{ij} \frac{\partial g}{\partial x_j} = \sum_j E_{sj} \frac{\partial g}{\partial x_j}$$

$$\{\bar{x}, g\} = E\, \bar{\nabla} g$$

$$\{\bar{x}, g\} = \begin{pmatrix} \{x_1, g\} \\ \vdots \\ \{x_{2n}, g\} \end{pmatrix}$$

$$\Rightarrow \quad \{p_h, g(\bar{p},\bar{q})\} = -\frac{\partial g}{\partial q_h}$$

$$\{q_h, g(\bar{p},\bar{q})\} = \frac{\partial g}{\partial p_h}$$

Se $g = H$

$$\dot{x}_i = \sum_j E_{ij} \frac{\partial H}{\partial x_j} = \{x_i, H\}$$

→ Eq. Ham. $\dot{x}_i = \{ x_i, H \}$ $i = 1, \dots, 2n$

es. particolare di $\frac{df}{dt} = \{ f, H \} + \frac{\partial f}{\partial t}$