PARENTESI DI POISSON & MOMENTO ANGOLARE

Prendiamo un corpo che si muove in $\mathbb{R}^3$, con coord. cartesiane $\bar{q}$, allora $\bar{p}$ è la quantità di moto.

$$\bar{M} = \bar{q} \times \bar{p} \qquad \leftarrow \text{ funz. delle } p_h \text{ e delle } q_h$$

$$M_i = \sum_{m,k=1}^{3} \epsilon_{imk}\, q_m p_k$$

$$\{p_\ell, M_i\} = -\frac{\partial M_i}{\partial q_\ell} = -\frac{\partial}{\partial q_\ell} \sum_{mk} \epsilon_{imk}\, q_m p_k =$$

$$= -\sum_{mk} \epsilon_{imk}\, \delta_{m\ell}\, p_k = -\sum_k \epsilon_{i\ell k}\, p_k$$

$$= \sum_k \epsilon_{\ell i k}\, p_k$$

$$\{p_\ell, \sum_{mk} \epsilon_{imk}\, q_m p_k\} \overset{\text{BILIN.}}{=} \sum_{mk} \epsilon_{imk}\, \{p_\ell, q_m p_k\} =$$

$$\overset{d}{=} \sum_{mk} \epsilon_{imk} \Big[\underbrace{\{p_\ell, q_m\}}_{= -\delta_{\ell m}} p_k + \underbrace{\{p_\ell, p_k\}}_{=0} q_m \Big] \overset{\text{P.P. fondam.}}{=}$$

$$= -\sum_k \epsilon_{i\ell k}\, p_k$$

$$\{q_\ell, M_i\} = \sum_m \epsilon_{\ell i m}\, q_m$$

$$\boxed{\{M_i, M_j\} = \sum_{k=1}^{3} \epsilon_{ijk} M_k}$$

[M_i soddisfano l'algebra di Lie di $SO(2)$]

$$M_i = \sum_{m,s=1}^{3} \epsilon_{ims}\, q_m P_s$$

$$\{M_i, M_j\} = \Big\{ \sum_{ms} \epsilon_{ims} q_m P_s, M_j \Big\} \overset{\text{BILIN.}}{=} \sum_{ms} \epsilon_{ims} \{q_m P_s, M_j\} \overset{c)}{=}$$

$$= \sum_{ms} \epsilon_{ims} \Big[q_m \underbrace{\{P_s, M_j\}}_{=\sum_r \epsilon_{sjr} P_r} + \underbrace{\{q_m, M_j\}}_{=\sum_r \epsilon_{mjr} q_r} P_s \Big]$$

$$= \sum_{msr} \epsilon_{ims} \big(\epsilon_{sjr} P_r q_m + \epsilon_{mjr} P_s q_r \big) =$$

$$= \sum_{msr} \epsilon_{ims} \epsilon_{sjr} P_r q_m + \sum_{msr} \epsilon_{ims} \epsilon_{mjr} P_s q_r$$

$$\sum_{smr} \epsilon_{ism} \epsilon_{sjr} P_m q_r = - \sum_{smr} \epsilon_{ims} \epsilon_{sjr} P_m q_r$$

$$= \sum_{msr} \epsilon_{ims} \underbrace{\epsilon_{sjr}}_{\epsilon_{jrs}} \big(P_r q_m - P_m q_r \big)$$

$$\delta_{ij}\delta_{mr} - \delta_{ir}\delta_{mj}$$

$$= \sum_{mr} \big(\delta_{ij}\delta_{mr} - \delta_{ir}\delta_{mj} \big) \big(P_r q_m - P_m q_r \big)$$

$$= \sum_m \delta_{ij} \left(\cancel{p_m} q_m - \cancel{p_m} q_m \right) - \left(p_i q_j - p_j q_i \right)$$

$$= q_i p_j - q_j p_i \qquad = \sum_k \epsilon_{ijk} M_k$$

$$\sum_k \epsilon_{ijk} \sum_{rs} \epsilon_{krs} q_r p_s =$$

$$= \sum_{rs} \left(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr} \right) q_r p_s =$$

$$= q_i p_j - q_j p_i \ //$$

$$\{M_i, M_j\} = \sum_k \epsilon_{ijk} M_k$$

$$\{M_1, M_2\} = \sum_k \epsilon_{12k} M_k = M_3$$

$$\{M_2, M_3\} = M_1 \qquad \{M_3, M_1\} = M_2$$

$$\{\bar{M}^2, M_i\} = 0$$

$$\left\{ \sum_e M_e^2, M_i \right\} \overset{b,c)}{=} \sum_e 2M_e \{M_e, M_i\} =$$

$$= 2 \sum_e M_e \sum_k \epsilon_{eik} M_k = -2 \sum_{ek} \underbrace{\epsilon_{iek}}_{\text{antisim. in } ek} \underbrace{M_e M_k}_{\text{sim. in } ek}$$

$$\left[\sum_{ek} \underset{\text{antisim}}{a_{ek}^{\,ij}} \underset{\text{sim.}}{S_{ek}^{\,ji}} = \sum_{ek} (-a_{ke}) S_{ke} = -\sum_{ij} a_{ij} S_{ij} = -\sum_{ek} a_{ek} S_{ek} \right.$$

$$\left. \Rightarrow \sum_{ek} a_{ek} S_{ek} = 0 \right]$$

$\downarrow$
Questa relazione è un caso particolare del fatto che $\{ M_i, f(\bar p, \bar q) \} = 0$ se f è invariante sotto rotazioni $SO(3)$.

Vediamo due casi particolari

$$\{ M_h, \bar p^2 \} = \{ M_h, \sum_j p_j^2 \} = \sum_j 2 p_j \{ M_h, p_j \} =$$

$$= 2 \sum_j p_j \sum_k \epsilon_{hjk} p_k = 0$$

$$\{ M_h, \frac{1}{r} \} = \sum_{rs} \epsilon_{rsh} \{ q_r p_s, \frac{1}{r(q)} \} \overset{c)}{=} \qquad \{ f(\bar q), g(\bar q) \} = 0$$

$$r = \sqrt{q_1^2 + q_2^2 + q_3^2}$$

$$= \sum_{rs} \epsilon_{rsh} \Big[q_r \underbrace{\{ p_s, \frac{1}{r} \}}_{= -\frac{\partial}{\partial q_s}\left(\frac{1}{r}\right)} + \underbrace{\cancel{\{ q_r, \frac{1}{r(q)} \}}}_{= \frac{\partial}{\partial p_r}\left(\frac{1}{r(q)}\right) = 0} p_s \Big]$$

$$= \sum_{rs} \epsilon_{rsh} q_r \left(\frac{1}{2r^3} 2 q_s \right) = \sum_{rs} \epsilon_{hrs} q_r q_s = 0$$

Vettore di Laplace-Runge-Lenz e Parentesi di Poisson

$$\bar A = \bar p \times \bar M - mk \frac{\bar q}{r} \qquad r = \sqrt{q_1^2 + q_2^2 + q_3^2}$$

indip. da t

$$H = \frac{\bar p^2}{2m} - \frac{k}{r}$$

Kepl.

$\rightarrow$ verifichiamo che $\bar A$ è cost. del moto, cioè $\{ A_i, H \} = 0$

$$A_i = \sum_{mn} \epsilon_{imn} p_m M_n - mk \frac{q_i}{r}$$

$$\{A_i, H\} = \sum_{mn} \epsilon_{imn} \left\{ p_m M_n, \frac{1}{2m} \sum_j p_j^2 - \frac{k}{r} \right\}$$

$$- mk \left\{ \frac{q_i}{r}, \frac{1}{2m} \sum_j p_j^2 - \frac{k}{r} \right\} =$$

$\{F(\bar{p}), G(\bar{p})\} = 0$

$$= \sum_{mn} \epsilon_{imn} \Big[p_m \cancel{\Big\{ M_n, \frac{1}{2m} \sum_j p_j^2 \Big\}} + \cancel{\Big\{ p_m, \frac{1}{2m} \sum_j p_j^2 \Big\}} M_n$$

inv. rotaz.

$$- p_m \cancel{\Big\{ M_n, \frac{k}{r} \Big\}} - \Big\{ p_m, \frac{k}{r} \Big\} M_n \Big]$$

$\{f(\bar{q}), g(\bar{q})\} = 0$

$$- mk \Big[\Big\{ \frac{q_i}{r}, \frac{1}{2m} \sum_j p_j^2 \Big\} - \cancel{\Big\{ \frac{q_i}{r}, \frac{k}{r} \Big\}} \Big]$$

$$= \sum_{mn} \epsilon_{imn} \underbrace{\Big\{ p_m, -\frac{k}{r} \Big\}}_{-\frac{q_m}{r^3}} M_n - \frac{k}{2} \sum_j \underbrace{\Big\{ \frac{q_i}{r}, p_j^2 \Big\}}_{2 \{ p_j, -\frac{q_i}{r} \} p_j}$$

$$-\frac{\partial}{\partial q_j} \left(\frac{q_i}{r} \right)$$

$$= \underbrace{-k \sum_{mn} \epsilon_{imn} \frac{q_m M_n}{r^3}}_{A} + \underbrace{k \sum_j p_j}_{B} \underbrace{\Big\{ p_j, \frac{q_i}{r} \Big\}}_{\frac{q_i q_j}{r^3} - \frac{\delta_{ij}}{r}}$$

$$\frac{\partial}{\partial q_m} \left(\frac{1}{r} \right) = \frac{\partial}{\partial q_m} \frac{1}{\sqrt{q_1^2 + q_2^2 + q_3^2}} = -\frac{1}{r^2} \frac{1}{2r} 2 q_m = \frac{-q_m}{r^3}$$

$$Ⓐ = -k\sum_{mn} \epsilon_{imn} \frac{q_m}{r^3} \sum_{ab} \epsilon_{nab} q_a p_b = -k\sum_{m\,ab} (\delta_{ia}\delta_{mb} - \delta_{ib}\delta_{ma}) \cdot \frac{1}{r^3}(q_m q_a p_b)$$

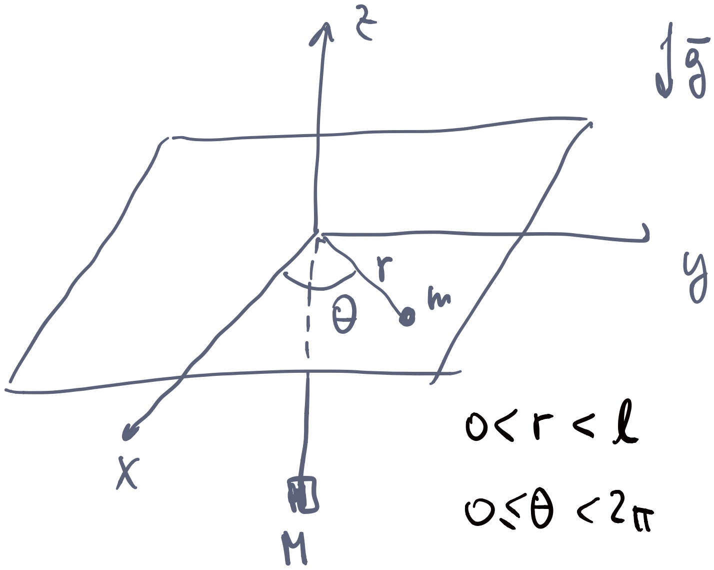
$$= -\frac{k}{r^3}\sum_m \left(q_m q_i p_m - \overbrace{q_m q_m}^{r^2} p_i \right) =$$

$$= k\left(\frac{1}{r} p_i - \frac{1}{r^3} (\bar{q}\cdot\bar{p}) q_i \right)$$

$$Ⓑ = k\sum_j p_j \cdot \left(\frac{q_i q_j}{r^3} - \frac{\delta_{ij}}{r} \right) = \frac{k(\bar{p}\cdot\bar{q})}{r^3} q_i - \frac{k}{r} p_i$$

$$Ⓐ + Ⓑ = 0 \quad //$$

ES. 18.02.18



$0 < r < \ell$

$0 \leq \theta < 2\pi$

$n = 2$

$x_m = r\cos\theta \qquad \dot x_m = \dot r\cos\theta - r\dot\theta\sin\theta$

$y_m = r\,\mathrm{sen}\,\theta \qquad \dot y_m = \dot r\sin\theta + r\dot\theta\cos\theta$

$z_m = 0$

$x_M = 0$

$y_M = 0$

$z_M = -\ell + r \qquad \dot z_M = \dot r$

1) $T = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) + \frac{1}{2}M\dot r^2$

trascuriamo la cost. in quanto irrilevante per la dinamica

$V = Mgz_M = Mgr - Mg\ell$

$L = T - V = \frac{1}{2}m(\dot r^2 + r^2\dot\theta^2) + \frac{1}{2}M\dot r^2 - Mgr \cancel{+ Mg\ell}$

2) $\frac{d}{dt}\frac{\partial L}{\partial \dot r} = \frac{d}{dt}(m\dot r + M\dot r) = (m+M)\ddot r$

$\frac{\partial L}{\partial r} = mr\dot\theta^2 - Mg \quad \rightarrow \quad \ddot r = \frac{m}{m+M}r\dot\theta^2 - \frac{M}{m+M}g$

$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = \frac{d}{dt}(mr^2\dot\theta) = 2mr\dot r\dot\theta + mr^2\ddot\theta$

$\frac{\partial L}{\partial \theta} = 0 \quad \rightarrow \quad \ddot\theta = -\frac{2\dot r\dot\theta}{r}$

3) Coord. ciclica: $\theta \rightarrow$ cost. moto $p_\theta = \frac{\partial L}{\partial \dot\theta} = mr^2\dot\theta$

4) $\tilde{p}_\theta \equiv m r^2 \dot{\theta} \rightarrow \dot{\theta} = \dfrac{\tilde{p}_\theta}{m r^2}$ $\quad \left(= \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} M \dot{r}^2 - Mgr\right.$

$$L^* = L - \dot{\theta} p_\theta \Big|_{\dot{\theta} = \frac{\tilde{p}_\theta}{m r^2}} = \frac{1}{2}(m+M)\dot{r}^2 - Mgr + \frac{1}{2} m r^2 \left(\frac{\tilde{p}_\theta}{m r^2}\right)^2 - \frac{\tilde{p}_\theta}{m r^2}\tilde{p}_\theta$$

$$= \frac{1}{2}(m+M)\dot{r}^2 - Mgr - \frac{1}{2}\frac{\tilde{p}_\theta^2}{m r^2}$$

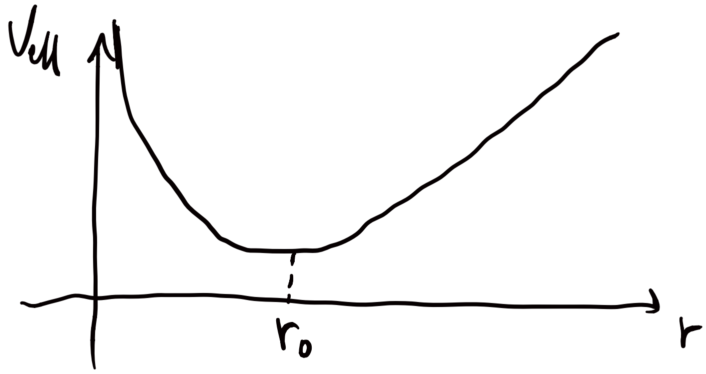
$$\rightarrow V_{eff} = Mgr + \frac{1}{2}\frac{\tilde{p}_\theta^2}{m r^2}$$

5) $V_{eff}' = Mg - \frac{1}{\cancel{2}} \cancel{2} \dfrac{\tilde{p}_\theta^2}{m r^3} = 0$

$$r_0^3 = \frac{\tilde{p}_\theta^2}{mMg} \rightarrow r_0 = \left(\frac{\tilde{p}_\theta}{mMg}\right)^{1/3}$$

$V_{eff}'' = 3\dfrac{\tilde{p}_\theta^2}{m}\dfrac{1}{r^4} > 0 \quad \forall r \Rightarrow r_0$ è MIN

$\rightarrow$ stab. per il sist. ridotto



6) Freq. piccole osc. p. probl. ridotto

$$\det(B - \lambda A) = 0$$

In qto caso B e A sono numeri

$$B = V_{eff}''(r_0) = \frac{3\tilde{p}_\theta^2}{m r_0^4} \qquad r_0 = \left(\frac{\tilde{p}_\theta^2}{mMg}\right)^{1/3}$$

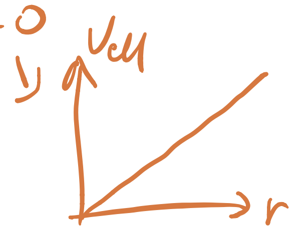
$$A = a(r_0) = m + M$$

$$B - \lambda A = 0 \rightarrow \lambda = \frac{B}{A}$$

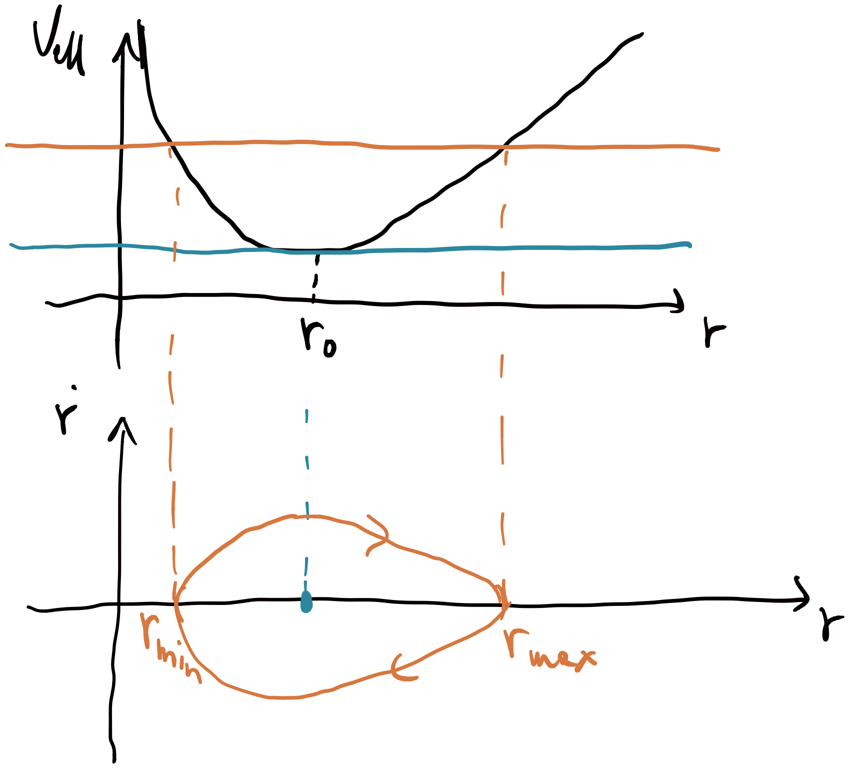
$$\omega^2 = \lambda = \frac{3\tilde{p}_\theta^2}{m r_0^4}\frac{1}{(m+M)} = \frac{3\tilde{p}_\theta^2}{m r_0^3 (m+M)}\frac{1}{r_0} =$$

$$= \frac{3\cancel{\tilde{p}_\theta^2}}{\cancel{m}(m+M)} \cdot \frac{\cancel{m}Mg}{\cancel{\tilde{p}_\theta^2}}\frac{1}{r_0} = \frac{3Mg}{m+M}\left(\frac{mMg}{\tilde{p}_\theta^2}\right)^{1/3}$$

7) ~~8.~~ Sì per $\tilde{p}_\theta > 0$, ma non c'è pto di equil. per $\tilde{p}_\theta = 0$



8)



-) circonferenza nel piano xy con velocità ang. cost. $\dot{\theta} = \dfrac{\tilde{p}_\theta}{m r_0^2}$

-) traiettorie limitate (tra r_{min} e r_{max})