MECCANICA RAZIONALE

Ing Civile & Ambientale
Novale

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Linearizzazione delle equazioni del moto.

→ coordinate libere
$q_i = q_i(t)$ lungo il moto

→ eq. differenziali per le variabili q_i
+ condizioni iniziali
= Tutta l'informazione di cui abbiamo bisogno per ottenere $q_i = q_i(t)$ lungo il moto

Come risolviamo le equazioni del moto?

- Tecniche numeriche
- analisi qualitativa
- metodi approssimati

c'è un regime dove possiamo ricondurre le eq. del moto a qualcosa di più semplice?

→ soluzione di equilibrio

→ piccole fluttuazioni intorno alla configurazione di equilibrio.

Eq. di moto → $q_i = q_{i,F} + \eta_i$

$$\underbrace{\qquad\qquad\qquad}$$

"vicino" = espansione in serie di Taylor fermandoci al primo ordine non banale

Caso conservativo :

$$\mathcal{L} = \frac{1}{2}\, \dot{\underline{q}} \cdot A(\underline{q}) \cdot \dot{\underline{q}} - V(\underline{q})$$

$$\downarrow \quad \underline{q} = \underline{q}_E + \eta \underline{x}$$

$$\tilde{\mathcal{L}} = \frac{1}{2}\, \dot{\underline{x}} \cdot A(\underline{q}_E) \cdot \dot{\underline{x}} - \frac{1}{2}\, \underline{x} \cdot \mathrm{Hess}\, V \Big|_{\underline{q}_E} \cdot \underline{x}$$

dove $\underline{q} \longrightarrow \underline{x}$

$$A(\underline{q}) \longrightarrow A(\underline{q}_E)$$

$$V(\underline{q}) \longrightarrow \frac{1}{2}\, \underline{x} \cdot \mathrm{Hess}\, V \Big|_{\underline{q}_E} \cdot \underline{x}$$

$$V(\underline{q} = \underline{q}_E + \eta \underline{x}) = V(\underline{q}_E) + dV \cdot \eta \underline{x}$$

$$+ \underbrace{\frac{1}{2}\, \eta^2\, \underline{x} \cdot \mathrm{Hess}\, V \Big|_{\underline{q}_E} \cdot \underline{x}} + \dots$$

$\to 0$ calcolato per $\underline{q}_E$

per trovare $\underline{q}_E$ risolviamo $dV = 0$

$$\frac{\partial V}{\partial x} = 0, \quad \frac{\partial V}{\partial y} = 0 \quad \dots$$

$$\operatorname{Hess} V_{ij} = \left(\frac{\partial^2 V}{\partial q_i \partial q_j} \right) \longrightarrow \operatorname{Hess} V \Big|_{q_E}$$

$$A(\underline{q}_E) \cdot \ddot{\underline{x}} + \underbrace{\operatorname{Hess} V \Big|_{\underline{q}_E}}_{C} \cdot \underline{x} = 0$$

(se ci sono Q che non sono del tipo $Q = -\frac{\partial V}{\partial q}$, allora

$$A(\underline{q}_E) \cdot \ddot{\underline{x}} + B \cdot \dot{\underline{x}} + C \underline{x} = 0$$

$$B = -\frac{\partial Q_i}{\partial \dot{q}_j} \Big|_{\underline{q}_E}$$

Troviamo un sistema di eq. diff accoppiate

Recap: eq. diff in $\mathbb{R}$, con una sola variabile $y(\tau)$

$$a \frac{d^2 y}{d\tau^2} + b \frac{dy}{d\tau} + c\, y = 0$$

prova $y(\tau) = e^{k\tau}$ $\quad$ k costante

sostituisci

$$\underbrace{(a k^2 + b k + c)}\, e^{k\tau} = 0$$

$$a k^2 + b k + c = 0$$ equazione caratteristica

$$k_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3 casi

- k_1, k_2 radici reali e distinte
  $b^2 - 4ac$ positivo

$e^{k_1 \tau}, e^{k_2 \tau}$ soluzioni

$$y(\tau) = c_1 e^{k_1 \tau} + c_2 e^{k_2 \tau}$$

- k_1, k_2 radici complesse coniugate

$b^2 - 4ac$ negativo

$$e^{k\tau} = e^{(\alpha + i\beta)\tau} = e^{\alpha\tau} e^{i\beta\tau}$$

$$= e^{\alpha\tau}(\cos\beta\tau + i\sin\beta\tau)$$

$$y(\tau) = e^{\alpha\tau}\left[c_1 \cos\beta\tau + c_2 \sin\beta\tau\right]$$

$$\alpha = -\frac{b}{2a} \qquad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

- $b^2 = 4ac \longrightarrow k_1 = k_2 = -\frac{b}{2a}$

$$y(\tau) = \left[c_1 + c_2\tau\right] e^{-\frac{b}{2a}\tau}$$

costanti arbitrarie fissate da

$$\begin{cases} y(\tau = 0) = y_0 \\ \dot{y}(\tau = 0) = y_1 \end{cases}$$

leri

$$\boxed{\frac{8}{3} m \ell^2 \ddot{x} + c \ell^2 (1-\gamma) x = 0}$$

↳ $a\ddot{x} + b\dot{x} + cx = 0$

$a = \frac{8}{3} m \ell^2$

$b = 0$ $\qquad c = c\ell^2(1-\gamma)$

↳ $a k^2 + \cancel{b}k + c = 0$ $\quad (b=0)$

$$k_{1,2} = \frac{\pm\sqrt{-4ac}}{2a}$$

carattere → $\left(\frac{8}{3} m \ell^2\right) \cdot \left(c \ell^2 (1-\gamma)\right)$

$(1-\gamma)$

$\gamma = \frac{mg}{c\ell}$ $\quad > 1$, < 1

$1-\gamma > 0 \qquad \gamma < 1$ → oscillazioni armoniche

cos( ) sin( ) $\quad k_{1,2} = \pm i\sqrt{\frac{3}{8}\,\frac{c(1-\gamma)}{m}}$

$1-\gamma < 0 \qquad \gamma > 1$

$$\to \quad c_1 e^{\nu t} + c_2 e^{-\nu t}$$

$$h_{1,2} = \pm \sqrt{\frac{3}{8}\, c\, \frac{(\gamma-1)}{m}}$$

$$\frac{8}{3} m l^2 \ddot{x} + c l^2 (1-\gamma) x + 4 \nu l^2 x = 0$$

Seconda parte

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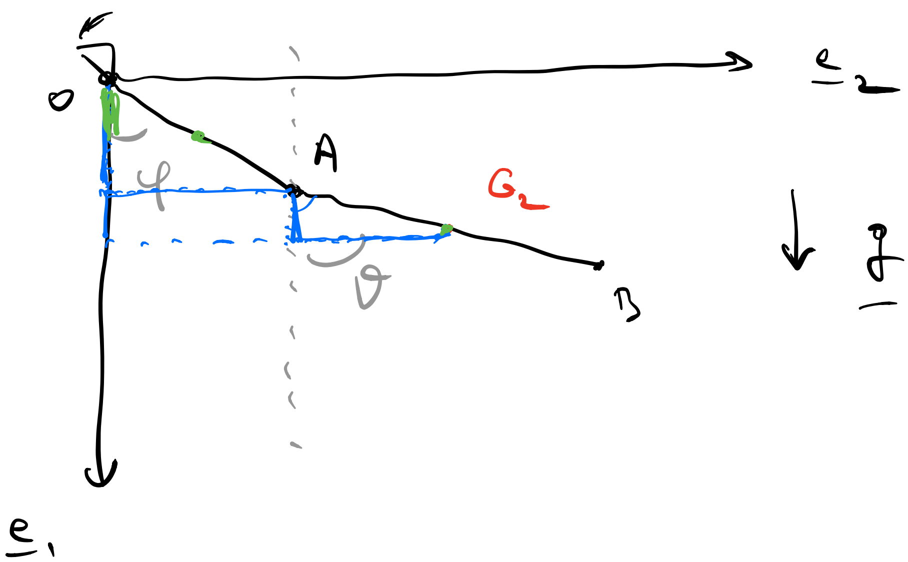

$\overline{OA} = l$

$\overline{AB} = 2l$

$m_{OA} = m$

$m_{AB} = 2m$

Eq. di Lagrange $\longrightarrow \mathcal{L} = K - V$

$$K = K_{OA} + K_{AB}$$

$$K_{OA} = \frac{1}{2} I^{OA}_{O,3}\, \dot{\varphi}^2 = \frac{1}{2} \left(\frac{m l^2}{3} \right) \dot{\varphi}^2$$

$$K_{AB} = \frac{1}{2}(2m)\,\|\underline{v}_{G_2}\|^2 + \frac{1}{2} I^{AB}_{G_2,3}\,\dot{\theta}^2$$

$$= \frac{1}{2}(2m)\,\|\underline{v}_{G_2}\|^2 + \frac{1}{2}\,\frac{(2m)(2\ell)^2}{12}\,\dot{\theta}^2$$

$$\underline{x}_{G_2} = \left(\ell \cos\varphi_{(t)} + \frac{2\ell}{2}\cos\theta_{(t)}\right)\underline{e}_1 +$$

$$+ \left(\ell \sin\varphi_{(t)} + \frac{2\ell}{2}\sin\theta_{(t)}\right)\underline{e}_2$$

$$\dot{\underline{x}}_{G_2} = \underline{v}_{G_2} = \left(-\ell\sin\varphi\,\dot{\varphi} - \ell\sin\theta\,\dot{\theta}\right)\underline{e}_1$$

$$+ \left(\ell\cos\varphi\,\dot{\varphi} + \ell\cos\theta\,\dot{\theta}\right)\underline{e}_2$$

$$\underline{v}_{G_2}\cdot\underline{v}_{G_2} = \left(-\ell\sin\varphi\,\dot{\varphi} - \ell\sin\theta\,\dot{\theta}\right)^2$$

$$+ \left(\ell\cos\varphi\,\dot{\varphi} + \ell\cos\theta\,\dot{\theta}\right)^2$$

$\underline{e}_1\cdot\underline{e}_2 = 0$

$$= \ell^2\left[\sin^2\varphi\,\dot{\varphi}^2 + \sin^2\theta\,\dot{\theta}^2 + 2\sin\varphi\sin\theta\,\dot{\varphi}\dot{\theta}\right.$$

$$\left. + \cos^2\varphi\,\dot{\varphi}^2 + \cos^2\theta\,\dot{\theta}^2 + 2\cos\varphi\cos\theta\,\dot{\varphi}\dot{\theta}\right]$$

$$= \ell^2\left[\dot{\varphi}^2 + \dot{\theta}^2 + 4\,\dot{\varphi}\dot{\theta}\cos(\varphi - \theta)\right]$$

$(\sin^2 + \cos^2) = 1$

Quindi:

$$K = \frac{1}{2} m \frac{\ell^2}{3} \dot\varphi^2 +$$

$$+ \frac{1}{2} (2m) \ell^2 \left[\dot\varphi^2 + \dot\theta^2 + 4 \dot\varphi \dot\theta \cos(\varphi - \theta) \right]$$

$$+ \frac{1}{2} (2m) \frac{(2\ell)^2}{12} \dot\theta^2$$

$$= \frac{1}{2} m \ell^2 \left[\frac{7}{3} \dot\varphi^2 + \frac{8}{3} \dot\theta^2 + 4 \dot\theta \dot\varphi \cos(\varphi - \theta) \right]$$

$$\begin{pmatrix} \frac{7}{3} & 2\cos(\varphi-\theta) \\ 2\cos(\varphi-\theta) & \frac{8}{3} \end{pmatrix}$$

$V \longrightarrow$ peso

$$V = - mg \frac{\ell}{2} \cos\varphi - 2mg \left(\ell \cos\varphi + \ell \cos\theta \right)$$

$$= - mg\ell \left(\frac{5}{2} \cos\varphi + 2 \cos\theta \right)$$

$$\mathcal{L} = \frac{1}{2} m \ell^2 \left(\frac{7}{3} \dot\varphi^2 + \frac{8}{3} \dot\theta^2 + 4 \dot\varphi \dot\theta \cos(\varphi - \theta) \right)$$

$$+ mg\ell \left(\frac{5}{2} \cos\varphi + 2 \cos\theta \right)$$

Equazioni di Lagrange:

$\varphi)$ $$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot\varphi} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$$\frac{d}{d\tau}\left(\frac{1}{\cancel{2}} m \ell^2 \left(\frac{7}{3}\cdot\cancel{2}\,\dot\varphi + \overset{2}{\cancel{4}}\,\dot\theta\cos(\varphi-\theta)\right)\right)$$

$$-\left(-\frac{1}{2} m \ell^2\, 4\,\dot\varphi\dot\theta\sin(\varphi-\theta) + \right.$$

$$\left. - mg\ell\,\frac{5}{2}\sin\varphi\right)$$

$$= m\ell^2\left(\frac{7}{3}\ddot\varphi + 2\ddot\theta\cos(\varphi-\theta) + \right.$$

$$\left. - 2\dot\theta\sin(\varphi-\theta)(\dot\varphi-\dot\theta)\right)$$

$$+ 2m\ell^2\dot\varphi\dot\theta\sin(\varphi-\theta) + \frac{5}{2}mg\ell\sin\varphi = 0$$

$$\boxed{= m\ell^2\left(\frac{7}{3}\ddot\varphi + 2\ddot\theta\cos(\varphi-\theta)\right) + 2m\ell^2\dot\theta^2\sin(\varphi-\theta) + \frac{5}{2}mg\ell\sin\varphi = 0}$$

$\theta)$ $$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot\theta} - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$\mathcal{L} = \frac{1}{2}m\ell^2\left[\frac{7}{3}\dot\varphi^2 + \frac{8}{3}\dot\theta^2 + 4\dot\varphi\dot\theta\cos(\varphi-\theta)\right] + mg\ell\left(\frac{5}{2}\cos\varphi + 2\cos\theta\right)$$

$$\frac{d}{dt}\left(m\ell^2\left(\frac{8}{3}\dot\theta + 2\dot\varphi\cos(\varphi-\theta)\right)\right) +$$

$$-\left(2m\ell^2\dot\varphi\dot\theta\sin(\varphi-\theta) - mg\ell\, 2\sin\theta\right)$$

$$= m\ell^2\left(\frac{8}{3}\ddot\theta + 2\ddot\varphi\cos(\varphi-\theta) +\right.$$

$$\left. - 2\dot\varphi\sin(\varphi-\theta)(\dot\varphi - \dot\theta)\right) +$$

$$- 2m\ell^2\dot\varphi\dot\theta\sin(\varphi-\theta) + mg\ell\,2\sin\theta$$

$$\boxed{= m\ell^2\left(\frac{8}{3}\ddot\theta + 2\ddot\varphi\cos(\varphi-\theta)\right) + - 2m\ell^2\dot\varphi^2\sin(\varphi-\theta) + 2mg\ell\sin\theta = 0}$$

Terza parte

Linearizziamo intorno:

- $\varphi = 0, \quad \theta = 0 \leftarrow$ ✓
- $\varphi = 0, \quad \theta = \pi \leftarrow$ ✓

$$V = -mgl\left(\frac{5}{2}\cos\varphi + 2\cos\vartheta\right)$$

$$\begin{cases} \dfrac{\partial V}{\partial \varphi} = 0 & \sin\varphi = 0 \\ \dfrac{\partial V}{\partial \vartheta} = 0 & \sin\vartheta = 0 \end{cases} \longrightarrow$$

Prendiamo $\varphi = 0$, $\vartheta = 0$

$$\varphi = \varphi_E + \eta\, z_1 = \eta\, z_1 \leftarrow$$

$$\vartheta = \vartheta_E + \eta\, z_2 = \eta\, z_2 \leftarrow$$

$$\mathcal{L} = \frac{1}{2} m l^2 \left(\frac{7}{3}\dot\varphi^2 + \frac{8}{3}\dot\vartheta^2 + 4\dot\varphi\dot\vartheta\cos(\varphi-\vartheta)\right)$$

$$+ mgl\left(\frac{5}{2}\cos\varphi + 2\cos\vartheta\right) = K - V$$

$$\tilde{\mathcal{L}} = \tilde{K} - \tilde{V}$$

$$\tilde{K} = \frac{1}{2} m l^2\left(\frac{7}{3}\dot z_1^2 + \frac{8}{3}\dot z_2^2 + 4\dot z_1 \dot z_2\right)$$

$$\cos(\varphi_E - \vartheta_E) = \cos(0-0) = 1$$

$$\tilde{V} = \frac{1}{2}\, \underline{z} \cdot \mathrm{Hess}\, V\Big|_{eq.} \underline{z} \qquad \underline{z} = (z_1\ \ z_2)$$

$$V = -mgl\left(\frac{5}{2}\cos\varphi + 2\cos\theta\right)$$

$$\frac{\partial V}{\partial \varphi} = mgl\,\frac{5}{2}\sin\varphi$$

$$\frac{\partial V}{\partial \theta} = mgl\,2\sin\theta$$

$$\to \frac{\partial^2 V}{\partial \varphi^2} = mgl\,\frac{5}{2}\cos\varphi$$

$$\to \frac{\partial^2 V}{\partial \varphi\,\partial \theta} = 0 = \frac{\partial^2 V}{\partial \theta\,\partial \varphi}$$

$$\to \frac{\partial^2 V}{\partial \theta^2} = mgl\,2\cos\theta$$

$$\mathrm{Hess}\,V = \begin{pmatrix} mgl\,\frac{5}{2}\cos\varphi & 0 \\ 0 & mgl\,2\cos\theta \end{pmatrix}$$

$$\mathrm{Hess}\,V\Big|_{q_E} = \begin{pmatrix} \frac{5}{2}mgl & 0 \\ 0 & 2mgl \end{pmatrix}$$

$$\tilde{V} = \frac{1}{2}\,\underline{x}\cdot \mathrm{Hess}\,V\Big|_{q_E}\cdot \underline{x} =$$

$$= \frac{1}{2} (z_1 \quad z_2) \begin{pmatrix} \frac{5}{2} mgl & 0 \\ 0 & 2mgl \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$= \frac{1}{2} mgl \left(\frac{5}{2} z_1^2 + 2 z_2^2 \right)$$

$$\text{Hess } V = \begin{pmatrix} \frac{\partial^2 V}{\partial \varphi^2} & \frac{\partial^2 V}{\partial \varphi \partial \theta} \\ \frac{\partial^2 V}{\partial \varphi \partial \theta} & \frac{\partial^2 V}{\partial \theta^2} \end{pmatrix}$$

$$\begin{matrix} (\varphi & \theta) \\ z_1 & z_2 \end{matrix} \begin{pmatrix} \quad \end{pmatrix}$$

$$\tilde{\mathcal{L}} = \frac{1}{2} m l^2 \left(\frac{7}{3} \dot{z}_1^2 + \frac{8}{3} \dot{z}_2^2 + 4 \dot{z}_1 \dot{z}_2 \right) +$$

$$- \frac{1}{2} mgl \left(\frac{5}{2} z_1^2 + 2 z_2^2 \right)$$

Eq. linearizzate

$$\frac{d}{dt} \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{z}_1} - \frac{\partial \tilde{\mathcal{L}}}{\partial z_1} = \frac{d}{dt} \left(\frac{1}{2} m l^2 \left(\frac{7}{3} 2 \dot{z}_1 + 4 \dot{z}_2 \right) \right)$$

$$+ mgl \frac{5}{2} z_1 =$$

$$\boxed{m l^2 \left(\frac{7}{3}\ddot{q}_1 + 2\ddot{q}_2 \right) + m g l \frac{5}{2} q_1 = 0}$$

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{q}_2} - \frac{\partial \tilde{L}}{\partial q_2} = \frac{d}{dt}\left(\frac{1}{2} m l^2 \left(\frac{8}{3} 2 \dot{q}_2 + 4 \dot{q}_1 \right)\right)$$

$$+ m g l \, 2 q_2$$

$$= \boxed{m l^2 \left(\frac{8}{3}\ddot{q}_2 + 2\ddot{q}_1 \right) + m g l \, 2 q_2 = 0}$$

$$\tilde{K} = \frac{1}{2} m l^2 \left(\frac{7}{3}\dot{q}_1^2 + \frac{8}{3}\dot{q}_2^2 + 4 \dot{q}_1 \dot{q}_2 \right) =$$

$$= \frac{m l^2}{2} \left(\dot{q}_1 \quad \dot{q}_2 \right) \overbrace{\begin{pmatrix} \frac{7}{3} & 2 \\ 2 & \frac{8}{3} \end{pmatrix}}^{A(q_B)} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$$

$$A(\underline{q}_{=}) \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} m g l \frac{5}{2} & 0 \\ 0 & 2 m g l \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0$$

$$\boxed{m l^2 \begin{pmatrix} \frac{7}{3} & 2 \\ 2 & \frac{8}{3} \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} m g l \frac{5}{2} & 0 \\ 0 & 2 m g l \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0}$$

$$\varphi_E = 0 \qquad\qquad \varphi = \eta z_1 \qquad \dot\varphi = \eta \dot z_1$$

$$\theta_E = \pi \longrightarrow \qquad \theta = \pi + \eta z_2 \qquad \dot\theta = \eta \dot z_2$$

$$K = \frac{1}{2} m l^2 \left(\frac{7}{3} \dot\varphi^2 + \frac{8}{3} \dot\theta^2 + 4 \dot\varphi \dot\theta \cos(\varphi - \theta) \right)$$

$$\tilde K = \frac{1}{2} m l^2 \underline{\left(\frac{7}{3} \dot z_1^2 + \frac{8}{3} \dot z_2^2 - 4 \dot z_1 \dot z_2 \right)}$$

$$\mathrm{Hess}\, V = \begin{pmatrix} mgl \frac{5}{2} \cos\varphi & 0 \\ 0 & 2mgl\cos\theta \end{pmatrix}$$

$$\mathrm{Hess}\, V \Big|_{\substack{q_E \\ \varphi_E = 0,\ \theta_E = \pi}} = \begin{pmatrix} mgl \frac{5}{2} & 0 \\ 0 & -2mgl \end{pmatrix}$$

$$A(q_E) = m l^2 \begin{pmatrix} \frac{7}{3} & -2 \\ -2 & \frac{8}{3} \end{pmatrix}$$

$$C = \begin{pmatrix} mgl \frac{5}{2} & 0 \\ 0 & -2mgl \end{pmatrix}$$

$$m l^2 \begin{pmatrix} \frac{7}{3} & -2 \\ -2 & \frac{8}{3} \end{pmatrix} \begin{pmatrix} \ddot z_1 \\ \ddot z_2 \end{pmatrix} + \underline{\begin{pmatrix} mgl \frac{5}{2} & 0 \\ 0 & -2mgl \end{pmatrix}} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} =$$

$$\begin{cases} m\ell^2\left(\frac{7}{3}\ddot{\theta}_1 - 2\ddot{\theta}_2\right) + mg\ell\,\frac{5}{2}\,\theta_1 = 0 \\ m\ell^2\left(-2\ddot{\theta}_1 + \frac{8}{3}\ddot{\theta}_2\right) - 2mg\ell\,\theta_2 = 0 \end{cases}$$

$$(\ddot{\theta}_1 \quad \ddot{\theta}_2)\begin{pmatrix} A_{11} & 0 \\ 0 & A_{21} \end{pmatrix}\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix}$$