$\mathcal{C}$ $\quad ob(\mathcal{C})$, $\ mor(\mathcal{C})$; $\ \forall A, B \in ob(\mathcal{C})$ $\quad Hom(A, B)$

$f : A \longrightarrow B$

$\mathcal{C}, \mathcal{C}'$ $\quad F : \mathcal{C} \longrightarrow \mathcal{C}'$ covariant functor

$A \in ob(\mathcal{C})$ $\quad F(A) \in ob(\mathcal{C}')$

$$\begin{array}{c} A \\ f \downarrow \\ B \end{array} \qquad \begin{array}{c} F(A) \\ \downarrow F(f) \\ F(B) \end{array} \qquad Hom_{\mathcal{C}}(A, B) \xrightarrow{F} Hom_{\mathcal{C}'}(F(A), F(B))$$

$F(1_A) = 1_{F(A)}$, $\quad F(g \circ f) = F(g) \circ F(f)$

F contravariant functor

$$\begin{array}{c} A \\ \downarrow f \\ B \end{array} \qquad \begin{array}{c} F(A) \\ \uparrow F(f) \\ F(B) \end{array} \qquad F(g \circ f) = F(f) \circ F(g)$$

$\mathcal{C}$ $\quad\quad$ $\mathcal{C}^{op}$ opposite category

$ob(\mathcal{C}^{op}) = ob(\mathcal{C})$; $\quad Hom_{\mathcal{C}^{op}}(A,B) \stackrel{def}{=} Hom_{\mathcal{C}}(B,A)$

$A \xrightarrow{f} B$ $\quad\quad$ $B \xrightarrow{f} A$

In $\mathcal{C}$ $\quad\quad$ In $\mathcal{C}^{op}$

$F: \mathcal{C} \to \mathcal{C}'$ contravariant : it can be seen as a covariant functor $\underline{\underline{\mathcal{C} \to (\mathcal{C}')^{op}}}$ or $\mathcal{C}^{op} \to \mathcal{C}'$

$$\begin{array}{ccccc}
A & & F(A) & & F(A) \\
f\downarrow & & F(f)\uparrow & = & \downarrow F(f) \\
B & & F(B) & & F(B) \\
\mathcal{C} & & \mathcal{C}' & & (\mathcal{C}')^{op}
\end{array}$$

Isomorphisms in a category:

Def. $f: A \to B$ in $\mathcal{C}$ is an isomorphism if $\exists\ g: B \to A$ in $\mathcal{C}$

s.t. $A \xrightarrow{f} B \xrightarrow{g} A$ $\quad g \circ f = 1_A$

$B \xrightarrow{g} A \xrightarrow{f} B$ $\quad f \circ g = 1_B$

If $\exists\ f: A \to B$ isom. $\Longrightarrow$ A, B are isomorphic in $\mathcal{C}$

$A \simeq B$

Ex. 1) If $f: A \to B$ is an isomorph. $\Longrightarrow$ $\exists !\ g: B \to A$

s.t. $g \circ f = 1_A$, $f \circ g = 1_B$: $g = f^{-1}$ inverse of f

2) If $f: A \to B$ is an isomorphism in $\mathcal{C}$,

$F: \mathcal{C} \to \mathcal{C}'$ functor $\Longrightarrow$ $F(f): F(A) \to F(B)$)

is an isomorphism

In particular if $A \simeq B$ $\Longrightarrow$ $F(A) \simeq F(B)$.

$\mathcal{C}$ ob($\mathcal{C}$) algebraic varieties over a fixed K
locally closed

Mor($\mathcal{C}$) : $Hom_{\mathcal{C}}(X, Y) = \{ f : X \longrightarrow Y$ regular maps $\}$

$\mathcal{C}$ a category : 1) composition is regular
2) identity $1_X : X \longrightarrow X$ exists
3) associativity

$\mathcal{C}'$ ob($\mathcal{C}'$) = {K-algebras}
mor($\mathcal{C}'$) = {K-homomorphisms}

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ & K \arrow[ul, "g"] \arrow[u, "g'"'] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "f"'] \arrow[dr] & \\ Y \arrow[r, "g"] & K \end{tikzcd}$$

$\mathcal{O} : \mathcal{C} \longrightarrow \mathcal{C}'$
$X \qquad \mathcal{O}(X)$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}(X) \\ \mathcal{O}(Y) \arrow[u, "f^* = \mathcal{O}(f)"'] \end{tikzcd}$$

We get a contravariant functor because
$1_{\mathcal{O}(X)} = (1_X)^*$, $(g \circ f)^* = f^* \circ g^*$

$\mathcal{O}(Y) = Hom(Y, A^1)$ $\qquad A^1 = K$
K-algebra $\qquad$ set

$\mathcal{O}$ can be interpreted as the contravariant representable functor $h_{A^1} = Hom(-, A^1)$

$\mathcal{O}(X) = Hom(X, A^1)$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}(X) \\ \mathcal{O}(Y) \arrow[u, "\text{composition w. } f"'] \end{tikzcd} \qquad h_{A^1}(f)$$

$\mathcal{C} \xrightarrow{\mathcal{O}} \mathcal{C}' \xrightarrow{U} Set \qquad U \circ \mathcal{O} = h_{A^1}$

composition of functors: it is a functor.

Natural transformations

$$\mathcal{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{C}'$$

F, G both covariant functors
or both contravariant

A natural transform. $\varphi : F \to G$ is:

1) $A \in ob(\mathcal{C})$ $\qquad F(A) \xrightarrow{\varphi_A} G(A)$

$\forall A \in ob(\mathcal{C})$ we have $\varphi_A \in Hom(F(A), G(A))$

2)
$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} F(A) \arrow[r, "\varphi_A"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowleft"] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\varphi_B"'] & G(B) \end{tikzcd} \quad *$$

$\forall f \in Mor(\mathcal{C})$ the diagram is commutative

We get a new category: $Hom(\mathcal{C}, \mathcal{C}')$ functors from $\mathcal{C}$ to $\mathcal{C}'$

$ob(Hom(\mathcal{C}, \mathcal{C}')) = \{$ functors $\mathcal{C} \to \mathcal{C}' \}$

$mor(Hom(\mathcal{C}, \mathcal{C}')) = \{$ natural transformations $\}$

$\forall F, G \qquad Hom(F, G) = \{ \varphi : F \to G$ nat. transf $\}$

is this a set?

One can check the axioms of a category.

Equivalence of categories

$\mathcal{C}, \mathcal{C}'$: an equivalence of $\mathcal{C}$ with $\mathcal{C}'$ is a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ s.t.

1) $\forall B \in ob(\mathcal{C}')$, $\exists A \in ob(\mathcal{C})$ s.t. $F(A) \simeq B$

surjectivity on the objects up to isomorph.

2) $\forall A, A' \in ob(\mathcal{C})$ $F : Hom_{\mathcal{C}}(A, A') \longrightarrow Hom_{\mathcal{C}'}(F(A), F(A'))$

is a bijection.

Other equivalent definition:

functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ s.t. $\exists$ $G : \mathcal{C}' \longrightarrow \mathcal{C}$?

and a natural transf. $G \circ F = 1_{\mathcal{C}}$ + something

Example $\mathcal{C}$ $\quad Ob(\mathcal{C}) = \{$affine alg varieties closed in some $\mathbb{A}^n_K\}$

$mor(\mathcal{C}) = \{$regular maps$\}$ full subcategory of alg. varieties

$\mathcal{C}' = \{$finitely generated reduced K-algebras$\}$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ = no nilpotent elem.

$\mathcal{O} : \mathcal{C} \longrightarrow \mathcal{C}'$ is an equivalence of categories if K is algebraically closed:

$X \in Ob(\mathcal{C})$ $\quad \mathcal{O}(X) = K[X]$ finitely gen. K-alg. reduced

$$K[X] = \frac{K[x_1 \ldots x_n]}{I(X)} \leftarrow \text{radical}$$

$\mathcal{O}$ is an equivalence:

i) $A \in Ob(\mathcal{C}')$ $\quad A = K[\xi_1, \ldots, \xi_n] \simeq \dfrac{K[x_1 \ldots x_n]}{\alpha}$

$\alpha = \ker(ev)$ $\quad ev : K[x_1 \ldots x_n] \longrightarrow A$

$\quad\quad\quad\quad\quad\quad F(x_1 \ldots x_n) \longrightarrow F(\xi_1 \ldots \xi_n)$

A reduced $\Longrightarrow \alpha = \sqrt{\alpha} = I(V(\alpha))$

$\underline{A \simeq K[X]}$ $\quad X = V(\alpha) \subset \mathbb{A}^n$

We can take another system of generators for A

$A = K[\eta_1, \ldots, \eta_r] \simeq \dfrac{K[x_1 \ldots x_r]}{(\beta)}$

$A \simeq K[V(\beta)]$

2) $X \xrightarrow{f} Y$ regul. $\quad Hom_{\mathcal{C}}(X,Y) \xrightarrow{\mathcal{O}} Hom_{\mathcal{C}'}(\mathcal{O}(Y), \mathcal{O}(X))$

is a bijection

$\xrightarrow{*}$
$\xleftarrow{\#}$

1) X algebraic variety $\qquad Op(X)$ open subsets of X

Def. a functor $\mathcal{O}_X : Op(X) \longrightarrow \mathcal{C}$

$\mathcal{C}'$ K-algebras + K-homom.

$$U \in Ob(Op(X)) \rightsquigarrow \mathcal{O}_X(U) = \{\varphi : U \to K \text{ regular}\}$$

$$\mathcal{O}_X(U) = \mathcal{O}(U)$$

$U \subseteq X$

$$\begin{tikzcd} U \arrow[d, hook] \\ V \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_X(U) \\ \mathcal{O}_X(V) \arrow[u, "\rho_{UV}"'] \end{tikzcd} \qquad \begin{tikzcd} U \arrow[d] \\ \varphi : V \arrow[r] & K \end{tikzcd}$$

ρ_{UV} = restriction

$\mathcal{O}_X$ is a contravariant functor: the <u>SHEAF</u> of regular functions on X

1) $U = \bigcup U_i \qquad \varphi_i \in \mathcal{O}_X(U_i) \rightsquigarrow \varphi \in \mathcal{O}_X(U)$

2) - . - -

$(X, \mathcal{O}_X)$ affine pair $\qquad$ SCHEME

2) X $\mathcal{O}(X) = K$ if X is projective

$K(X)$

$\mathcal{D}$ ob($\mathcal{D}$) algebraic varieties : irreducible

mor($\mathcal{D}$) $\operatorname{Hom}_{\mathcal{D}}(X,Y) = \{$ rational map from X to Y $\}$

Then functor from $\mathcal{D}$ to a $\mathcal{D}'$: objects fields extensions of K, morphisms K-homomorphisms.

RATIONAL MAPS

X, Y irred. alg. varieties over K

Def: a rational map $\varphi : X \dashrightarrow Y$ is a germ of a regular map from $U \subseteq X$, open $\neq \emptyset$, to Y

$U \xrightarrow{\psi} Y$ regular

$S = \{ (U, \psi) \mid U \subseteq X, \text{ open} \neq \emptyset, \ \psi : U \to Y \}$

Relation in S: $(U, \psi) \sim (U', \psi')$ if

$\psi|_{U \cap U'} = \psi'|_{U \cap U'}$

This is an equivalence relation: so we can consider $S/\sim$: its elements are rational maps from X to Y.

Pf $\sim$ is reflexive, symmetric.

Transitivity: $U \xrightarrow{\psi} Y$, $U' \xrightarrow{\psi'} Y$, $U'' \xrightarrow{\psi''} Y$

$\psi|_{U \cap U'} = \psi'|_{U \cap U'}$, $\psi'|_{U' \cap U''} = \psi''|_{U' \cap U''} \Longrightarrow$

$\psi|_{U \cap U' \cap U''} = \psi''|_{U \cap U' \cap U''}$. We need to deduce

that $\psi|_{U \cap U''} = \psi''|_{U \cap U''}$

We need to prove a Lemma:

ass: $\varphi, \psi : X \longrightarrow Y$ φ, ψ regular, X irred. st.
$\varphi|_U = \psi|_U$ where $U \subseteq X$ open $\neq \emptyset$. Then $\varphi = \psi$

Pf: $P \in X$ if $P \in U$ $\varphi(P) = \psi(P)$; ass. $P \notin U$
$\varphi(P), \psi(P) \in Y$ $Y \subseteq \mathbb{P}^n = U_0 \cup U_1 \cdots \cup U_n$.

(*) If $\exists i$ s.t. $\varphi(P), \psi(P) \in U_i$, consider
$\overline{U} : \varphi^{-1}(Y \cap U_i) \cap \psi^{-1}(Y \cap U_i)$ open in X

$\varphi|_{\overline{U}}, \psi|_{\overline{U}} : \overline{U} \longrightarrow \underline{\underline{Y \cap U_i}}$ work w. non homog. coord.

Now $\varphi|_{\overline{U}}, \psi|_{\overline{U}}$ are given by an n-tuple of regular functions on $\overline{U}$ $\varphi|_{\overline{U}} = (\varphi_1, \ldots, \varphi_n)$,
$\psi|_{\overline{U}} = (\psi_1, \ldots, \psi_n)$, $\varphi_1 \ldots \varphi_n, \psi_1 \ldots \psi_n \in \mathcal{O}(\overline{U})$

We know by assump. $\forall$ index j $\varphi_j|_{\overline{U} \cap U} = \psi_j|_{\overline{U} \cap U} \Longrightarrow$
$\varphi_j|_{\overline{U}} = \psi_j|_{\overline{U}} \Longrightarrow \varphi(P) = \psi(P)$

If (*) is not satisfied, we can change coordinates:

$\varphi(P), \psi(P) \in \mathbb{P}^n$: $\exists\, H \subset \mathbb{P}^n$ hyperplane s.t.

$\varphi(P), \psi(P) \notin H \iff \varphi(P), \psi(P) \in \mathbb{P}^n \setminus H$

why H exists? Otherwise,

{hyperplanes passing thru $\varphi(P)$} = linear system of hyperplane = hyperplane in $(\mathbb{P}^n)^\vee$

{hyperplanes thru $\psi(P)$} ~ hyp. in $(\mathbb{P}^n)^\vee$

$\Rightarrow (\mathbb{P}^n)^\vee \neq H_1 \cup H_2$ it is not reducible.

$\Rightarrow$ such H exists. Then choose a syst. of coord. s.t. $H: x_0 = 0$.

More in general, given finitely many pts in $\mathbb{P}^n$

$P_1, \dots, P_r$

$\downarrow \qquad \downarrow$

$H_1 \qquad H_r$ hyperplanes in $(\mathbb{P}^n)^\vee$

$H_1 \cup \dots \cup H_r \subsetneq (\mathbb{P}^n)^\vee$

Therefore in $\mathbb{P}^n$ $\exists\, H$ st. $P_1, \dots, P_r \notin H$

$P(a_0, \dots, a_n)$ $\qquad a_0x_0 + \dots + a_nx_n = 0$: equation in $(\mathbb{P}^n)^\vee$ of the hyperplanes passing thru P

$S_d \subseteq \mathbb{P}^n \rightsquigarrow S^\vee$ of dim $n-d-1$

$P \qquad P^\vee \qquad n-1$