X, Y quasi-projective var.

$\varphi : X \dashrightarrow Y$ rational map : φ is the germ of a reg. map

$U \xrightarrow{\varphi_1} Y$, $U \subseteq X$ open

$V \xrightarrow{\varphi_2} Y$ $\qquad \varphi_1|_{U \cap V} = \varphi_2|_{U \cap V}$

$\operatorname{dom} \varphi = \bigcup_{[U_i, \varphi_i] = \varphi} U_i$ union of the open subsets U_i s.t. (U_i, φ_i) is a pair which represents φ

If X, Y are affine, $Y \subseteq \mathbb{A}^n \Longrightarrow \varphi = (\varphi_1, \dots, \varphi_n)$

$\varphi_1, \dots, \varphi_n$ rational functions on $X \subseteq \mathbb{A}^m$

$\varphi_i = \frac{F_i}{G_i}$ quotient of poly in m variables

In gen. if $Y \subseteq \mathbb{P}^n$: $\varphi : \underset{X}{U} \longrightarrow Y \subseteq \mathbb{P}^n$, $U \subseteq X$

φ is locally given by $[F_0, \dots, F_n]$, homog. of the same degree

If φ is given by $F_0, \dots, F_n$ on U, φ is completely determined by $F_0, \dots, F_n$

$(U, [F_0, \dots, F_n])$: $\forall P \in U$ at least one of $F_0, \dots, F_n$ must be $\neq 0$

$F_0, \dots, F_n \in K[x_0, \dots, x_n]$: if at least one is not the zero pol. $[F_0, \dots, F_n]$ defines a rational map on $\mathbb{P}^m$, if $F_i \in K[x_0, \dots, x_m]$

To define a category with objects quasi proj. varieties over K, and morphisms the rational maps, we need a composition of rational maps.

$$X \overset{\varphi}{-\,-\to} Y \overset{\psi}{-\,-\to} Z$$

$\psi \circ \varphi$ can be applied only if $P \in \underbrace{\varphi^{-1}(\operatorname{dom}\psi) \cap \operatorname{dom}\varphi}_{\text{could be empty}}$

$$\underset{\operatorname{dom}\varphi}{\overset{\cap}{P}} \longrightarrow \varphi(P)$$

$$\boxed{\varphi(\operatorname{dom}\varphi) \cap \operatorname{dom}\psi = \emptyset}$$

.. this can happen; in this case $\psi \circ \varphi$ is not def.

Def. $\varphi : X -\,-\to Y$ is dominant if $\underline{\varphi(\operatorname{dom}\varphi)}$ is dense in Y

image of φ by def.

If φ is dominant, $\forall\, \psi : Y -\,-\to Z \Longrightarrow$

$\psi \circ \varphi$ is well defined as rational map

We can consider a category $\mathcal{C}$ with morphisms the rational dominant maps

Fact $X \overset{\varphi}{\dashrightarrow} Y \overset{\psi}{\dashrightarrow} Z$ If φ, ψ are dominant $\Longrightarrow$ also $\psi \circ \varphi$ is dominant.

$X \xrightarrow{1_X} X$, associativity holds

Isomorphisms in this category:

$X \overset{\varphi}{\dashrightarrow} Y$ dominant such that

$Y \overset{\psi}{\dashrightarrow} X$ dominant : such tat

$\underline{\psi \circ \varphi} = 1_X$ as rational map = they are equal where they are both regular

$\varphi \circ \psi = 1_Y$ " "

$X \subseteq \mathbb{A}^2$ cuspidal cubic $\qquad\qquad k$

$\mathbb{A}^1$

$X \not\cong \mathbb{A}^1$ $\qquad \varphi : \mathbb{A}^1 \longrightarrow X$ $\qquad$ bijective, cont.

$\qquad\qquad t \longmapsto (t^2, t^3)$

$\varphi^{-1} : X \longrightarrow \mathbb{A}^1$

$(x,y) \longrightarrow \frac{y}{x}$ not reg. at $(0,0)$

φ is regular

$\psi : X \dashrightarrow \mathbb{A}^1$ rational dominant map

$\operatorname{dom}\psi = X \setminus \{(0,0)\}$

$$X \dashrightarrow \mathbb{A}^1 \qquad U = X \setminus \{(0,0)\} \longrightarrow \mathbb{A}^1 \xrightarrow{\varphi} X$$

$$(x,y) \longrightarrow \frac{y}{x} \qquad \underset{\substack{\neq \\ (0,0)}}{(x,y)} \longrightarrow \underset{\neq 0}{\frac{y}{x}} \longrightarrow \underset{\substack{\parallel \\ (x,y)}}{\left(\frac{y^2}{x^2}, \frac{y^3}{x^3}\right)}$$

$\operatorname{Im}(\varphi \circ \psi) = U \qquad \underset{\substack{\parallel \\ 1_U}}{\varphi\circ\psi} : U \longrightarrow U$ not def. at $(0,0)$

but it defines a rational map $X \dashrightarrow X$

which is the identity

$$\mathbb{A}^1 \xrightarrow{\varphi} X \overset{\psi}{\dashrightarrow} \mathbb{A}^1$$

$t \longrightarrow (t^2, t^3)$

$\psi \circ \varphi$ is not def. at 0

it is reg. only on $U' = \mathbb{A}^1 - \{0\}$

On U' $\quad \psi \circ \varphi = 1_{U'}$ $\Rightarrow$ as rat. map

$\psi \circ \varphi = 1_{\mathbb{A}^1}$

X and $\mathbb{A}^1$ are "isomorphic" in this new category

Def: isomorphisms in the category are called BIRATIONAL MAPS or BIRATIONAL TRANSFORMATIONS

If $X \dashrightarrow Y$ birational: X, Y are birationally equivalent or birational.

$\mathbb{A}^1$, X cusp cubic: birationally equivalent

$\varphi: X \dashrightarrow Y$ dominant rat map $\Rightarrow \exists$ the comorph.

$\varphi^*: K(Y) \longrightarrow K(X)$

$f \in K(Y) \longrightarrow f \circ \varphi \in K(X)$ well def. because φ is domin.

φ^* is a field homom., fixing K, injective

$\Rightarrow \varphi^* K(Y) \simeq K(X)$:

Identifying $K(Y)$ with $\varphi^* K(Y) \Longrightarrow$

$K(X)$ is an extension of $K(Y)$

$\Rightarrow \dim X \geqslant \dim Y$.

X, Y quasi-projective varieties

$u : K(Y) \longrightarrow K(X)$ $\quad$ K-homom. (injective)

then $\exists\ u^{\#} : X \dashrightarrow Y$ rat. dominant s.t. $u = (u^{\#})^*$

$Y = \bigcup_{\alpha \in I} Y_\alpha$ $\quad$ Y_α open affine : affine covering of Y

$K(Y) \simeq K(Y_\alpha)$ $\quad \forall \alpha$

$\quad \| $

$K(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n$ are coord. functions on a closed subset $Z \subseteq \mathbb{A}^n$, $Z \simeq Y_\alpha$

$u : K(Y) \longrightarrow K(X)$ $\quad$ $u(t_1), \ldots, u(t_n) \in K(X)$ rational fns.

$\quad \cong$

$K(t_1 \ldots t_n)$

$V = \bigcap_{i=1}^{n} \operatorname{dom} u(t_i) = \text{open in } X$

$V \xrightarrow[u(t_1), \ldots, u(t_n)]{} \mathbb{A}^n$ $\quad$ The image is contained in Z

$V \longrightarrow Z \subseteq \mathbb{A}^n$: the germ of this map gives a rational map $X \overset{u^{\#}}{\dashrightarrow} Y$

$\quad \cong$

$Y_\alpha \subseteq Y$

The def. doesn't depend on the choices.

$\operatorname{Hom}_{\delta}(X, Y) \xleftrightarrow[\text{bijection}]{} \operatorname{Hom}(K(Y), K(X))$