$\varphi : X \dashrightarrow Y$ If φ is dominant, $\varphi^* : K(Y) \longrightarrow K(X)$
K-homom.

$X \overset{\varphi}{\dashrightarrow} Y \overset{\psi}{\dashrightarrow} Z$ φ domin. $\Rightarrow$ $\psi \circ \varphi$ is a rational map

Def. φ is birational if φ is dominant, $\exists\, \psi : Y \dashrightarrow X$ domin. s.t. $\psi \circ \varphi = 1_X$
$\varphi \circ \psi = 1_Y$ as rational maps.

If $u : K(Y) \longrightarrow K(X)$ is a K-homom. $\Rightarrow \exists\, u^\# : X \dashrightarrow Y$ rat. domin. s.t. $u = (u^\#)^*$

Thm $\varphi : X \dashrightarrow Y$ is a rational map.
The following are equivalent:
1) φ is birational
2) $K(Y) \underset{\varphi^*}{\xrightarrow{\sim}} K(X)$ is an isomorphism

Pf. φ birat. $\Rightarrow \exists\, \psi$ - - - - -
φ^*, ψ^* give an isom. $K(Y) \simeq K(X)$
If $\varphi^* : K(Y) \longrightarrow K(X)$ is an isom. : we have functoriality of the construction of #
$\Rightarrow$ φ is also an isom.

Thm. X, Y quasi-proj. varieties. TFAE:

1) X, Y are birational

2) $\boxed{K(X), K(Y)}$ are isomorphic

3) $\exists$ $U \subseteq X$, $V \subseteq Y$ open $\neq \emptyset$ s.t. $U \simeq V$: isomorphic.

(U, V are dense) ✓

Pf 1) $\Longleftrightarrow$ 2)

1) $\Rightarrow$ 3) $X \overset{\varphi}{\dashrightarrow} Y$ birational; $Y \overset{\psi}{\dashrightarrow} X$ $\psi = \varphi^{-1}$

$U' = \mathrm{dom}(\varphi) \subseteq X$, $\mathrm{dom}\,\psi = V' \subseteq Y$

$U' \xrightarrow{\varphi} Y$

$U = \varphi^{-1}(V') \xrightarrow{\varphi} V' \xrightarrow{\psi} X$ the composition is well-defined and is the identity

$U \subseteq U'$, $V' \subseteq Y$

$\Rightarrow$ we have an isom. between $U = \varphi^{-1}(V')$ and $\varphi(U)$ $\quad U \simeq \varphi(U)$

Similarly if we ~~start~~ start from $\psi: Y \dashrightarrow X$

3) $\Rightarrow$ 2) $U \simeq V$ open non-empty $\quad K(U) \simeq K(X)$

$U \subseteq X$, $V \subseteq Y$ $\quad K(U) \cong K(V) \simeq K(Y)$

$\Rightarrow K(X) \simeq K(Y)$.

<u>Consequence</u> $\mathbb{P}^n_K$, $\mathbb{A}^n_K$ are birational because
$\mathbb{A}^n_K \simeq U_0 \subseteq \mathbb{P}^n_K$ open in $\mathbb{P}^n_K$

<u>Examples</u>
$\varphi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ rational map (here instead of $\mathbb{P}^n$ we could take X quasi-proj.)

<u>Then</u> $\operatorname{dom} \varphi = \mathbb{P}^1$ i.e. φ is in fact <u>regular</u>

<u>Pf.</u> $\exists$ an open subset of $\mathbb{P}^1$ U where $\varphi = [F_0, \dots, F_n]$, $F_0, \dots, F_n$ homog. pol. of deg $d \geq 1$ in $\underline{K[x_0, x_1]}$; we can assume $F_0, \dots, F_n$ don't have any common factor.

$P \in \mathbb{P}^1$ / at least one F_i $F_i(P) \neq 0 \implies \varphi_i$ is regular at P
\ $\boxed{F_0(P) = \dots = F_n(P) = 0} \implies F_i \in I(P) =$
$= \langle a_1 x_0 - a_0 x_1 \rangle$

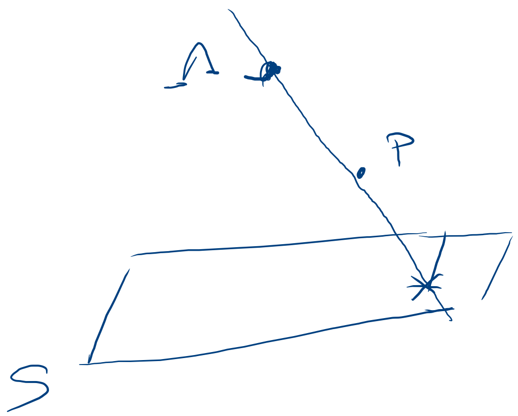
$P[a_0, a_1]$ $\qquad \operatorname{rk}\begin{pmatrix} a_0 & a_1 \\ x_0 & x_1 \end{pmatrix} < 2$

$a_1 x_0 - a_0 x_1$: homogeneization of the eq. of P in A^1
$\implies F_0, \dots, F_n$ are all multiples of $a_1 x_0 - a_0 x_1$: contradiction because $F_0, \dots, F_n$ are coprime.

This can be extended to rational maps whose domain is a smooth curve

2) Projections

$\mathbb{P}(V) = \mathbb{P}^n$ Fix a linear subspace $\Lambda \subseteq \mathbb{P}^n$, fix a linear space complementary to Λ : S



Ex. Λ a point, S a hyperplane : $\Lambda \notin S$

$\Lambda = \mathbb{P}(W)$, $S = \mathbb{P}(U)$

$V = W \oplus U$ $n+1 = r+1+s+1$

$\dim \Lambda = r$, $\Longrightarrow$ $\dim S = n - r - 1 = s$

$\langle \Lambda \cup S \rangle = \mathbb{P}^n$

π_Λ projection of centre Λ to S is

$\pi_\Lambda : \mathbb{P}^n \setminus \Lambda \longrightarrow S$

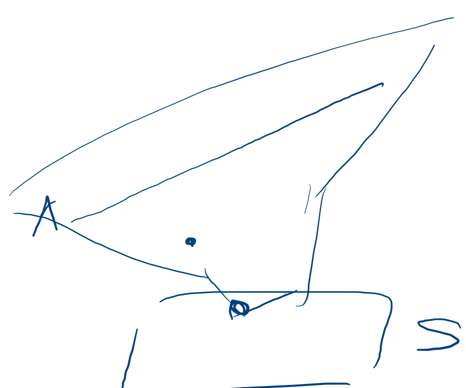
$P \longrightarrow \langle \Lambda \cup P \rangle \cap S$: one point because of Grassmann relation

$\dim \langle \Lambda \cup P \rangle = r+1$

$\dim \langle \Lambda \cup P \rangle \cap S = \dim \langle \Lambda \cup P \rangle + \dim S - \dim \langle \Lambda \cup P \cup S \rangle$

$= r+1+s-n =$

$= r+1+(n-r-1)-n = 0$



$\pi_\Lambda : \mathbb{P}^n \setminus \Lambda \dashrightarrow \mathbb{P}^{n-r-1}$ $\quad r = \dim \Lambda$

this defines a rational map $\pi_\Lambda : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r-1}$

we have to check that $\pi_\Lambda : \mathbb{P}^n \setminus \Lambda \longrightarrow \mathbb{P}^{n-r-1}$ is regular

We can fix homogeneous coordinates s.t.

$\Lambda = \langle E_{n-r}, \dots, E_n \rangle$: $\boxed{x_0 = x_1 = \dots = x_{n-r-1} = 0}$

$S = \langle E_0, \dots, E_{n-r-1} \rangle$: $x_{n-r} = \dots = x_n = 0$

$\pi_\Lambda([a_0, \dots, a_n]) = [\underbrace{a_0, a_1, \dots, a_{n-r-1}}, 0 \dots 0]$

$\mathbb{P}^n \setminus \Lambda \longrightarrow S \quad \cancel{\Lambda}$

$P[a_0, \dots, a_n] \qquad \langle \Lambda \cup P \rangle$

$\Lambda = \{ [0 \dots \underset{n-r-1}{0}, x_{n-r}, \dots, x_n] \}$

$\{ [\underline{\lambda a_0, \dots, \lambda a_{n-r-1}}, \lambda a_{n-r} + x_{n-r}, \dots, \lambda a_n + x_n] \} \cap S =$

$= \{ [\lambda a_0, \dots, \lambda a_{n-r-1}, 0 \dots 0] \}$

π_Λ is regular on $\mathbb{P}^n \setminus \Lambda$ because def. by homog. polynom. $x_0, \dots, x_{n-r-1}, 0 \dots 0$

In general: Λ is def by $L_0 = \dots = L_{n-r-1} = 0$

where $L_0, \dots, L_{n-r-1}$ are linearly indep linear forms

we can take S: $L_{n-r} = \dots = L_n = 0$ where

$L_0, \dots, L_{n-r-1}, \dots, L_n$ are linearly indep. or

a basis for V^*

$$\pi_\Lambda : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r-1} \simeq S$$

$$P \longrightarrow [L_0(P), \dots, L_{n-r-1}(P)]$$

$$P \notin \Lambda$$

On S $L_0, \dots, L_{n-r-1}$ can be interpreted as coordinates.

If we have a rational map $m \leq n$

$\varphi : \mathbb{P}^n \overset{[L_0 \dots L_m]}{\dashrightarrow} \mathbb{P}^m$ φ given by $m+1$ linear forms linearly

indep. $\Longrightarrow$ φ is a projection; the centre is

Λ, the set of zeros of $L_0, \dots, L_m$

$\mathbb{P}^n$

$\varphi: \mathbb{P}^n \xrightarrow{[L_0 \dots L_m]} \mathbb{P}^m$ given by linear forms $L_0, \dots, L_m$

If $L_0, \dots, L_m$ are lin. dep., we can assume that $L_0, \dots, L_s$ are lin. indep. and a basis for the linear subspace of V^* gen. by $L_0, \dots, L_m$.

$L_{s+1}, \dots, L_m$ are linear combinations of $L_0 \dots L_s$

$$\begin{tikzcd}
\mathbb{P}^n \arrow[rr, dashed, "{[L_0 \dots L_m]}"] \arrow[dr, dashed, "{[L_0 \dots L_s]}"'] & & \mathbb{P}^m \\
& \mathbb{P}^s \arrow[ur, "\text{regular injective}"'] &
\end{tikzcd}$$

$$[x_0 \dots x_s] \mapsto [x_0 \dots x_s, a_{10}x_0 + \dots + a_{1s}x_s, \dots, a_{m-s,0}x_0 + \dots + a_{m-s,s}x_s]$$

where

$$\begin{cases} L_{s+1} = a_{10}L_0 + \dots + a_{1s}L_s \\ \vdots \\ L_m = a_{m-s,0}L_0 + \dots + a_{m-s,s}L_s \end{cases}$$

$m = s + (m-s)$

$$\begin{tikzcd} \mathbb{P}^2 \arrow[rr, dashed, "Q"] \arrow[dr, "\nu_{2,2}"'] & & \mathbb{P}^2 \\ & \mathbb{P}^5 \arrow[ur, dashed] & \end{tikzcd}$$

$[x_0, x_1, x_2] \longrightarrow [x_1x_2,\ x_0x_2,\ x_0x_1]$

composition of Veronese map and a projection; the center is a 2-plane of equations $z_1 = z_2 = z_4 = 0$

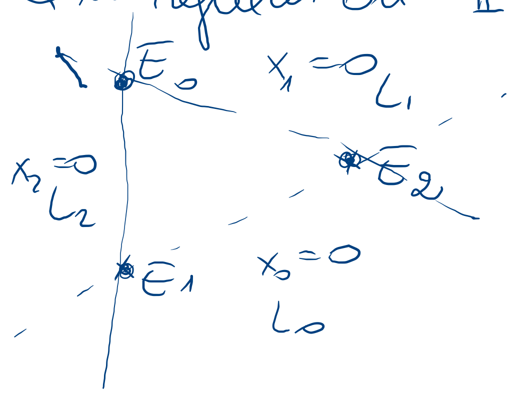
$[z_0, \dots, z_5]$, $z_0 = x_0^2, \dots, z_5 = x_2^2$

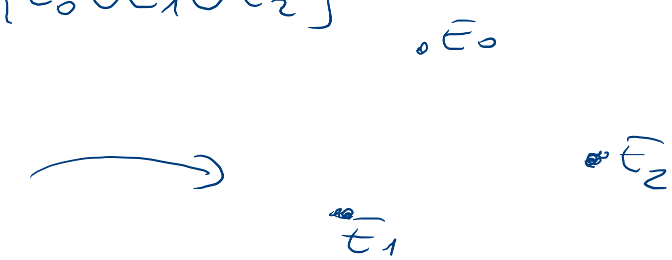
$$\begin{cases} x_1x_2 = 0 \\ x_0x_2 = 0 \\ x_0x_1 = 0 \end{cases}$$

$x_1 = 0 \implies x_2 = 0$ or $x_0 = 0$

The solutions are $E_0[1\ 0\ 0]$, $E_1[0\ 1\ 0]$, $E_2[0\ 0\ 1]$

Q is regular on $\mathbb{P}^2 \setminus \{E_0 \cup E_1 \cup E_2\}$





$[x_0, x_1, x_2]$

$x_0x_1x_2 \neq 0$

$U = \mathbb{P}^2 \setminus \{L_0 \cup L_1 \cup L_2\}$ $\quad Q: U \longrightarrow U$ regular

$Q \circ Q : [x_0, x_1, x_2] \longrightarrow [x_1x_2, x_0x_2, x_0x_1] \longrightarrow [x_0^2x_1x_2, x_0x_1^2x_2, x_0x_1x_2^2] =$ $[x_0, x_1, x_2]$

$\quad \in U$

$Q \circ Q = 1_{\mathbb{P}^2}$ as rational map

$\implies Q$ is birational, $Q^{-1} = Q$

$Q: U \xrightarrow{\sim} U$ isomorphism

X quasi-projective variety $\qquad \mathrm{Bir}(X) = \{ \varphi : X \dashrightarrow X \}$ birational domin. maps

$\mathrm{Bir}(X)$ group for composition
$\cup$
$\mathrm{Aut}(X)$

$\mathrm{Bir}(\mathbb{P}^2) \supsetneq \mathrm{Aut}(\mathbb{P}^2) = PGL(3,K)$

Q standard quadratical map

Max Noether: $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $PGL(3,K)$ and Q
Enriques $\qquad$ $=$ the Cremona group

$\mathrm{Bir}(\mathbb{P}^3)$, $\mathrm{Bir}(\mathbb{P}^n)$ ___ . ___

Def. X quasi-projective variety
X is rational if it is birational to a projective space. Equivalently: $K(X) \simeq K(\mathbb{P}^n)$ or
$X \supseteq U$ open $\qquad U \simeq V \subseteq \mathbb{P}^n$
$\ne \emptyset \qquad \ne \emptyset$

$\dim X = n$ $\qquad K(X) \simeq K(\mathbb{P}^n) \simeq K(\mathbb{A}^n) = Q(K[x_1 \dots x_n])$
$= K(x_1, \dots, x_n)$ = pure transcendental extension of K
$\underbrace{x_1, \dots, x_n}$ n algebraically indip. elem.

$\varphi : X \dashrightarrow \mathbb{P}^n$ domin. $\}$ compos. is ident.
$\boxed{\psi : \mathbb{P}^n \dashrightarrow X}$ domin.
$X \subseteq \mathbb{P}^m$

ψ is a birationally parametrization of X

Weaker definition: X is UNIRATIONAL if

$\exists\ \varphi : \mathbb{P}^n \dashrightarrow X$ rational dominant

LÜROTH Theorem: for curves "unirational" = "rational"

LÜROTH Problem: (1880) can we extend Lüroth thm to dim > 1?

Guido Castelnuovo: yes for surfaces if char $k = 0$,
1894

$n = 3$? UGO MORIN, FANO, ...

NO 1971 3 groups gave different counterexamples

Iskovskih - Manin

Clemens - Griffiths $\to$ cubic hypersurface in $\mathbb{P}^4$

$\to$ Artin - Mumford