OBSERVATION

OFTEN IN ADVANCED Q.M. THE PLANCK CONSTANT $\hbar$ IS SET $=1$ AS WELL AS THE SPEED OF THE LIGHT IN THE FREE SPACE IS SET $c=1$, THESE ARE KNOWN AS NATURAL UNITS. CONNECTING THE NATURAL UNITS TO C.g.S IT IS SUFFICIENT TO MULTIPLY BY THE APPROPRIATE COMBINATION OF $\hbar$ AND c FACTORS.
(WITH C.g.S. WE MEAN THE RATIONALISED GAUSSIAN UNITS WHERE THE ELECTRICAL PERMITTIVITY OF THE FREE SPACE IS SET $=1$).
FOR ISTANCE THE MASS OF THE PROTON IS $m_p \simeq 1.67 \times 10^{-27}$ Kg. IN NATURAL UNITS IT CAN BE EXPRESSED AS AN ENERGY, THE UNITS OF ENERGY ARE $J = Kg\ m^2/s^2$ WE NEED TO MULTIPLY THE VALUE BY A COMBINATION OF FACTORS THAT HAS DIMENSION $m^2/s^2 \Rightarrow c^2$. OF COURSE THIS PROCEDURE IS DERIVED FROM THE RELATIVISTIC MASS-ENERGY RELATION $E = mc^2$.

EXAMPLE

$$m_p = 1.67 \times 10^{-27}\,Kg\Big|_{cgs} \Rightarrow 1.67 \times 10^{-27} \times c^2\,Kg\Big|_{nu} \Rightarrow$$

$$1.50 \times 10^{-10}\,J\Big|_{nu}.$$

FOR SUB-NUCLEAR PHYSICS THE PREFERRED UNIT FOR ENERGY IS GeV. THE CONVERSION BETWEEN JOULE AND GeV IS: $1\ GeV = 10^9\ eV = 1.60 \times 10^{-19} \times 10^9\ CV \Rightarrow$
$= 1.60 \times 10^{-10}\ J$

$\Rightarrow m_p \simeq 0.938\,GeV|_{n.u.}$

ALSO A SECOND AND A m IN c.g.s. CAN BE GIVEN IN TERMS OF ENERGY IN n.u.

$$1\,s|_{cgs} \rightarrow \frac{1s}{\hbar} = 0.952\times 10^{34}\,J^{-1}|_{n.u.} = 1.52\times 10^{24}\,GeV^{-1}|_{n.u.}$$

$$1\,m|_{cgs} \rightarrow \frac{1m}{\hbar c}\bigg|_{n.u.} = 0.317\times 10^{26}\,J^{-1}|_{n.u.} = 0.507\times 10^{16}\,GeV^{-1}|_{n.u.}$$

$\Rightarrow$ IN n.u. TIME AND SPACE HAVE THE DIMENSIONS OF THE INVERSE OF AN ENERGY (OR THE INVERSE OF A MASS), NOTE THAT IN n.u. THE TYPICAL SIZE OF A NUCLEON IS $\sim 1\,fm = 10^{-15}\,m \approx 1/(200\,MeV)$

- THE e^2 ($e \equiv$ ELECTRON CHARGE) IS $\approx 2.56\times 10^{-38}\,C^2$, IN S.I. UNITS IT HAS THE DIMENSION OF $kg\,m^3/s^2$. TO CONVERT TO c.g.s UNITS IT SHOULD BE DIVIDED BY $\varepsilon_0 \simeq 2.89\times 10^{-25}\,kg\,m^3/s^2$.
THE FINE STRUCTURE CONSTANT IS A PURE NUMBER SO IT DOES NOT CHANGE

$$\alpha = \frac{e^2}{4\pi\hbar c}\bigg|_{cgs} = \frac{e^2}{4\pi}\bigg|_{n.u.} \approx \frac{1}{137}$$

- BOHR RADIUS $r_B \simeq 0.5\times 10^{-10}\,m = 1/(4\,keV)$ $\Rightarrow$

$$\alpha/r_B \approx 30\,eV.$$

- CROSS-SECTION, IN n.u. THEY ARE GIVEN OFTEN IN GeV^{-2}; IN cgs THEY HAVE THE DIMENSION OF A SURFACE $\Rightarrow$ THE ARE GIVEN IN m^2 (OR MORE COMMONLY IN $barn = 10^{-28}\,m^2$), THE CONVERSION IS

$\sigma = N\,GeV^{-2}\big|_{n.u.} \Rightarrow N\,GeV^{-2}(\hbar c)^2\big|_{cgs} \Rightarrow$

$N\,GeV^{-2}(1.973\times 10^{-16})^2\,GeV^2 m^2\big|_{cgs} = N\;0{,}389\;mbarn\big|_{cgs}$

- KLEIN-GORDON AND DIRAC EQS IN TENSORIAL FORM

THE K-G EQUATION

$$\left(-\frac{1}{c^2}\partial_t^2+\nabla^2\right)\psi=\frac{m^2c^2}{\hbar^2}\psi \Rightarrow \Box^2\psi+\frac{m^2c^2}{\hbar^2}\psi=0$$

BY SETTING $\hbar=c=1$ BECOMES

$$\left(\Box^2\psi+m^2\right)\psi=0 \quad \left(\Box^2\equiv\partial_t^2-\nabla^2\right)$$

IN THE RELATIVITY TENSORIAL FORMALISM THE D'ALAMBERTIAN OPERATORS IN THE COVARIANT NOTATION $\equiv x_\mu=(-ct,\bar{x})$

$$\Box^2\equiv\frac{1}{c^2}\partial_t^2-\nabla^2=g^{\mu\nu}\partial_\nu\partial_\mu=\partial^\mu\partial_\mu,$$ BEING

$g^{\mu\nu}$ IS THE MINKOWSKI METRIC

$(g^{00}=1,\ g^{11}=g^{22}=g^{33}=0$ FOR $\mu\neq\nu) \Rightarrow$

$$\boxed{\left(\partial^\mu\partial_\mu+m^2\right)\psi=0}$$

- THE DIRAC EQ. IN THE COVARIANT NOTATION

NOWADAYS NOBODY USES THE $\bar{\alpha}$ AND β MATRICES INSTEAD THE γ- MATRICES ARE USED, AS WE HAVE SEEN ON PAGE 189, $\gamma^0=\beta$, $\gamma^j=\beta\alpha_j$ $(j=1,2,3)$. THE DIRAC EQUATION, NOT DERIVED HERE BECOMES $\boxed{i\gamma^0\partial_t\psi=m\psi-i\gamma^j\bar{\nabla}_j\psi}$

OBSERVATIONS ON THE DIRAC EQ.

HISTORICALLY DIRAC WAS LOOKING FOR A COVARIANT WAVE EQUATION THAT WAS FIRST-ORDER IN TIME, TO AVOID THE PROBLEM PRESENTED BY THE K-G. EQ.

1 - THE CHARGE DENSITY PROBABILITY (NOT DISCUSSED HERE) IS NOT NECESSARELY POSITIVE,

2 - THE K-G IS A SECOND-ORDER DIFF. EQ. IN t
$\Rightarrow$ NEED TO KNOW BOTH ψ AND $\partial_t \psi$ AT $t=0$ IN ORDER TO SOLVE FOR ψ AT $t>0$. THUS THERE IS AN EXTRA DEGREE OF FREEDOM, NOT PRESENT IN THE S-EQ.

3 - AS MENTIONED BEFORE THE EQ. ON WHICH THE K-G EQUATION IS BASED $E^2 = p^2c^2 + m^2c^4$ HAS BOTH POSITIVE AND NEGATIVE SOLUTION.

WE WANT NOW TO PROVE THAT THE DIRAC EQ. IS CONSISTENT WITH THE K-G EQ. TO DO SO WE USE THE $\bar{\alpha}$ AND $\bar{\beta}$ MATRICES NOTATION (WE LEAVE THE γ^μ MATRIX NOTATION FOR MORE ADVACES LECTURES). AGAIN WE SET $c = \hbar = 1$

THE DIRAC-EQ IS

$$i\partial_t \psi = \bar{\beta} m\psi - ic\bar{\alpha}\cdot\bar{\nabla}\psi$$

BY SQUERING THIS EQ WE OBTAIN

$$-i\partial_t^2 \psi = \bar{\beta}^2 m\psi - im(\bar{\beta}\bar{\alpha} + \bar{\alpha}\bar{\beta})\bar{\nabla}\psi + -(\bar{\alpha}\cdot\bar{\nabla})^2\psi$$

REMEMBERING THAT $\beta^2 = \alpha_x^2 = \alpha_y^2 = \alpha_z^2 = 1$ AND

$\beta\alpha_j + \alpha_j\beta = \alpha_j\alpha_k + \alpha_k\alpha_j = 0$ FOR ALL $j \neq k = x, y, z$,

184 - 187

WE OBTAIN THE K-G-EQ: $-\partial_t^2 \psi = m^2 \psi - \nabla^2 \psi$.

- OBSERVATION A SUITABLE REPRESENTATION.

$$\beta \equiv \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \equiv \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array}\right)$$

$$\alpha_j \equiv \left(\begin{array}{c|c} 0 & \sigma_j \\ \hline \sigma_j & 0 \end{array}\right)$$ WHERE σ_i ARE THE PAULI SPIN MATRICES

$$\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

AS WE HAVE SEEN ψ IS REPRESENTED BY A 4-COMPONENTS OBJECT CALLED SPINOR (NOT A 4-VECTOR). EACH COMPONENT SATISFIES THE K-G EQ.

- FOR A PARTICLE AT REST $\psi = \phi e^{-(imc^2t/\hbar)}$

THE D-EQ. $\Rightarrow \phi = \beta \phi \Rightarrow$

$$\phi \equiv \begin{vmatrix} \phi_1 \\ \phi_2 \\ 0 \\ 0 \end{vmatrix}$$ WHERE ϕ_1 AND ϕ_2 ARE THE SPIN ORIENTATION.

- FOR ANTI-PARTICLES AT REST $\psi = \phi e^{(imc^2t/\hbar)}$

$\Rightarrow \phi = -\beta \phi \Rightarrow$

$$\phi \equiv \begin{vmatrix} 0 \\ 0 \\ \phi_3 \\ \phi_4 \end{vmatrix}$$ WITH ϕ_3, ϕ_4 GIVING THE SPIN ORIENTATION.

SPIN OF DIRAC PARTICLES (OPTIONAL)

HOW DO WE PROVE THAT DIRAC-EQ CORRESPONDS TO SPIN 1/2? WE MUST SHOW THAT THERE EXISTS AN OPERATOR $\hat{S}'$ SUCH THAT $\hat{J} = \hat{L} + \hat{S}'$ IS A CONSTANT OF MOTION AND ($\hbar = 1$) $S'^2 = S(S+1) = \frac{3}{4}$

- WE NOTE FIRST THAT $\bar{L} = \bar{r} \times \bar{p}$ IS NOT A CONSTANT OF MOTION i.e. DIH.

$$H = \beta m + \bar{\alpha} \cdot \bar{p}$$ (NOT DEMON. HERE, DIRAC HAMILTONIAN)

$$[L_z, H] = [x, H] p_y - [y, H] p_x = i\alpha_x p_y - i\alpha_y p_x$$

IN GENERAL $[\bar{L}, H] = i\bar{\alpha} \times \bar{p} \neq 0 \Rightarrow$

$[S, H] = -i\bar{\alpha} \times \bar{p}$. THIS IS TRUE IF $\bar{S} = \frac{1}{2} \Sigma'_i$

WHERE $\Sigma'_j = \left(\begin{array}{c|c} \sigma_i & 0 \\ \hline 0 & \sigma_i \end{array} \right) = i(\alpha_x \alpha_y \alpha_z)\bar{\alpha} \Rightarrow$

$$S^2 = \frac{1}{4}\left(\Sigma'^2_x + \Sigma'^2_y + \Sigma'^2_z \right) = \frac{3}{4}$$ PROVING THAT

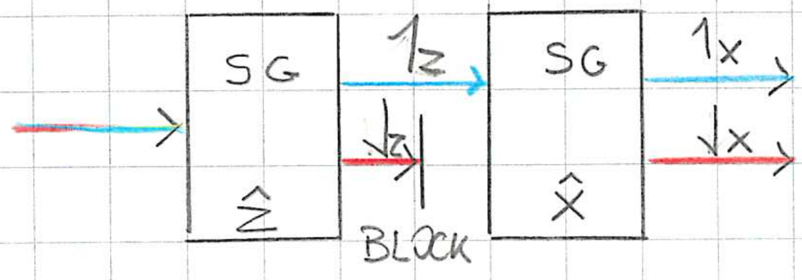
$$S = \frac{1}{2}$$

STERN-GERLACH SEQUENTIAL MEASUREMENTS

- SUPPOSE A $S = \pm\frac{1}{2}$ $L=0$ ATOM IS PASSED TROUGH A S-G. DEVICE AND SELECT THE SPIN UP ($\uparrow$) IN THE $\hat{z}$. IF WE PASS IT THROUGH A SECOND S-G DEVICE IT WILL STILL BE FOUND WITH $\uparrow$ IN THE $\hat{z}$ DIRECTION.
- LET'S TILT NOW THE S-G OF $\pi/2$ SO THE $\hat{x}$ COMPONENT IS MEASURED, WE WILL FIND A DISCRETE COMPONENT OF THE SPIN, EITHER $\uparrow$ AND $\downarrow$

BUT IN THE $\hat{X}$.

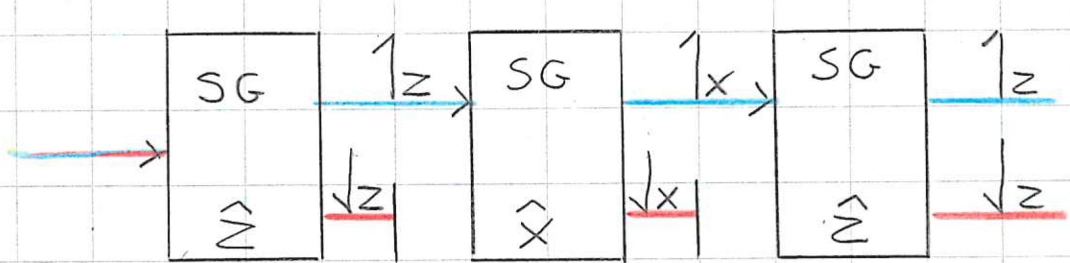
- THIS SUGGESTS THAT THE S-G APPARATUS CAN BE ROTETED OF AN ARBITRARY ANGLE θ AROUND THE SPIN-PARTICLE BEAM DIRECTION AND FOUND TWO SPIN COMPONENTS, $\uparrow$ AND $\downarrow$.
- LET'S PERFORM NOW THE FOLLOWING EXPERIMENT



IN THIS CASE WE HAVE $\uparrow_X$ $\downarrow_X$ WITH EQUAL PROBABILITIES.

IF WE HAD BLOCK $\uparrow_Z$ THE OUTCOME WOULD HAVE BEEN THE SAME.

- SUPPOSE NOW THAT WE MEASURE $\uparrow_Z$ AND THEN $\downarrow_X$. WHAT HAPPENS IF WE MEASURE THE Z COMPONENTS AGAIN?



IN THIS CASE WE GET $\uparrow_Z$ $\downarrow_Z$ WITH EQUAL PROBABILITIES. MEASURING $\hat{X}$ HAS ERASED OUR ORIGINAL MEASUREMENT OF $\hat{Z}$. SIMILAR RESULTS ARE OBTAINED WITH THE SEQUENCE $\hat{X} \to \hat{Z} \to \hat{X}$. BY A MORE COMPLEX ARRANGMENT WE CAN MEASURE THE $\uparrow_{\hat{y}}$ AND $\downarrow_{\hat{y}}$. THIS BRINGS US TO THE FOLLOWING CONCLUSION

- MEASURING $\hat{y}$ ERASES $\hat{x}$ OR $\hat{z}$.
  MEASURING $\hat{x}$ ERASES $\hat{z}$ OR $\hat{y}$ $\Rightarrow$
  MEASURING $\hat{z}$ ERASES $\hat{y}$ OR $\hat{x}$

THE $\hat{x}, \hat{y}, \hat{z}$ COMPONENTS OF THE SPIN ARE ALL COMPLEMENTARY VARIABLES, $\Rightarrow$ KNOWING ONE OF THE THREE PRECLUDES KNOWING THE OTHER TWO. THEY ARE NOT ALL SIMULTANEOUSLY WELL DEFINED. IF A GIVEN VARIABLE IS NOT WELL-DEFINED FOR A GIVEN STATE OF THE SYSTEM THE MEASURE GIVES RANDOM RESULTS.

- SUPPOSE $\uparrow_z$ BY MEASURING THE SPIN AT θ RESPECT TO $\hat{z}$, WE FIND THE $\uparrow_\theta$ WITH A PROBABILITY $P_{\uparrow\theta} = \cos^2(\theta/2)$. THE SAME FOR THE $\hat{x}$ AND $\hat{y}$ DIRECTION. MEANWHILE $\downarrow_\theta$ GIVES $P_{\downarrow\theta} = \sin^2(\theta/2)$.

- MATHEMATICAL DESCRIPTION OF THE SPIN

LET'S CHOOSE BY CONVENTION THE $\hat{z}$. THERE ARE TWO SPIN STATES $\uparrow_{\hat{z}}$ AND $\downarrow_{\hat{z}}$. AT AN ANGLE $\theta_{\hat{z}}$ THE $\uparrow_\theta$ COULD BE

$$\psi(\theta) = \uparrow_{\hat{z}} \cos(\theta/2) + \downarrow_{\hat{z}} \sin(\theta/2)$$

- THE VALUE $\cos(\theta/2)$ AND $\sin(\theta/2)$ ARE THE AMPLITUDES IN THE $\uparrow_\theta$ AND $\downarrow_\theta$ DIRECTIONS. IF WE MEASURE THE $\uparrow_{\hat{z}}$ AND $\downarrow_{\hat{z}}$ $\hat{z}$ COMPONENTS OF THE SPIN

$\uparrow_{\hat{z}}$ OR $\downarrow_{\hat{z}}$ THE PROBABILITIES ARE $\cos^2(\theta/2)$ AND $\sin^2(\theta/2)$ (THE AMPLITUDE SQUARED).

- IF THE SPIN IS $\uparrow_{\hat{x}}$ (OR $\downarrow_{\hat{x}}$) ($\theta = \pi/2$) THE

$$\psi_{\uparrow_{\hat{x}}} = \left(\uparrow_{\hat{z}} + \downarrow_{\hat{z}}\right)/\sqrt{2}$$

$$\psi_{\downarrow_{\hat{x}}} = \left(\uparrow_{\hat{z}} - \downarrow_{\hat{z}}\right)/\sqrt{2}$$

AND THE SAME FOR THE OTHER CONFIGURATIONS
BUT WHAT ABOUT THE $\hat{y}$ DIRECTION? THAT IS ALSO $\pi/2$ RESPECT TO $\hat{z}$ BUT $\uparrow_{\hat{y}}$ AND $\downarrow_{\hat{y}} \neq \uparrow_{\hat{x}}$ AND $\downarrow_{\hat{x}}$. WE AVOID THE PROBLEM BY LETTING THE AMPLITUDES BE COMPLEX NUMBERS.

$$\uparrow_{\hat{y}} = \left(\uparrow_{\hat{z}} + i\downarrow_{\hat{z}}\right)/\sqrt{2}$$

$$\downarrow_{\hat{y}} = \left(\uparrow_{\hat{z}} - i\downarrow_{\hat{z}}\right)/\sqrt{2}$$

THE MOST GENERAL STATE THAN CAN BE WRITTEN

$$\psi = \tilde{\alpha}\uparrow_{\hat{z}} + \tilde{\beta}\downarrow_{\hat{z}}$$

WHERE $\tilde{\alpha}$ AND $\tilde{\beta}$ ARE COMPLEX NUMBERS SUCH THAT

$$|\tilde{\alpha}|^2 + |\tilde{\beta}|^2 = 1$$

THIS IS THE NORMALIZATION CONDITION.

- WE INTERPRET $\uparrow_{\hat{z}}$ AND $\downarrow_{\hat{z}}$ AS BEING THE TWO BASIS VECTORS FOR A 2D COMPLEX

VECTOR SPACE.

- IN TERMS OF THIS BASIS WE CAN WRITE ANY STATE AS A COLUMN VECTOR
$$[\psi]_{\hat{z}} = \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix}$$

WE MAKE THE ASSUMPTION THAT $\uparrow_{\hat{z}}$ AND $\downarrow_{\hat{z}}$ ARE ORTHOGONAL VECTORS OF UNIT LENGTH, IN OTHER WORDS THIS BASIS IS ORTHONORMAL.
THE SAME ARGUMENTS ARE VALID FOR THE $\hat{x}$ AND $\hat{y}$.

- THE $\hat{z}$ BASIS CAN BE CHANGED TO A DIFFERENT BASIS (LET SAY $\hat{x}$) BY APPLYING A LINEAR TRANSFORMATION
$$[\psi]_{\hat{x}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tilde{\alpha}+\tilde{\beta} \\ \tilde{\alpha}-\tilde{\beta} \end{pmatrix}$$

IN GENERAL THE MATRIX $\hat{U}$ THAT IS USED CHANGE BETWEEN TWO ORTHONORMAL BASIS IS A UNITARY MATRIX $\hat{U}\hat{U}^* = \hat{U}^*\hat{U} = I$
THE PROBABILITY TO MEASURE $\uparrow_{\hat{z}}$ OR $\downarrow_{\hat{z}}$ IS GIVEN BY $|\tilde{\alpha}|^2$ AND $|\tilde{\beta}|^2$.

- THE PAULI MATRICES RIVISITED

SINCE $\uparrow_{\hat{z}}$ AND $\downarrow_{\hat{z}}$ ARE ORTHOGONAL WE CAN FIND A HERMITIAN MATRIX $\hat{Z} = \hat{Z}^+$ SUCH THAT $\uparrow_{\hat{z}}$ AND $\downarrow_{\hat{z}}$ ARE EIGENVECTORS WITH EIGENVALUES ± 1. SIMILAR MATRICES CAN BE FOUND FOR $\hat{X}$ AND $\hat{Y}$. IN THE STANDARD Z BASIS, THESE MATRICES ARE.

$$\hat{X} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \hat{Y} \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \hat{Z} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

THESE ARE OBSERVABLE PHYSICAL QUANTITIES, AND THEY ARE KNOWN AS PAULI MATICES, SHOWING A REMARKABLE NUMBER OF IMPORTANT MATHEMATICAL PROPERTIES.

- PAULI MATRICES ARE
  - ANTICOMMUTATIVE

$$\hat{X}\hat{Y} = -\hat{Y}\hat{X} = i\hat{Z}$$
$$\hat{Y}\hat{Z} = -\hat{Z}\hat{Y} = i\hat{X}$$
$$\hat{Z}\hat{X} = -\hat{X}\hat{Z} = i\hat{Y}$$

  - TRACELESS AND IDEMPOTENT

$$\mathrm{Tr}\{\hat{X}\} = \mathrm{Tr}\{\hat{Y}\} = \mathrm{Tr}\{\hat{Z}\} = 0$$

$$\hat{X}^2 = \hat{Y}^2 = \hat{Z}^2 = \hat{I}$$ $\Rightarrow$ IN ADDITION OF BEING HERMITIAN THEY ARE ALSO UNITARY.

- BESIDES IDENTITY PROPERTIES THE PAULI MATRICES FORM A BASIS FOR ALL 2X2 MATRICES. ANY 2X2 MATRIX $\hat{O}$ CAN BE WRITTEN AS A LINEAR COMBINATION

$$\hat{O} = \tilde{a}\hat{I} + \tilde{b}\hat{X} + \tilde{c}\hat{Y} + \tilde{d}\hat{Z}$$

WHERE $\tilde{a}, \tilde{b}, \tilde{c}$ AND $\hat{d}$ ARE COMPLEX NUMBERS. IT CAN ALSO BE SHOWN FROM THE ALGEBRIC PROPERTIES OF $\hat{X}$, $\hat{Y}$ AND $\hat{Z}$ THAT

$$\tilde{a} = \frac{1}{2}\mathrm{Tr}\{\hat{O}\}, \quad \tilde{b} = \frac{1}{2}\mathrm{Tr}\{\hat{X}\hat{O}\}, \quad \hat{c} = \frac{1}{2}\mathrm{Tr}\{\hat{Y}\hat{O}\}$$
$$\hat{d} = \frac{1}{2}\mathrm{Tr}\{\hat{Z}\hat{O}\}$$

SPIN AND PHOTON POLARIZATION: AN INTERESTING ANALOGY

CLASSICAL E.M. FIELD CARRIES LINEAR MOMENTUM DENSITY, S/c^2 WHERE $\bar{S} = (\bar{E} \times \bar{B}) \frac{1}{\mu_0}$ IS THE POYTING VECTOR. THE TOTAL MOMENTUM IS

$$\bar{P} = \int_{\tau} \frac{1}{c^2} \bar{S} \, d^3x$$

FROM THE FIELDS-POTENTIAL IN THE FREE SPACE

$$= -\varepsilon_0 \int \dot{A}(\bar{x},t) \times (\bar{\nabla} \times A(\bar{x},t)) d^3x$$

AFTER THE QUANTIZATION OF $\bar{A}(\bar{x},t)$

$$\hat{P} = \sum_{k,s}' \hbar \bar{k} \, \hat{a}^+_{ks} \hat{a}_{ks}$$

i.e. $\hat{P}|k,s\rangle = \hat{P} \hat{a}_{k,s} |\omega\rangle = \hbar \bar{k} |\bar{k},s\rangle$ (FOR $s=1,2$)

- THE ANGULAR MOMENTUM $\bar{L} = \bar{x} \times \bar{P}$ INCLUDES INTRINSIC COMPONENT,

$$\bar{M} = -\int_{\tau} (\dot{\bar{A}} \times \bar{A}) d^3x \rightarrow$$

$$\hat{M} = -i\hbar \sum_{k}' \hat{e}_k \left[\hat{a}^+_{k1} \hat{a}_{k2} - \hat{a}^+_{k2} \hat{a}_{k1} \right]$$

DEFINING CREATION OPERATORS FOR RIGHT/LEFT CIRCULAR POLARIZATION (R, L)

$$\hat{a}^+_{kR} = \frac{1}{\sqrt{2}} (\hat{a}^+_{k1} + \hat{a}_{k2}) \; ; \; \hat{a}^+_{kL} = \frac{1}{\sqrt{2}} (\hat{a}^+_{k1} - \hat{a}_{k2})$$

WE FIND THAT
$$\hat{\bar{M}} = \sum_{\bar{k}}{}' \hbar \hat{e}_k \left[\hat{a}^{+}_{kR} \hat{a}_{kR} - \hat{a}^{+}_{kL} \hat{a}_{kL} \right]$$

$\Rightarrow$ THEREFORE, SINCE $\hat{e}_k \cdot \hat{\bar{M}} |\bar{k}, R/L\rangle = \pm\hbar |\bar{k}, R/L\rangle$ WE CONCLUDE THAT PHOTONS CARRY INTRINSIC ANGULAR MOMENTUM $\pm\hbar$ (KNOWN AS HELICITY) ORIENTED PARALLEL OR ANTI-PARALLEL TO THE DIRECTION OF THE MOMENTUM PROPAGATION.