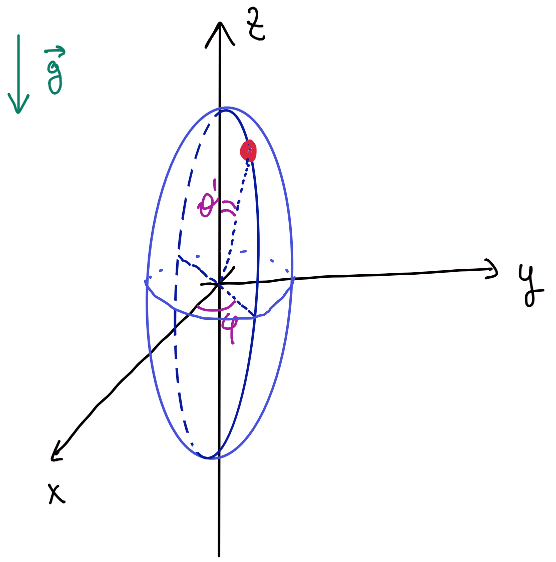
# ESERCIZI

## ES. del 13/07/2020

$$\operatorname{tg}\theta' = \frac{b}{a}\operatorname{tg}\theta$$




$F(q)$:

$$x = a\,\operatorname{sen}\theta\cos\varphi$$
$$y = a\,\operatorname{sen}\theta\,\operatorname{sen}\varphi$$
$$z = b\cos\theta$$

$$\dot{x} = a\dot{\theta}\cos\theta\cos\varphi - a\dot{\varphi}\,\operatorname{sen}\theta\,\operatorname{sen}\varphi$$
$$\dot{y} = a\dot{\theta}\cos\theta\,\operatorname{sen}\varphi + a\dot{\varphi}\,\operatorname{sen}\theta\cos\varphi$$
$$\dot{z} = -b\dot{\theta}\,\operatorname{sen}\theta$$

1) $$T = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) = \frac{1}{2}m\left(a^2\dot{\theta}^2\cos^2\theta + a^2\dot{\varphi}^2\operatorname{sen}^2\theta + b^2\dot{\theta}^2\operatorname{sen}^2\theta\right)$$

$$= \frac{m}{2}\left[a^2\operatorname{sen}^2\theta\,\dot{\varphi}^2 + \left(a^2\cos^2\theta + b^2\operatorname{sen}^2\theta\right)\dot{\theta}^2\right]$$

$$V = mgz = mgb\cos\theta$$

$$L = \frac{m}{2}\left(a^2\operatorname{sen}^2\theta\,\dot{\varphi}^2 + \left(a^2\cos^2\theta + b^2\operatorname{sen}^2\theta\right)\dot{\theta}^2\right) - mgb\cos\theta$$

$$a = \begin{pmatrix} ma^2\operatorname{sen}^2\theta & 0 \\ 0 & m\left(a^2\cos^2\theta + b^2\operatorname{sen}^2\theta\right) \end{pmatrix} \leftarrow \text{Matrice cinetica}$$

3) $L$ non dipende da $\varphi$ $\rightarrow$ $\varphi$ è ciclica

$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = ma^2\operatorname{sen}^2\theta\,\dot{\varphi}$$

$$\ell = ma^2\operatorname{sen}^2\theta\,\dot{\varphi} \rightarrow \dot{\varphi} = \frac{\ell}{ma^2\operatorname{sen}^2\theta}$$

$$L^* = L - p_\varphi \dot\varphi \Big|_{\dot\varphi = \frac{\ell}{ma^2 \operatorname{sen}^2\theta}} = \frac{m}{2}\left(a^2\cos^2\theta + b^2\operatorname{sen}^2\theta\right)\dot\theta^2 - mgb\cos\theta + $$

$$+ \frac{m}{2} a^2 \operatorname{sen}^2\theta \frac{\ell^2}{(ma^2\operatorname{sen}^2\theta)^2} - \frac{\ell^2}{ma^2\operatorname{sen}^2\theta}$$

$$L^* = \frac{m}{2}\left(a^2\cos^2\theta + b^2\operatorname{sen}^2\theta\right)\dot\theta^2 - mgb\cos\theta - \frac{\ell^2}{2ma^2\operatorname{sen}^2\theta}$$

5) $$\frac{d}{dt}\left(\frac{\partial L^*}{\partial\dot\theta}\right) = \frac{d}{dt}\left( m\dot\theta\left(a^2\cos^2\theta + b^2\operatorname{sen}^2\theta\right)\right) =$$

$$= m\ddot\theta\left(a^2\cos^2\theta + b^2\operatorname{sen}^2\theta\right) + m\dot\theta^2\Big(\underbrace{-2a^2\cos\theta\operatorname{sen}\theta + 2b^2\operatorname{sen}\theta\cos\theta}_{2(b^2-a^2)\operatorname{sen}\theta\cos\theta}\Big)$$

$$\frac{\partial L^*}{\partial\theta} = \frac{m}{\cancel{2}}\left(\cancel{2}(b^2-a^2)\operatorname{sen}\theta\cos\theta\right)\dot\theta^2 + mgb\operatorname{sen}\theta + \frac{\ell^2\cos\theta}{ma^2\operatorname{sen}^3\theta}$$

$$m\ddot\theta\left(a^2\cos^2\theta + b^2\operatorname{sen}^2\theta\right) + m\dot\theta^2(b^2-a^2)\operatorname{sen}\theta\cos\theta - mgb\operatorname{sen}\theta - \frac{\ell^2\cos\theta}{ma^2\operatorname{sen}^3\theta} = 0$$

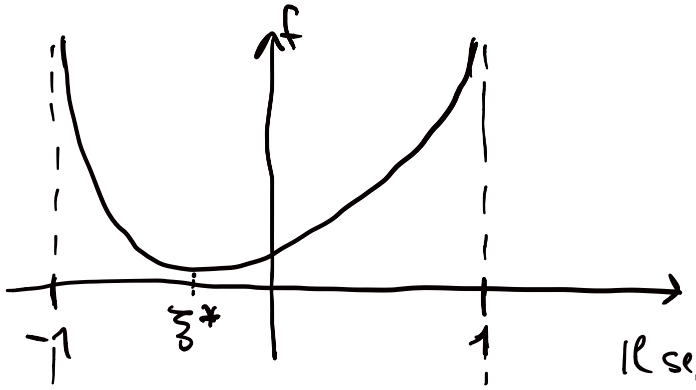
6) $V_{eff} = f(\cos\theta)$ $\qquad \xi \equiv \cos\theta$

$\theta \in [0,\pi] \to \xi \in [-1,1]$

$\uparrow$ in questo intervallo $\cos\theta$ è monotona

$$V_{eff} = mgb\cos\theta + \frac{\ell^2}{2ma^2\operatorname{sen}^2\theta}$$

$$f(\xi) = mgb\,\xi + \frac{\ell^2}{2ma^2(1-\xi^2)}$$ $\leftarrow$ def. positiva




Il segno di qsto termine è determinato da $\xi$.

$$f'(\xi) = mgb + \underbrace{\frac{l^2}{2ma^2}\,\frac{2\xi}{(1-\xi^2)^2}}$$

Ha sempre uno zero in $\xi^* \in [-1, 0]$

$$\updownarrow$$

$\theta^* \in [\frac{\pi}{2}, \pi]$

7) $p_\varphi$ è una cost. del moto

$p_\varphi(\varphi, \dot\varphi, \theta, \dot\theta) = l \leftarrow$ insieme di livello

$\dot\varphi(0) = 0$ condiz. iniziale per una traiettoria che sta in un insieme di livello

$t=0$

$p_\varphi = \frac{\dot\varphi}{ma^2 \sin^2\theta}$

$\Downarrow$

$l = 0$

$$\rightarrow \quad L^*_{(l=0)} = \frac{m}{2}\left(a^2\cos^2\theta + b^2 \underbrace{\sin^2\theta}_{1-\cos^2\theta}\right)\dot\theta^2 - mgb\cos\theta$$

$\Rightarrow V_{eff} = mgb\cos\theta \rightarrow$ min. per $\theta = \pi$

$$\theta = \pi + \delta\theta \qquad \dot\theta = \delta\dot\theta \qquad \cos(\pi+\delta\theta) = -\cos\delta\theta$$

$$L^* = \frac{m}{2}\left(\left(a^2-b^2\right)\cos^2\delta\theta + b^2\right)\delta\dot\theta^2 + mgb\cos\delta\theta$$

Espando $L^*$ attorno $\delta\theta \sim 0$.

trascuro cost.

$$\hat{L}^* = \frac{m}{2}\left(\left(a^2-b^2\right)\left(1-\frac{\delta\theta^2}{2}\right)^2 + b^2\right)\delta\dot\theta^2 + mgb\left(\cancel{1}-\frac{\delta\theta^2}{2}\right)$$

$$= \frac{m}{2}\left(a^2-\cancel{b^2}+\cancel{b^2}\right)\delta\dot\theta^2 - mgb\frac{\delta\theta^2}{2}$$

$$= \frac{ma^2}{2}\delta\dot\theta^2 - \frac{mgb}{2}\delta\theta^2 \longleftrightarrow \frac{1}{2}A\delta\dot\theta^2 - \frac{1}{2}B\delta\theta^2$$

$$\lambda = \frac{B}{A} = \frac{\text{coeff }\delta\theta^2}{\text{coeff }\delta\dot\theta^2} = \frac{mgb}{ma^2} = g\frac{b}{a^2} = \omega^2$$

# Es. 2 del 29.01.20

$$\bar{B} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} \qquad \bar{A} = \frac{B}{2}\begin{pmatrix} -q_2 \\ q_1 \\ 0 \end{pmatrix} \qquad \bar{B} = \bar{\nabla} \times \bar{A}$$

$$L = \frac{1}{2} m \dot{\bar{q}}^2 + e\, \dot{\bar{q}} \cdot \bar{A}$$

1) $r, \varphi$, $z$

$$\begin{cases} q_1 = r\cos\varphi \\ q_2 = r \,\mathrm{sen}\,\varphi \\ q_3 = z \end{cases} \qquad \begin{cases} \dot{q}_1 = \dot{r}\cos\varphi - r\dot{\varphi}\,\mathrm{sen}\,\varphi \\ \dot{q}_2 = \dot{r}\,\mathrm{sen}\,\varphi + r\dot{\varphi}\cos\varphi \\ \dot{q}_3 = \dot{z} \end{cases}$$

$$T_2 = \frac{1}{2} m \left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right) = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2\right)$$

$$\bar{A} = \frac{B}{2}\begin{pmatrix} -r\,\mathrm{sen}\,\varphi \\ r\cos\varphi \\ 0 \end{pmatrix} \qquad \dot{\bar{q}} = \begin{pmatrix} \dot{r}\cos\varphi - r\dot{\varphi}\,\mathrm{sen}\,\varphi \\ \dot{r}\,\mathrm{sen}\,\varphi + r\dot{\varphi}\cos\varphi \\ \dot{z} \end{pmatrix}$$

$$e\,\bar{A}\cdot\dot{\bar{q}} = \frac{eB}{2}\Big[ -r\,\underline{\mathrm{sen}\,\varphi}\left(\cancel{\dot{r}\cos\varphi} - r\dot{\varphi}\,\underline{\mathrm{sen}\,\varphi}\right) + r\,\underline{\cos\varphi}\left(\cancel{\dot{r}\,\mathrm{sen}\,\varphi} + r\underline{\dot{\varphi}\cos\varphi}\right) \cdot \cancel{0\,\dot{z}} \Big]$$

$$= \frac{eB}{2} r^2 \dot{\varphi}$$

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{eB}{2} r^2\dot{\varphi} + \frac{1}{2} m \dot{z}^2$$

2) $z$ è ciclica $\longrightarrow$ $p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$ è cost. del moto $\rightsquigarrow$ sim. traslazione lungo l'asse $z$

$\uparrow$ $B$ è cost.

$\varphi$ è ciclica $\longrightarrow$ $p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2\dot{\varphi} + \frac{eB}{2} r^2$ $\rightsquigarrow$ sim. rotaz. attorno asse $z$

è cost. del moto

$\bar{B} = B\bar{e}_z$

3) $V = V_0 \, e^{q_3^2/\ell^2}$

$$L = \frac{1}{2} m \left( \dot r^2 + r^2 \dot\varphi^2 \right) + \frac{eB}{2} r^2 \dot\varphi + \frac{1}{2} m \dot z^2 - V_0 \, e^{z^2/\ell^2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) = \frac{d}{dt}\left( m \dot r \right) = m \ddot r$$

$$\frac{\partial L}{\partial r} = m r \dot\varphi^2 + eB r \dot\varphi \quad \rightarrow \quad \ddot r = r \dot\varphi^2 + \frac{eB}{m} r \dot\varphi$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot \varphi}\right) = \frac{d}{dt}\left( m r^2 \dot\varphi + \frac{eB}{2} r^2 \right) = 2 m r \dot r \dot\varphi + m r^2 \ddot\varphi + eB r \dot r$$

$$\frac{\partial L}{\partial \varphi} = 0 \quad \rightarrow \quad \ddot\varphi = -2 \frac{\dot r \dot\varphi}{r} - \frac{eB}{m r} \dot r$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot z}\right) = m \ddot z$$

$$\frac{\partial L}{\partial z} = -V_0 \frac{2z}{\ell^2} e^{z^2/\ell^2} \quad \rightarrow \quad \ddot z = -\frac{V_0}{m} \frac{2z}{\ell^2} e^{z^2/\ell^2}$$

4) $\varphi$ è ciclica $\quad p_\varphi = m r^2 \dot\varphi + \frac{eB}{2} r^2 = \tilde p_\varphi$ cost. $\rightarrow$

$$\xrightarrow[\text{inversione}]{} \quad \dot\varphi = \left( \tilde p_\varphi - \frac{eB}{2} r^2 \right) \cdot \frac{1}{m r^2}$$

$$L^* = L - \dot\varphi \, p_\varphi \Big|_{\dot\varphi = \dots} = \frac{1}{2} m \dot r^2 + \frac{1}{2} m \dot z^2 - V_0 \, e^{z^2/\ell^2} + \frac{1}{2} \cancel{m r^2} \frac{1}{(m r^2)^{\cancel{2}}} \left( p_\varphi - \frac{eB r^2}{2} \right)^2$$

$$\underbrace{- \frac{\tilde p_\varphi}{m r^2} \left( \tilde p_\varphi - \frac{eB}{2} r^2 \right) + \frac{eB}{2} r^2 \frac{1}{m r^2} \left( \tilde p_\varphi - \frac{eB}{2} r^2 \right)}_{-\left( \tilde p_\varphi - \frac{eB}{2} r^2 \right)\left( \tilde p_\varphi - \frac{eB}{2} r^2 \right) \frac{1}{m r^2}}$$

$$= - \left( \tilde p_\varphi - \frac{eB}{2} r^2 \right)^2 \frac{1}{m r^2}$$

$$L^* = \frac{1}{2} m \dot{z}^2 + \frac{1}{2} m \dot{r}^2 - V_0 e^{z^2/\ell^2} - \frac{1}{2mr^2}\left(\tilde{p}_\varphi - \frac{eB}{2} r^2\right)^2$$

5) $$V_{eff} = V_0 e^{z^2/\ell^2} + \frac{1}{2mr^2} \overbrace{\left( \tilde{p}_\varphi^2 - eB\tilde{p}_\varphi r^2 + \left(\frac{eB}{2}\right)^2 r^4 \right)}^{\left(\tilde{p}_\varphi - \frac{eB}{2} r^2\right)^2}$$

$$= V_0 e^{z^2/\ell^2} + \frac{\tilde{p}_\varphi^2}{2mr^2} + \frac{e^2B^2}{8m} r^2 - \frac{eB\tilde{p}_\varphi}{2m} \quad \text{cost.}$$

$$\frac{\partial V_{eff}}{\partial r} = -\frac{\tilde{p}_\varphi^2}{\cancel{m} r^3} + \frac{e^2B^2}{4\cancel{m}} r = 0 \rightarrow r^{\cancel{4}\,2} = \left(\frac{2\tilde{p}_\varphi}{eB}\right)^{\cancel{2}} \equiv r_0^2 \rightarrow r_0 = \sqrt{\frac{2\tilde{p}_\varphi}{eB}}$$

$$\frac{\partial V_{eff}}{\partial z} = V_0 e^{z^2/\ell^2} \frac{2z}{\ell^2} = 0 \longrightarrow z = 0$$

Pto equil: $(r, z) = (r_0, 0) \leftarrow$ MIN (stab.)

Moto 3d: $\dot{\varphi} = 0 \rightarrow$ pto è fermo a dist. $r_0$ da origine

(manifesto dalla forma del potenziale efficace).

6) $$\frac{1}{2} \dot{\bar{q}} \cdot A \dot{\bar{q}} - \frac{1}{2} \bar{q} \cdot B \bar{q}$$

$$r = r_0 + \delta r \qquad z = 0 + \delta z$$

$$L^* = \frac{m\dot{z}^2}{2} + \frac{m\dot{r}^2}{2} - V_0 e^{z^2/\ell^2} - \frac{1}{2mr^2}\left(\tilde{p}_\varphi - \frac{eB}{2} r^2\right)^2$$

$$= \frac{m\,\delta\dot{z}^2}{2} - V_0\left(1 + \frac{\delta z^2}{\ell^2} + \dots\right) + \frac{m\,\delta\dot{r}^2}{2} - \frac{e^2B^2}{2m}\delta r^2$$

$$\frac{1}{2mr^2}\left(\tilde{p}_\varphi - \frac{eB}{2} r^2\right)^2 = \frac{e^2B^2}{8mr^2}\left(r^2 - \underbrace{\frac{2\tilde{p}_\varphi}{eB}}_{=r_0^2}\right)^2 = \frac{e^2B^2}{8mr^2}(r-r_0)^2(r+r_0)^2 =$$

$$= \frac{e^2B^2}{8m(r_0+\delta r)^2}\,\delta r^2 (2r_0+\delta r)^2 = \frac{e^2B^2}{8m\cancel{r_0^2}}\,4\cancel{r_0^2}\,\delta r^2 + \ldots$$

termini di ordine superiore in $\delta r$

$$\hat{L}^{*} = \underbrace{\frac{m\,\delta\dot z^2}{2} - \frac{1}{2}\,2V_0\,\frac{\delta z^2}{\ell^2}}_{\text{dip. solo da } z,\dot z} + \underbrace{\frac{m\,\delta\dot r^2}{2} - \frac{e^2B^2}{2m}\,\delta r^2}_{\text{dip. solo da } r,\dot r}$$

$$\omega_z = \frac{2V_0}{\ell^2}\cdot\frac{1}{m} = \frac{2V_0}{\ell^2 m}$$

$$\omega_r = \frac{e^2B^2}{m}\,\frac{1}{m} = \left(\frac{eB}{m}\right)^2$$

7) Eq. di Lagr. di $\hat{L}^{*}$ sono le eq. di due oscillatori armonici disaccoppiati

$$x(t) = A\cos(\omega t + \phi)$$

$$\downarrow$$

$$r(t) = r_0 + A_r\cos(\omega_r t + \phi_r)$$

$$z(t) = A_z\cos(\omega_z t + \phi_z)$$

$$\bar{A} = \frac{B}{2}\begin{pmatrix} -q_2 \\ q_1 \\ 0 \end{pmatrix} \qquad \dot{q} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix}$$

Problema si può risolvere anche in coord. cartesiane

$$e\,\dot{q}\cdot\bar{A} = \frac{eB}{2}\left(q_1\dot{q}_2 - q_2\dot{q}_1\right)$$

$$L = \frac{1}{2}m\left(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2\right) + \frac{eB}{2}\left(q_1\dot{q}_2 - q_2\dot{q}_1\right)$$

Eq. Lagr.:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = m\ddot{q}_1 - \frac{eB}{2}\dot{q}_2 \qquad \frac{\partial L}{\partial q_1} = \frac{eB}{2}\dot{q}_2 \qquad m\ddot{q}_1 = eB\,\dot{q}_2$$

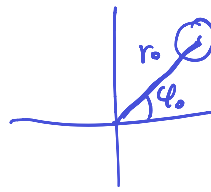
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} = m\ddot{q}_2 + \frac{eB}{2}\dot{q}_1 \qquad \frac{\partial L}{\partial q_2} = -\frac{eB}{2}\dot{q}_1 \qquad m\ddot{q}_2 = -eB\dot{q}_1$$

$$\dot{\eta}_1 = \frac{eB}{m}\eta_2 \qquad \eta_2 = \frac{m}{eB}\dot{\eta}_1 \qquad \eta_2 = -A\,\mathrm{sen}(\omega t+\phi)$$

$$\dot{\eta}_2 = -\frac{eB}{m}\eta_1 \quad \rightarrow \quad \ddot{\eta}_1 = -\underbrace{\left(\frac{eB}{m}\right)^2}_{\omega}\eta_1 \rightarrow \eta_1 = A\cos(\omega t+\phi)$$

$$q_1 = \frac{A}{\omega}\cos(\omega t+\phi) + r_0\cos\varphi_0 \qquad \rightarrow \text{circonferenze}$$

$$q_2 = \frac{A}{\omega}\,\mathrm{sen}(\omega t+\phi) + r_0\,\mathrm{sen}\varphi_0$$




$\delta r$ ha moto armonico