X quasi-projective variety : rational if X is birational to a projective space or to an affine space. If X is rat. of dim n, X is birational to $\mathbb{P}^n$.
X unirational if $\exists$ rat. domin. map $\mathbb{P}^n \dashrightarrow X$

$X \subseteq \mathbb{P}^3$ quadric surface in $\mathbb{P}^3$ of rk 4, if $K = \overline{K}$ is algebraically closed and char $K \neq 2$ $\Longrightarrow$ all quadrics of rk 4 are projectively equivalent.
A 4×4 over K : $\exists$ M invertible 4×4 s.t.
$${}^tMAM = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} = E_4 \iff \exists\ \varphi : \mathbb{P}^3 \longrightarrow \mathbb{P}^3$$
projectivity s.t. $\varphi(X)$ has equation $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$.
Another representative of this class of projective equivalence is $x_0 x_3 - x_1 x_2 = 0$

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$$

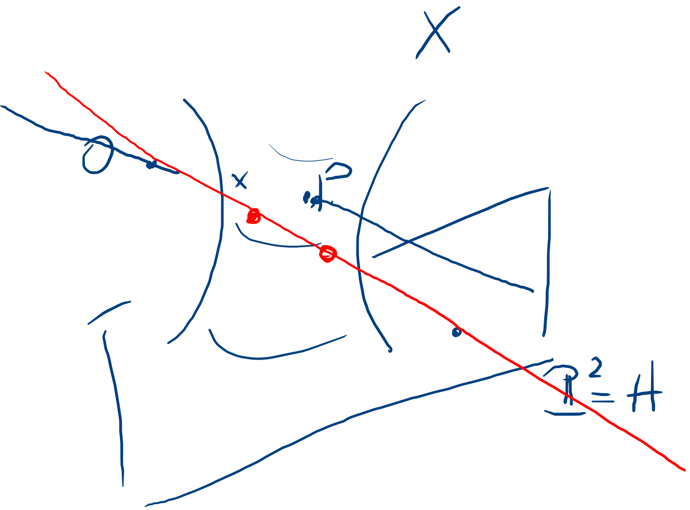
We will construct a birational map between $X : x_0 x_3 - x_1 x_2 = 0$ and $\mathbb{P}^2$

$\pi_O : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ projection of centre O

$\cup|$

X

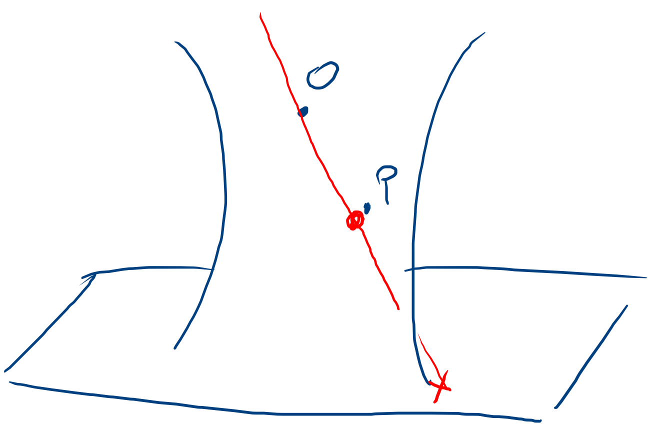
If $O \notin X$ $\qquad \pi_O|_X : X \longrightarrow \mathbb{P}^2$



$$OP \cap X = \begin{cases} \{P\} & \text{with mult. } 2 \\ \{P, P'\} & : \pi_O(P) = \pi_O(P') \\ OP \subseteq X \end{cases}$$

$\pi_O : X \xrightarrow{2:1} \mathbb{P}^2$

If $O \in X$ $\qquad \pi_O|_X : X \dashrightarrow \mathbb{P}^2$ not regular



$O[1,0,0,0] \in X \qquad x_0x_3 - x_1x_2 = 0$

$x_1 = x_2 = x_3 = 0$

$\pi_O : [x_0, x_1, x_2, x_3] \longrightarrow [x_1, x_2, x_3]$

$\updownarrow$

$\mathbb{P}^2 \longleftrightarrow H : x_0 = 0 \qquad [0, x_1, x_2, x_3]$

$$[y_1, y_2, y_3] \xrightarrow[\varphi]{?\ \pi_0^{-1}} [y_1 y_2, y_1 y_3, y_2 y_3, y_3^2]$$

$$\downarrow$$

$$[0, y_1, y_2, y_3]$$

Claim this defines the inverse of $\pi_0 /_X$

$$\mathbb{P}^2 \dashrightarrow X \xrightarrow{\pi_0} \mathbb{P}^2$$

$$[y_1, y_2, y_3] \to [y_1 y_2, y_1 y_3, y_2 y_3, y_3^2] \longrightarrow [y_1 y_3, y_2 y_3, y_3^2]$$

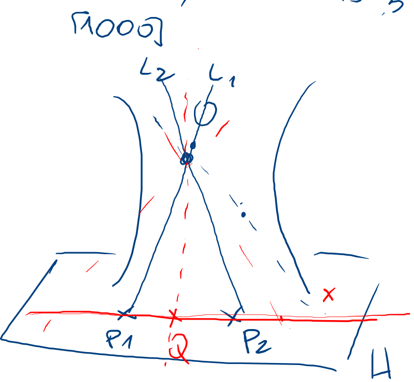
where $y_3 \neq 0$ $\quad \pi_0 \circ \varphi = 1_{\mathbb{P}^2}$

$$X \xrightarrow{\pi_0} [x_1, x_2, x_3] \xrightarrow{\varphi} [x_1 x_2, x_1 x_3, x_2 x_3, x_3^2]$$

$$[x_0 - x_3] \qquad \qquad x_1 x_2 = x_0 x_3 \qquad [x_0 - x_3]$$

$$\varphi \circ \pi_0 = id_X$$

π_0 is birational and X is rational.



$x_0x_3 - x_1x_2 = 0$

$\exists$ two lines on X, thru O,

$L_1: x_1 = x_3 = 0$

$L_2: x_2 = x_3 = 0$

π_0 contracts L_1, L_2 to 2 points

$\underbrace{[x_0, 0, x_2, 0]}_{\in L_1} \xrightarrow{\pi_0} [0,1,0] \in \mathbb{P}^2$

$\underset{\substack{\Uparrow \\ L_2}}{[x_0, x_1, 0, 0]} \xrightarrow{\pi_0} [1,0,0] \in \mathbb{P}^2$

L_1, L_2 are the only lines $\subseteq X$ thru O (easy)

$\pi_0|_U$ $\quad U = X \setminus \{L_1 \cup L_2\}$ is injective

$\pi_0^{-1}: [y_1, y_2, y_3] \longrightarrow [y_1y_2, y_1y_3, y_2y_3, y_3^2]$

$V_p(y_1y_2, \underbrace{y_1y_3, y_2y_3, y_3^2}) \qquad \begin{cases} y_3 = 0 \\ y_1y_2 = 0 \end{cases} \quad P_1 \cup P_2$

π_0^{-1} is regular on $\mathbb{P}^2 - \{P_1 \cup P_2\}$

Line P_1P_2: $y_3 = 0$

$[y_1 y_2 0] \longrightarrow [1\,0\,0\,0] = O$

The plane OP_1P_2 is the tg plane to X at O.

$\underset{\substack{\cap \\ X}}{U} = X \setminus \{L_1 \cup L_2\} \overset{\pi_0}{\longleftrightarrow} \mathbb{P}^2 \setminus \{P_1, P_2\}$ isomorphic

$X \subseteq \mathbb{P}^n$, $F \in K[x_0, \dots, x_n]_d$, $D(F) = \mathbb{P}^n \setminus V_p(F)$

var. proiettiva $d \geq 1$ open in $\mathbb{P}^n$

$$X \cap D(F) \text{ open in } X \quad \begin{cases} = X & \text{when } X \subseteq D(F) \\ \subsetneq X & \text{when } X \not\subseteq D(F) \end{cases}$$

$$X \cap D(F) = X \setminus V_p(F)$$

In particular, if $X = \mathbb{P}^n$, $D(F)$.

$X \cap D(F) = X \setminus V_p(F)$: is it affine?

We will see that $X \cap D(F)$ is affine if $\neq X$.

1) $\mathbb{P}^n \setminus V_p(F)$: if $F = x_i$, $\mathbb{P}^n \setminus V_p(x_i) \cong \mathbb{A}^n$

If F is linear, wlog, $\mathbb{P}^n \setminus V_p(F) \cong \mathbb{P}^n \setminus V_p(x_i)$ change coordinates

Ass. $\deg F = d \geq 1$

$v_{d,n} : \mathbb{P}^n \longrightarrow \mathbb{P}^N \quad N = \binom{n+d}{d} - 1$

$\mathbb{P}^n \simeq V_{d,n}$

$V_p(F) \xrightarrow{\sim} V_{d,n} \cap H$

$\mathbb{P}^n \setminus V_p(F) \simeq V_{d,n} \setminus (V_{d,n} \cap H)$

H: def. by the linear equat. associated to F

$$= \underbrace{(\mathbb{P}^n \setminus H)}_{\substack{\wr\wr \\ \mathbb{A}^n \simeq U_0}} \cap V_{d,n}$$

: it is affine.

$$X \cap D(F) = X \cap \underbrace{(\mathbb{P}^n \setminus V_p(F))}_{\text{aff. var.} \simeq \mathbb{A}^n} \overset{v_{d,n}}{\simeq} v_{d,n}(X) \cap (V_{d,n} \setminus H)$$

$$= v_{d,n}(X) \cap \underbrace{(\mathbb{P}^n \setminus H)}_{\substack{\wr\wr \\ \mathbb{A}^n}}$$

proj.

$\Longrightarrow$ closed in $\mathbb{A}^n$ up to isom.

Products of algebraic varieties.

$\sigma : \mathbb{P}^n_k \times \mathbb{P}^m_k \longrightarrow \mathbb{P}^{(n+1)(m+1)-1} = \mathbb{P}^N$

$([x_0, \dots, x_n], [y_0, \dots, y_m]) \xrightarrow{\sigma} [x_0y_0, x_0y_1, \dots, x_0y_m, x_1y_0, \dots, x_ny_m]$

σ is well defined: if $x_i \neq 0$, $y_j \neq 0 \Rightarrow x_iy_j \neq 0$

σ is injective: as in the case of $\mathbb{P}^1 \times \mathbb{P}^1$

$\sigma(\mathbb{P}^n \times \mathbb{P}^m) = \Sigma_{n,m}$ is closed in $\mathbb{P}^N$

Pf. coordinates in $\mathbb{P}^N$: w_{ij} with $i = 0, \dots, n$, $j = 0, \dots, m$

$\sigma([\underline{x}], [\underline{y}]) = [\underline{w}]$ s.t. $w_{ij} = x_iy_j$

Quadratic equations satisfied by $\Sigma_{n,m}$:

$w_{ij} = x_iy_j$ $\qquad$ $w_{ij}w_{hk} = (x_iy_j)(x_hy_k) = (x_iy_k)(x_hy_j) =$

$w_{hk} = x_hy_k$ $\qquad$ $= w_{ik}w_{hj}$

$\Sigma_{n,m} \subseteq V_P(w_{ij}w_{hk} - w_{ik}w_{hj})_{\substack{i,h=0,\dots,n \\ j,k=0,\dots,m}}$

Let $[\underline{w}] \in V_p(w_{ij} w_{hk} - w_{ik} w_{hj})$:

Ass. $w_{hk} \neq 0$

$[w_{00}, \dots, w_{nm}] = [w_{00} w_{hk}, \dots, w_{ij} w_{hk}, \dots, w_{nm} w_{hk}] =$

$w_{00} w_{hk} = w_{0k} w_{h0}$, $w_{ij} w_{hk} = w_{ik} w_{hj}$, $w_{nm} w_{hk} = w_{nk} w_{hm}$

$= \sigma([w_{0k}, \dots, w_{nk}] [w_{h0}, \dots, w_{hm}])$

If $w_{hk} \neq 0$, $w_{rs} \neq 0$: since σ is injective we get two pairs whose image in σ is $[w]$ but they must coincide.

$$\mathbb{P}^n \times \mathbb{P}^m \xleftrightarrow{\sigma} \sigma(\mathbb{P}^n \times \mathbb{P}^m) = \Sigma_{n,m}$$ Segre variety on biprojective space

σ Segre embedding

$\mathbb{P}^n \times \mathbb{P}^m$ is identified with $\Sigma_{n,m}$.

$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i \qquad U_i \simeq \mathbb{A}^n \quad , \qquad \mathbb{P}^m = \bigcup_{j=1}^{m} V_j \qquad V_j \simeq \mathbb{A}^m$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad V_j = \mathbb{P}^m - H_j$

$\mathbb{P}^N = \bigcup_{\substack{i=0,\dots,n \\ j=0,\dots,m}} W_{ij} \quad , \quad W_{ij} \simeq \mathbb{A}^N$

$U_i \times V_j \longleftrightarrow \mathbb{A}^n \times \mathbb{A}^m \longleftrightarrow \mathbb{A}^{n+m}$

$\sigma|_{U_i \times V_j} : U_i \times V_j \longrightarrow \Sigma_{n,m} \cap W_{ij}$

$([x_0 \dots x_n][y_0 \dots y_m]) \xrightarrow{\sigma} [\dots, x_i y_j, \dots]$

$x_i \neq 0 \qquad y_j \neq 0 \qquad\qquad\qquad x_i y_j = w_{ij} \neq 0$

$\mathbb{A}^{n+m} \longrightarrow \Sigma_{n,m} \cap \mathbb{A}^N$

$\sigma|_{U_i \times V_j}$ can be expressed in non-homog. coord.

Q: is $\sigma|_{U_i \times V_j} : \mathbb{A}^{n+m} \longrightarrow \Sigma_{n,m} \cap \mathbb{A}^N$ regular?

is it an isomorphism?

<u>Thm</u> The answer is yes.

Pf To simplify the notation: $i = j = 0$

$$\sigma\big|_{U_0\times V_0} : \underset{\mathbb{A}^{n+m}}{\underset{\wr\!\!\mid}{U_0\times V_0}} \longrightarrow \Sigma_{n,m}\cap W_{00} \longleftrightarrow \Sigma_{n,m}\cap \mathbb{A}^N$$

On U_0: $\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) = (u_1, \dots, u_n)$

On V_0 $\left(\frac{y_1}{y_0}, \dots, \frac{y_m}{y_0}\right) = (v_1, \dots, v_m)$

On W_{00} $\left(\frac{w_{01}}{w_{00}}, \dots, \frac{w_{nm}}{w_{00}}\right) = (z_{01}, \dots, z_{nm})$

$$(\underline{u_1, \dots, u_n}, v_1, \dots, v_m) \xrightarrow{\sigma} (v_1, \dots, v_m, \overset{z_{10}}{u_1}, u_1v_1, \dots \dots, u_n v_m)$$

$$([1, u_1, \dots, u_n], [1, v_1, \dots, v_m]) \longrightarrow [1, v_1, \dots, v_m, u_1, u_1v_1, \dots, u_1 v_m, \dots u_n v_m]$$

The components of $\sigma\big|_{U_0\times V_0}$ are of the form: $u_1, \dots, u_n$

$v_1 \dots v_m$

$u_i v_j \dots$

$\underline{\text{regular}}$

$$\Sigma_{n,m}\cap W_{00} \xrightarrow{\sigma^{-1}} \mathbb{A}^{n+m} \qquad \boxed{\text{regular}}$$

$$(z_{01}, \dots z_{ij} \dots z_{nm}) \longrightarrow (z_{10}, \dots, z_{n0}, z_{01}, \dots, z_{0m})$$

$$\mathbb{P}^n \times \mathbb{P}^m \xleftrightarrow{\sigma} \Sigma_{n,m}$$

$$U_i \times V_j \simeq \Sigma_{n,m} \cap W_{ij} \quad \text{: isomorphism of varieties}$$

<u>Corollary</u>

1) $\Sigma_{n,m}$ is irreducible

2) birational to $\mathbb{P}^{n+m}$, in particular it is rational.

<u>Pf.</u> if $T = \bigcup_{i \in I} A_i$, T top. sp., open covering,

A_i is irreducible $\forall i$

$A_i \cap A_j \neq \emptyset \implies T$ is irred.

$$\Sigma_{n,m} = \bigcup_{i,j} (\Sigma_{n,m} \cap W_{ij}), \quad \Sigma_{n,m} \cap W_{ij} \simeq \mathbb{A}^{n+m} \Rightarrow \text{irred.}$$

$$\Sigma_{n,m} \cap \underline{W_{ij}} \cap \underline{W_{hk}} = \sigma(U_i \times V_j) \cap \sigma(U_h \times V_k)$$

$$= \sigma((U_i \times V_j) \cap (U_h \times V_k)) =$$

$$= \sigma(\underbrace{(U_i \cap U_h)}_{\neq \emptyset} \times \underbrace{(V_j \cap V_k)}_{\neq \emptyset}) \neq \emptyset$$

$\implies \Sigma_{n,m}$ is irreducible.

$\Sigma_{n,m}$ irred. proj. variety

$\cup$

$\Sigma_{n,m} \cap W_{ij} \simeq \mathbb{A}^{n+m}$

$\Longrightarrow \Sigma_{n,m}$ is birational to $\mathbb{A}^{n+m}$: they contain isom. open subsets

$\Sigma_{n,m}$ is birational to $\mathbb{P}^{n+m}$: ——— - - -

<u>Cor.</u> dim $\Sigma_{n,m} = n+m$

$\mathbb{P}^1 \times \mathbb{P}^1 \xleftrightarrow{\sigma} X : x_0x_3 - x_1x_2 = 0$ in $\mathbb{P}^3$

$w_{00}w_{11} - w_{01}w_{10} = 0$

$\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ birational

<u>Rmk</u> $\mathbb{P}^1 \times \mathbb{P}^1 \not\simeq \mathbb{P}^2$: this follows from Bézout theorem over K alg. closed

In $\mathbb{P}^2$ 2 projective curves always have intersection $\neq \emptyset$

In $\mathbb{P}^1 \times \mathbb{P}^1$ there skew lines..