$$\sigma : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \Sigma_{n,m} \subseteq \mathbb{P}^N \qquad N = (n+1)(m+1) - 1$$

$$\Sigma_{n,m} = \bigcup \big(\underbrace{\Sigma_{n,m} \cap W_{ij}}_{\substack{\cong \\ \mathbb{A}^{n+m}}} \big) \quad , \quad \Sigma \text{ irreducible, rational}$$

$X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$ quasi-projective varieties

$X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$ $\qquad \sigma : X \times Y \longrightarrow \Sigma$

$\sigma(X \times Y) \longleftrightarrow X \times Y$

<u>Claim</u> X, Y projective $\Longrightarrow$ $X \times Y$ $(= \sigma(X \times Y))$ is also projective

X, Y quasi-proj. $\Longrightarrow$ $X \times Y$ is quasi-proj.

~~Pf~~ - X, Y proj. : we want to prove that $\sigma(X \times Y)$ is closed in $\mathbb{P}^N$.

Fact : T top. sp., $T = \bigcup U_i$ open cover, $Z \subseteq T$

Z closed $\Longleftrightarrow$ $\forall i$ $(Z \cap U_i)$ is closed in U_i.

$$\Sigma = \bigcup_{i,j} (\Sigma \cap W_{ij}) = \bigcup_{i,j} \left(\sigma(U_i \times V_j)\right)$$

X closed in $\mathbb{P}^n$, $X \cap U_i$ closed in $U_i \simeq \mathbb{A}^n$

Y — — $\mathbb{P}^m$, $Y \cap V_j$ closed in $V_j \simeq \mathbb{A}^m$ $\Longrightarrow$

$(X \cap U_i) \times (Y \cap V_j)$ is closed in $U_i \times V_j$

$\quad \parallel$

$(X \times Y) \cap (U_i \cap V_j)$

$\sigma\left((X \times Y) \cap (U_i \cap V_j)\right)$ is closed in $\sigma(U_i \times V_j) = \Sigma \cap W_{ij}$

$$\sigma(X \times Y) = \bigcup \underbrace{\left(\sigma(X \times Y) \cap (U_i \times V_j)\right)}_{\text{closed in } U_i \times V_j} \quad \text{open covering}$$

$\Longrightarrow \sigma(X \times Y)$ is closed in Σ closed in $\mathbb{P}^N$

If $X = X_1 \setminus X_2$, $Y = Y_1 \setminus Y_2$, X_1, X_2, Y_1, Y_2 closed

$$X \times Y = \underbrace{X_1 \times Y_1}_{\text{closed}} \setminus \left[\underbrace{(X_1 \times Y_2) \cup (X_2 \times Y_1)}_{\text{closed}}\right] : \text{locally closed}$$

the images in σ

If X, Y are irreducible $\Longrightarrow \sigma(X \times Y)$ is irreducible

X, Y quasi-projective or locally closed in $\mathbb{P}^n$, $\mathbb{P}^m$

$$\begin{tikzcd} & \sigma(X \times Y) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & X \times Y \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \quad \begin{tikzcd} & (x, y) \arrow[dl] \arrow[dr] & \\ x & & y \end{tikzcd}$$

__Exercise__ p_1, p_2 are regular maps. : work locally.

__Universal property for the product:__

if Z is quasi-proj. with 2 regular maps, reg. $\Longrightarrow$

$\exists!\ X \times Y \longrightarrow Z$ regular st. the diagrams are commutative.

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[r, phantom, "\circlearrowleft"] & {} & Y \arrow[l, phantom, "\circlearrowleft"] \\ & Z \arrow[ul, "f_1"] \arrow[ur, "f_2"'] \arrow[uu, "f"] & \end{tikzcd}$$

$$f_1 = p_1 \circ f, \quad f_2 = p_2 \circ f.$$

This is a uniqueness property for Segre embedding.

$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ Σ quadric of rk 4

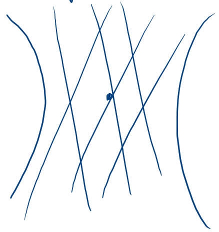
$w_{00}w_{11} - w_{01}w_{10} = 0$ $[x_0, x_1, x_2, x_3]$ $x_0x_3 - x_1x_2 = 0$

Σ contains two families of lines parametrized by $\mathbb{P}^1$

1) $\{\sigma(\{P\} \times \mathbb{P}^1)\}_{P \in \mathbb{P}^1}$

2) $\{\sigma(\mathbb{P}^1 \times \{Q\})\}_{Q \in \mathbb{P}^1}$

2 lines of the same family are disjoint, 2 lines of different families meet at a point $\sigma((P,Q))$



There are no other lines in Σ:

1) if thru $A \in \Sigma$ there are 3 lines in Σ, and they are coplanar $\Longrightarrow$ $H \cap \Sigma$: conic in Σ $(H \not\subset \Sigma)$,

a conic cannot contain 3 lines.

2) $\begin{cases} x_0 = \lambda a_0 + \mu b_0 \\ \vdots \\ x_3 = \lambda a_3 + \mu b_3 \end{cases}$ line contained in Σ

$$(\lambda a_0 + \mu b_0)(\lambda a_3 + \mu b_3) - (\lambda a_1 + \mu b_1)(\lambda a_2 + \mu b_2) = 0$$

must be identical in λ, μ $\Longrightarrow$

coeff of $\lambda^2, \lambda\mu, \mu^2$:

λ^2: $[a_0 \dots a_3] \in \Sigma$

μ^2: $[b_0 \dots b_3] \in \Sigma$

$\lambda\mu$: the point B satisfies a linear equat. whose coeff depend on $a_0, \dots, a_3$

fixed $[a_0 \dots a_3]$, B can only vary in a plane.

$$x_0x_3 - x_1x_2 = \begin{vmatrix} x_0 & x_1 \\ x_2 & x_3 \end{vmatrix} = 0$$

If $A[a_0 - a_3] \in \Sigma$ $\quad [a_0, a_1] = [a_2, a_3]$

$$[a_0, a_2] = [a_1, a_3]$$

$\Rightarrow \exists$ constants s.t. $\lambda_1(a_0, a_1) = \lambda_2(a_2, a_3)$

$$\mu_1(a_0, a_2) = \mu_2(a_1, a_3)$$

$\forall$ choice of $[\lambda_1, \lambda_2] \in \mathbb{P}^1$ $\quad \lambda_1(x_0, x_1) = \lambda_2(x_2, x_3)$ represents points on the quadric : a line

$\forall$ $[\mu_1, \mu_2]$ $\quad \mu_1(x_0, x_2) = \mu_2(x_1, x_3)$: line in Σ.

Tensors V, W vector spaces over K of finite dimension $\longrightarrow V \otimes W$

$$V \otimes W = \frac{K(V \times W)}{U}$$

U subvector space

~~generated by all pairs of the form~~

$K(V \times W) = \{$ formal linear combinations with coeff. in K of elem. in $V \times W \}$

$$(v, w) + (v', w) - (v + v', w)$$

$$(v, w) + (v, w') - (v, w + w')$$

$$\lambda(v, w) - (\lambda v, w), \quad \lambda(v, w) - (v, \lambda w)$$

$$[(v, w)] = v \otimes w$$

$$v \otimes w + v' \otimes w = (v + v') \otimes w$$

$\vdots$

$V \otimes W$ is a K-vector space generated by all elements $v \otimes w$

Any elem. in $V \otimes W$ is a linear combin.

$$\sum_{\text{fin}} \lambda_i (v_i \otimes w_i) = \sum (\underbrace{\lambda v_i}_{\in V}) \otimes w_i = \sum v_i \otimes (\lambda w_i)$$

$\Longrightarrow$ a gen. elem. has an expr. $\sum_{\text{fin}} v_i \otimes w_i$.

$V \otimes W$ tensor product of V, W

$v \otimes w$ decomposable tensors

An elem. of $V \otimes W$ is a <u>2-tensor</u>:

$$\omega = v_1 \otimes w_1 + v_1 \otimes w_2 + \dots + w_r \otimes w_r$$

the minimal r s.t. ω has an expr. as sum of r decomposable tensors is the rank of ω.

$\mathcal{B} = (e_1, \dots, e_n)$
$\mathcal{B}' = (e_1', \dots, e_m')$ bases $\Longrightarrow$

A basis for $V \otimes W$ is formed by

$e_1 \otimes e_1', \dots, e_i \otimes e_j', \dots, e_n \otimes e_m'$: nm elem.

$\Longrightarrow \dim V \otimes W = \dim V \dim W$

$v = x_1 e_1 + \dots + x_n e_n$

$w = y_1 e_1' + \dots + y_m e_m'$ $\Longrightarrow v \otimes w$

$$v \otimes w = \sum_{i,j} x_i y_j \, e_i \otimes e_j'$$

$\otimes : V \times W \longrightarrow V \otimes W$ is a bilinear map

$$\begin{tikzcd}
(v,w) \arrow[r, mapsto] \arrow[d, "f"'] & v \otimes w \arrow[dl, "\bar{f}"] \\
U &
\end{tikzcd}$$

universal property: $\forall\, U$ vector space with f bilinear

$\Longrightarrow \exists!\, \bar{f}$ linear s.t. $f = \bar{f} \circ \otimes$

From $\otimes$ we can projectivize

$\mathbb{P}(V) \times \mathbb{P}(W) \longrightarrow \mathbb{P}(V \otimes W)$

$([v],[w]) \longrightarrow [v \otimes w]$

has coord. the products

$v = x_1 e_1 + \dots + x_n e_n$

$w = y_1 e_1' \dots e_m'$

"$\otimes$ is the Segre map"

We have the map even without fixing bases!

"coordinate free description of Segre"

$\dim V \otimes W = nm = \dim M(m \times n, K)$

Fixed bases $\longrightarrow$ isomorphism

$$e_i \otimes e'_j \longleftrightarrow \begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & & & \\ \cdot & \cdots & & \cdots & 0 \end{pmatrix} \rightarrow ij$$

$$\omega = \sum w_{ij} \, e_i \otimes e'_j \longleftrightarrow \begin{pmatrix} - & & \\ & w_{ij} & \\ - & & \end{pmatrix}$$

Equations of Segre variety: $w_{ij} w_{hk} - w_{ik} w_{hj} = 0$: the 2×2 minor of the matrix

$$\begin{pmatrix} w_{11} & - & - & - & w_{1n} \\ \vdots & & & & \\ & w_{ij} & w_{ik} & & \\ & w_{hj} & w_{hk} & & \end{pmatrix}$$

Image of σ: matrices of rk 1

$\Sigma \subseteq \mathbb{P}^N = \mathbb{P}(V \otimes W)$

$\downarrow$ matrices of rk 1

Rank of a tensor ω = rank of the corresponding matrix. a matrix of rk r is sum of matrices of rk 1

$\boxed{\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}\cdots\times\mathbb{P}^{n_k}}$ r projective spaces

$$\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}\times\mathbb{P}^{n_3} \overset{def}{=} \underline{(\mathbb{P}^{n_1}\times\mathbb{P}^{n_2})\times\mathbb{P}^{n_3}}$$

Segre images

Fact: $(\mathbb{P}^{n_1}\times\mathbb{P}^{n_2})\times\mathbb{P}^{n_3} \simeq \mathbb{P}^{n_1}\times(\mathbb{P}^{n_2}\times\mathbb{P}^{n_3})$

$$V_1\otimes V_2\otimes\cdots\otimes V_k$$

$$\mathbb{P}^{n_1}\times\mathbb{P}^{n_2}\cdots\times\mathbb{P}^{n_k}\subseteq\mathbb{P}(\underbrace{V_1\otimes V_2\otimes\cdots\otimes V_k}_{k\text{-tensors}})$$

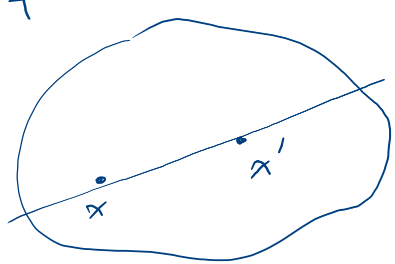
$$\sum_i v_1^i\otimes v_2^i\cdots\otimes v_k^i$$

$$\dim V_1\otimes\cdots\otimes V_k = \dim V_1 \dim V_2 \cdots \dim V_k$$

→ multidimensional matrices

rank of a k-tensor:

X



Variety of secant lines of X: closure of the set $\bigcup_{x\neq x',\, x,x'\in X} \overline{xx'}$: S_2X

$S_2(\Sigma_{n,m}) \longleftrightarrow$ 2-tensors of rank 2

$V \quad \forall d \geq 1 \quad \underbrace{V \otimes V \otimes \cdots \otimes V}_{d \text{ copies}}$

$v_1, v_2, \ldots, v_d \qquad v_1 \otimes v_2 \otimes \cdots \otimes v_d$

$\tau \in S_d \qquad \neq$

$v_{\tau(1)} \otimes v_{\tau(2)} \otimes \cdots \otimes v_{\tau(d)}$

$U \subseteq V \otimes V \otimes \cdots \otimes V$

$\| $

$\langle v_1 \otimes \cdots \otimes v_d - v_{\tau(1)} \otimes \cdots \otimes v_{\tau(d)} \mid \tau \in S_d \rangle$

$V \otimes \cdots \otimes V / U = S^d V$ d-th symmetric of V

$[\sum v_1 \otimes \cdots \otimes v_d] = \sum [v_1 \otimes \cdots \otimes v_d]$

$[v_1 \otimes \cdots \otimes v_d] = v_1 v_2 \cdots v_d$

Basis for $S^d V$: $e_{i_1} e_{i_2} \cdots e_{i_d}, \quad i_1 \leq i_2 \leq \cdots \leq i_d$

$\dim S^d V = \#\{$monomial of degree d in n elements$\}$

$= \binom{n+d-1}{d} = \dim K[x_1 \ldots x_n]_d$

$S^d V \simeq K[x_1 \ldots x_n]_d$

$SV = \bigoplus_{d \geq 0} S^d V = K \oplus V \oplus S^2 V \oplus \cdots$

we can def. a product $\Rightarrow$ K-algebra

the symmetric algebra of V

$SV \simeq K[x_1 \ldots x_n]$

$e_1 \ldots e_n \longleftrightarrow x_1 \ldots x_n$

$$V \longrightarrow S^d V \qquad \text{linear map}$$

$$v \longmapsto v^d = [v \otimes \cdots \otimes v]$$

$v = x_1 e_1 + \dots + x_n e_n$

$$v^d = (x_1 e_1 + \dots + x_n e_n)^d =$$

$$= \begin{pmatrix} \vdots \end{pmatrix} x_1^{i_1} \cdots x_n^{i_n} \, e_1^{i_1} \cdots e_n^{i_n}$$

coeff.

$\mathbb{P}(V) \longrightarrow \mathbb{P}(S^d V)$ is the Veronese map up to a projectivity in $\mathbb{P}(S^d V)$

Coordinate free description of Veronese map

$d=2 \implies$ interpretation in terms of symmetric square matrices

$d \geq 2$ multidimensional symmetric tensors

Identifiability

Antisymmetric tensors $\longleftrightarrow$ GRASSMANNIANS

$V, \; n, \; r \leq n \quad \{ W \subseteq V \mid \dim W = r \}$