$X, Y \subseteq \mathbb{P}^n_k$ quasi-projective varieties (irreducible)

$X \cap Y$ is a locally subset of $\mathbb{P}^n$, can be $\emptyset$

Assume $X \cap Y \neq \emptyset$: what about $\dim(X \cap Y)$?

If $X, Y \subset \mathbb{P}^n$ are linear subvarieties $\Rightarrow$ $X \cap Y$ is again linear $\dim(X \cap Y) = \dim X + \dim Y - \boxed{\dim \langle X \cup Y \rangle}$

In partic.: $X \cap Y \neq \emptyset$ if $\dim X + \dim Y - \dim \langle X \cup Y \rangle \geq 0$

If $X, Y \subset \mathbb{A}^n$ are affine lin. subvar. $\Rightarrow$ $X \cap Y$ is linear, but $X \cap Y$ can be $\emptyset$ even if is ≥ 0, if $X \cap Y \neq \emptyset$ then the dim is given by the formula alone.

If we don't know $\dim \langle X \cup Y \rangle$, $\dim \langle X \cup Y \rangle \leq n$

$$\boxed{\dim X \cap Y \geq \dim X + \dim Y - n.}$$

Theorem (thm. of the intersection). Ass. K is algebraically closed, X, Y quasi-proj. subvarieties of $\mathbb{P}^n_K$, $\dim X = r$, $\dim Y = s$. Ass. $X \cap Y \neq \emptyset$:

if Z is any irreducible component of $X \cap Y$ then

$\dim Z \geq r + s - n$.

. For proj. varieties, if $r+s-n \geq 0$ then $X \cap Y \neq \emptyset$.

$r+s-n$: expected dimension for $X \cap Y$

Pf: It is based on the

Knull principal ideal Theorem (Hauptideal theorem)

Ass. R is a commutative ring noetherian

$a \in R$ a non invertible element so (a) is proper

let P be a prime ideal minimal over (a) : P is minimal in the set of prime ideals containing (a)

Then $\operatorname{ht} P \leq 1$: $\operatorname{ht} P = \sup\{$lengths of finite chains of prime ideals contained in $P\}$.

Moreover if a is not a zero divisor, then $\operatorname{ht} P = 1$.

Rmk 1: we can assume X, Y are proj. irred. varieties taking the closure.

X, Y are open in some proj. var., taking closures dim remains the same. $\overline{X}, \overline{Y}$, ~~$\overline{X \cap Y} = \overline{X} \cap \overline{Y}$~~

Z irred comp. of $X \cap Y \longrightarrow \overline{Z}$ irred. comp. of $\overline{X} \cap \overline{Y}$.

Rmk 2 X, Y proj. var., $X \cap Y \neq \emptyset$, Z irred comp.

$\mathbb{P}^n = U_0 \cup \dots \cup U_n$, $Z \cap U_i \neq \emptyset \Longrightarrow$ we can work with $X \cap U_i$, $Y \cap U_i$, $(X \cap Y) \cap U_i$: the dim remain the same: $X \cap U_i$, $Y \cap U_i$ are affine varieties, and the irred components of $(X \cap U_i) \cap (Y \cap U_i)$ correspond to some of the irred. comp. of $X \cap Y$

$\overline{X} \cap \overline{Y} = W_1 \cup \dots \cup W_m$

$X \cap Y = \underbrace{V_1}_{} \cup \dots \cup \underbrace{V_p}_{}$ $\quad \overline{V_i} \subset \overline{X} \cap \overline{Y}$

$\overline{V_i} \subseteq \underbrace{W_{j_i}}_{}$ irred. and max in $\overline{X} \cap \overline{Y}$

$\underbrace{W_{j_i} \cap (X \cap Y)}_{\text{open in } W_{j_i} \text{: irred.}} \supseteq \underline{\underline{V_i}}$ irred. max in $X \cap Y$

closed in W_{j_i}

$\Longrightarrow V_i = \underbrace{W_{j_i} \cap (X \cap Y)}_{\dim = W_{j_i}}$

We can reduce to the case $X, Y \subseteq \mathbb{A}^n$ irred. affine varieties.

Case 1 X is a hypersurface : $\dim X = n-1$

$\underset{n-1}{X}$, $\underset{s}{Y}$: claim $\forall Z$ ($\dim Z \geq (n-1)+s-n = s-1$)

a) $Y \subseteq X \Rightarrow X \cap Y = Y$: $\dim Y = s \geq s-1$

b) $Y \not\subseteq X$ $I(X) = (F)$ F irred. polyn.

$I(X \cap Y) = \sqrt{I(X) + I(Y)} = \sqrt{I(Y) + (F)} \subseteq K[x_1 \dots x_n]$

$\mathcal{O}(Y) \doteq K[Y] = \dfrac{K[x_1 \dots x_n]}{I(Y)}$

Z irred. comp. of $X \cap Y$: Z is a maximal irred. closed subset of $X \cap Y \longleftrightarrow I(Z)$ is a minimal prime ideal containing $I(X \cap Y)$

consider $I_Y(X \cap Y) \trianglelefteq \mathcal{O}(Y)$ $Z \subseteq X \cap Y \subseteq Y$

$I_Y(X \cap Y) = \dfrac{I(X \cap Y)}{I(Y)}$ $I(Z) \supseteq I(X \cap Y) \supseteq I(Y)$

$I_Y(Z)$ is a min. prime ideal of $\mathcal{O}(Y)$ containing $I_Y(X \cap Y) = \dfrac{\sqrt{I(Y) + (F)}}{I(Y)} = \sqrt{(f)}$ where $f = \overline{F}$ function on Y def. by the pol. F

$f \neq 0$ because $X \not\supseteq Y$

A prime ideal contains $\sqrt{(f)} \Longleftrightarrow$ it contains (f).

$\sqrt{(f)}$ is the intersection of the prime ideals containing (f).

$I_Y(Z)$ is a min prime ideal containing (f).

$\Rightarrow \operatorname{ht}_Y I(Z) \leq 1$.

(f) f it is not invertible in $\mathcal{O}(Y)$: otherwise $(f) = \mathcal{O}(Y)$ $\quad V(f) = \emptyset$ but $f = 0$ defines $X \cap Y$

$V(F) = X \quad V(f) = X \cap Y \neq \emptyset$

$\Rightarrow \operatorname{ht} I_Y(Z) = 1$

f is not zero-divisor because $f \neq 0$ and $\mathcal{O}(Y)$ is integral domain.

$\mathcal{O}(Y) \supseteq I_Y(Z) \qquad \mathcal{O}(Z) = \dfrac{\mathcal{O}(Y)}{I_Y(Z)}$

$$\mathcal{O}(Z) = \frac{K[x_1 \dots x_n]}{I(Z)} \simeq \frac{\dfrac{K[x_1 \dots x_n]}{I(Y)}}{\dfrac{I_Y(Z)}{I(Y)}}$$

$\dim Z = \dim \mathcal{O}(Z) = \dim \mathcal{O}(Y) - \underline{\operatorname{ht} I_Y(Z)} =$

$= \dim Y - 1.$

$\dim X = n-1$, Y $\dim s$ $\Rightarrow Z \begin{cases} s & \text{if } Y \subseteq X \\ s-1 & \text{if } Y \not\subseteq X \end{cases}$

Case 2 $\dim X = r$ Ass. $I(X) = (F_1, \dots, F_{n-r})$

This implies that $X = V(F_1) \cap \dots \cap V(F_{n-r})$:
$n-r$ is the minimal possible number of generators for $I(X)$:

$\dim X = n-1 \implies n-r = 1$

$\dim X = n-2$, the ideal cannot be generated by only 1 pol, otherwise $\dim X = n-1$: we need at least two

$\dim X = n-3$, if $I(X) = (F_1, F_2)$: $V(F_1)$ has dim $n-1$
$V(F_1) \cap V(F_2)$ the dim can decrease only by 1 because of case 1)

etc.

$X = V(F_1) \cap \dots \cap V(F_{n-r})$: $\boxed{|V(F_1) \cap V(F_2)|} \geq 2$ irred. comp.

$V(F_1)$ all its irred. comp. are irred. hypersurfaces :

$V(F_1) = V(G_1) \cup \dots \cup V(G_a)$

$\forall i$ $\underbrace{V(G_i)}_{\text{irred hyp.}} \cap V(F_2)$ $\begin{cases} n-1 & \text{if } V(G_i) \supseteq V(F_2) \text{: only if } F_2, G_i \text{ differ by an invertible elem.} \\ n-2 & \text{otherwise} \end{cases}$

✓ is the union of all irred components of $V(G_i) \cap V(F_2)$: the irred comp. of $V(F_1) \cap V(F_2)$ are among them : each has dim $\geq n-2$

$\forall$ irred. comp. of $V(F_1) \cap V(F_2)$: W, consider
$W \cap \underbrace{V(F_3)} = W \cap (\underbrace{V(H_1) \cup \dots V(H_t)}_{\text{irred hyp.}}) \implies$ apply case 1) for each $W \cap V(H_i)$: dim $\begin{cases} n \\ n-1 \\ n-2 \\ n-3 \end{cases}$

By induction we get that $\dim Z \geq s - (n-r) = s + r - n$

Case 3 $\quad X \cap Y \cong (X \times Y) \cap \underbrace{\Delta_{\mathbb{A}^{2n}}}$

$\Delta_{\mathbb{A}^{2n}} = \{(x_1 \dots x_n\ x_1 \dots x_n) \mid (x_1 \dots x_n) \in \mathbb{A}^n\}$ diagonal: affine linear subspace of $\mathbb{A}^{2n}$ of dim n

$\updownarrow$ isomorph.

$\mathbb{A}^n$ $\qquad$ dim $\Delta_{\mathbb{A}^{2n}} = n$, $(x_1 \dots x_n\ x_{n+1}, \dots x_{2n})$ coordinates in $\mathbb{A}^{2n}$

n equations $\begin{cases} x_1 - x_{n+1} = 0 \\ \vdots \\ x_{n+1} - x_{2n} = 0 \end{cases}$ equations for $\Delta_{\mathbb{A}^{2n}}$

dim $X \times Y$ = dim X + dim Y, irreducible (exercise)

In $\mathbb{A}^{2n}$: $\quad X \times Y \quad$ dim $r+s$ $\qquad (X \times Y) \cap \Delta_{\mathbb{A}^{2n}} \simeq X \cap Y$

$\qquad \Delta_{\mathbb{A}^{2n}} \quad$ dim n

In the isom. irred. components $\longleftrightarrow$ irred. comp.

For $(X \times Y) \cap \Delta_{\mathbb{A}^{2n}}$ we are in the situat. of case 2)

$\forall Z \quad$ dim $Z \geq$ dim$(X \times Y)$ + dim $\Delta - 2n$ =

$= r+s+n-2n = r+s-n$.

$X, Y \qquad X \cap Y$

$r \quad s$

$X \subseteq \mathbb{P}^3$ skew cubic $\quad I_h(X) = \langle F_1, F_2, F_3 \rangle$

F_1, F_2, F_3 homog of deg 2 $\in K[x_0 \dots x_3]_2$ vector space

F_1, F_2, F_3 are linearly indep.

the syzygies are all generated by syzygies of deg 1

$\Rightarrow$ no relations of deg 0.

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

$X \not\subseteq H$, H a plane: otherwise $H: a_0x_0 + \dots + a_3x_3 = 0$

$H \supseteq X \Rightarrow \forall\, \underline{[\lambda,\mu] \in \mathbb{P}^1}$ $\; a_0\lambda^3 + a_1\lambda^2\mu + a_2\lambda\mu^2 + a_3\mu^3 = 0$

$\Rightarrow$ it is the zero polyn. in $\lambda, \mu \Rightarrow a_0 = \dots = a_3 = 0$.

$\Rightarrow I_h(X)$ cannot be generated by only 2 polynomials

$V_p(F_1) \cap V_p(F_2)$ all irred. comp. have dim 1.

one is X, the other is a line

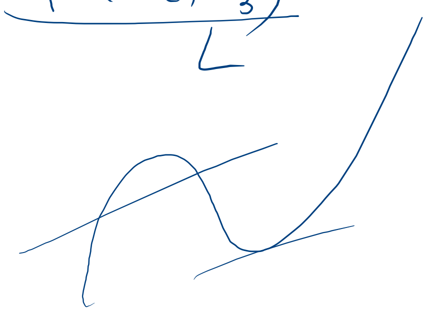
$V(x_1x_3 - x_2^2, x_0x_3 - x_1x_2) \supseteq \underbrace{V_p(x_2, x_3)}_{L}$

$V_p(F_1) \cap V_p(F_2) = X \cup L$

$V_p(F_1) \cap V_p(F_3) = X \cup L'$

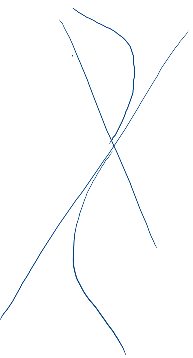
— — . L''

L, L', L'' are skew 2 by 2



But $X = V(F_1) \cap V(G)$ intersection of two hypersurfaces

$$G = \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_0 \end{vmatrix} \qquad F_1 = \begin{vmatrix} x_0 & x_1 \\ x_1 & x_2 \end{vmatrix}$$



$V(F_1)$ and $V(G)$ are tangent along X

$G = x_0 F_1 - x_3 F_2 + x_2 F_3 \in I_h(X)$

$V(G) \supseteq X$ $\qquad X \subseteq V(G) \cap V(F_1)$

$V(F_1) \supseteq X$

<u>Def</u>. $X \subseteq \mathbb{P}^n$ proj. variety of dim r
($X \subseteq \mathbb{A}^n$ aff. var. of — — ; —)

is a COMPLETE INTERSECTION if

$I_h(X)$ is generated by $n-r$ homog. pol.
$I(X)$ is generated by $n-r$ polynom.

In partic. X is intersection of $n-r$ hypersurfaces " with multiplicity 1 "

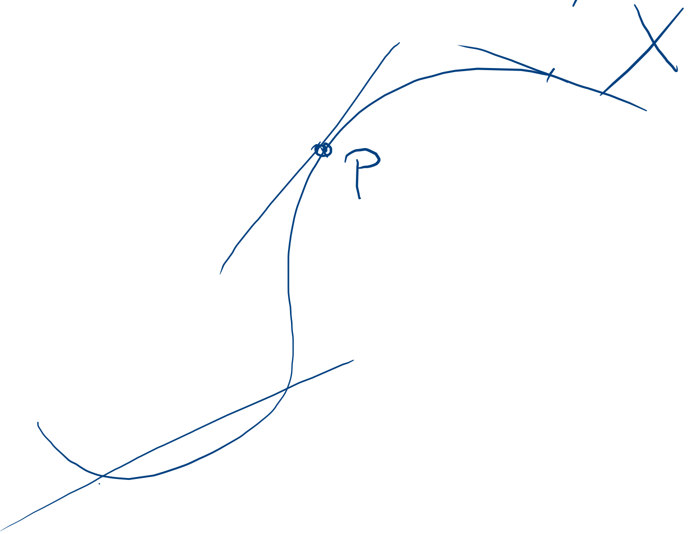
X skew cubic : is set-theoretically intersection of 2 hypersurfaces

<u>Def</u>. X of dim r is set-theoretically complete intersection if it the intersection of <u>$n-r$ hypersurfaces</u> (K alg closed)

$I_h(X) = \sqrt{(F_1, \dots, F_{n-r})}$

or $I(X)$

X is <u>locally</u> complete intersection



$$J_u(X) = (F_1, F_2, F_3)$$

$P \rightsquigarrow U_p$ ~~the~~

$$V_p(F_1) \cap V_p(F_2) \cap V_p(F_3) \cap U_p = X \cap U_p$$