$$\boxed{V_p(F_1)\cap V_p(F_2) = X \cup L}$$

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

$L: x_1 = x_0 = 0$

$F_1 = x_0 x_2 - x_1^2$

$F_2 = x_1 x_2 - x_0 x_3$

$F_3 = x_1 x_3 - x_2^2$

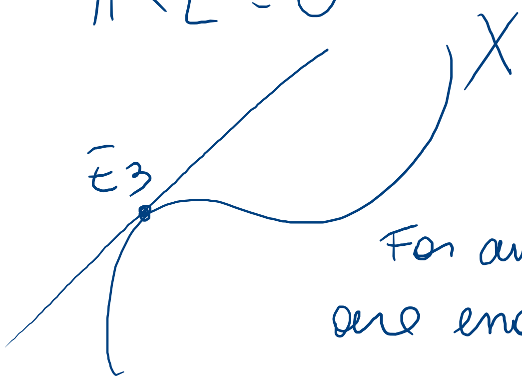
$L \cap X = \{[0,0,\underset{}{0},1]\} = \{E_3\}$

$\mathbb{P}^3 \setminus L = U$

$X \cap U = X \setminus \{E_3\}$

$\underline{V_p(F_1,F_2)} \cap \underline{U} = (X \cup L)\cap U$

$= X \cap U = X \setminus \{E_3\}$

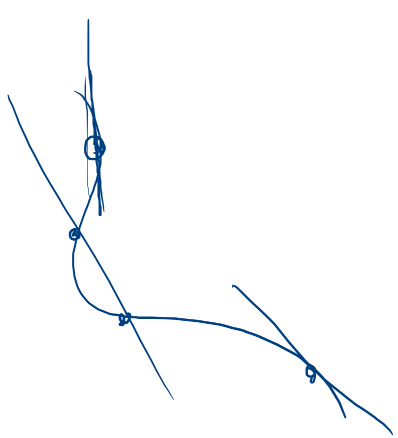


For any $P \neq E_3$, two equations are enough to cut X locally

$V_p(F_1)\cap V_p(F_3) = X \cup \underbrace{V_p(x_1,x_2)}_{= L'}$, $X \cap L' = \{\underbrace{[1,0,0,0]}_{= E_0}, \underbrace{[0,0,0,1]}_{E_3}\}$

On $U' = \mathbb{P}^3 \setminus L'$, X is def. by $F_1 = F_3 = 0$.

$V_p(F_2)\cap V_p(F_3) = X \cup \underbrace{V_p(x_2,x_3)}_{= L''}$ $X \cap L'' = [1,0,0,0]$



U, U', U'' $X \subseteq U \cup U' \cup U''$

$\Rightarrow \forall P \in X$ $\exists\, U_P$ open nbhd of P

s.t. $X \cap U_P = V_p(F_i) \cap V_p(F_j) \cap U_P$

for some i,j

Example of a locally complete intersection variety.

k alg closed

$X, Y \subseteq \mathbb{P}^n$ projective varieties, $\dim X = r$, $\dim Y = s$.

If $X \cap Y \neq \emptyset$, $\forall\, Z$ irred. comp. of $X \cap Y$

$\dim Z \geq r + s - n$.

<u>Thm.</u> If $r + s - n \geq 0 \implies X \cap Y \neq \emptyset$.

Pf $\pi : \mathbb{A}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n$

$(x_0, \dots, x_n) \longrightarrow [x_0, \dots, x_n]$

$C(X)$, $C(Y)$ $\qquad C(X) = \pi^{-1}(X) \cup \{0\}$

$C(X) \cap C(Y) \neq \emptyset$ because $0 \in C(X) \cap C(Y)$

$r + s - n \geq 0$ $\qquad \boxed{\text{Fact } \dim C(X) = \dim X + 1}$

$\dim C(X) = r + 1$, $\dim C(Y) = s + 1$

$\forall\, W$ irred. comp. of $C(X) \cap C(Y)$

$\underline{\dim W} \geq (r+1) + (s+1) - (n+1) = r + s - n + 1 \ \underline{\geq 1}$

W is not only one point

$\forall\, P \in C(X) \cap C(Y)$, $P \neq 0$

$\pi(P) \in X \cap Y \implies X \cap Y \neq \emptyset$

<u>Cor.</u> In $\mathbb{P}^2$, X, Y proj. curve $\implies$

$X \cap Y \neq \emptyset$

Similarly, $X, Y, Z \subset \mathbb{P}^3$ surfaces

$\implies X \cap Y \cap Z \neq \emptyset$

Def. $X, Y \subseteq \mathbb{P}^n$, with $\dim X = r$, $\dim Y = s$. The intersection $X \cap Y$ is proper if $\dim(X\cap Y) = r+s-n$

Z irred. of $X \cap Y \implies \dim Z \geq r+s-n$

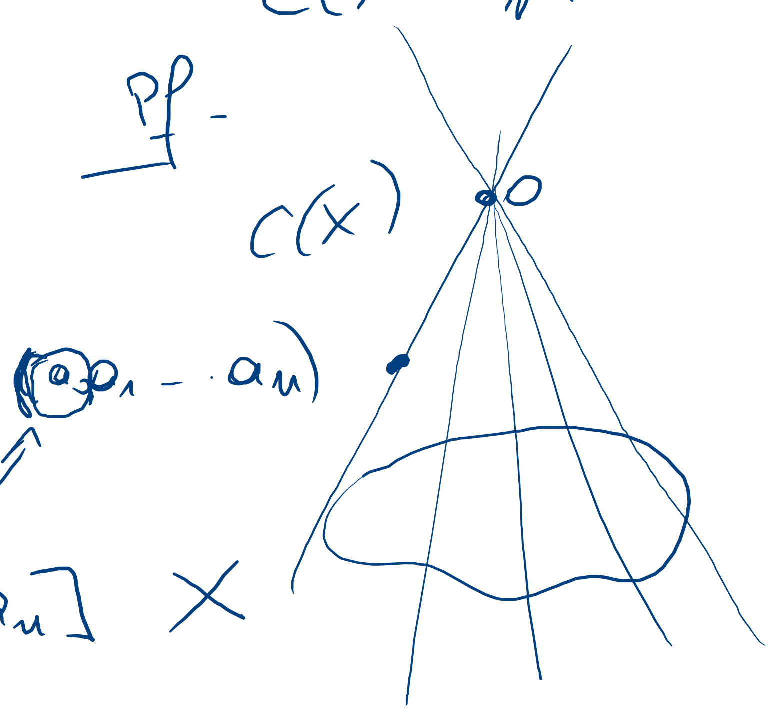
If $X\cap Y$ is proper: every Z has $\dim r+s-n$.
X, Y intersect properly

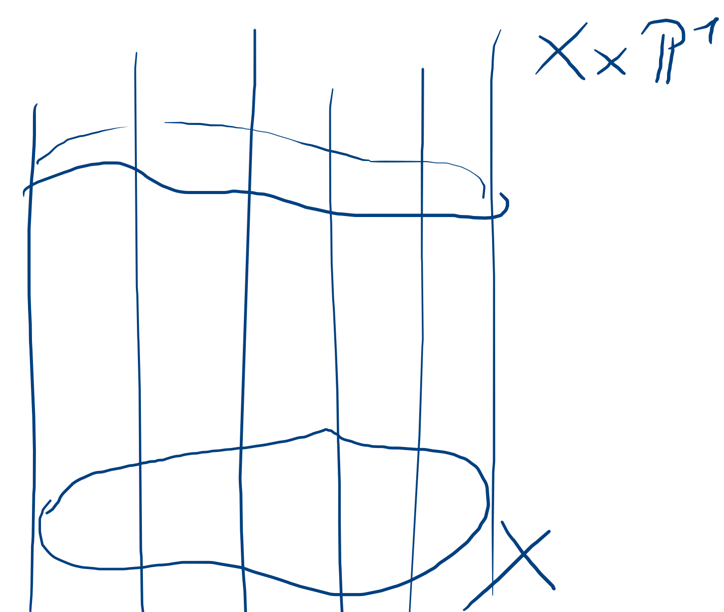
$r+s-n$ = expected dimension of $X \cap Y$

Prop. $X \subseteq \mathbb{P}^m$ proj. var.

$C(X) \subseteq \mathbb{A}^{m+1} \implies \dim C(X) = \dim X + 1$

Pf.





$[a_0, a_1, \dots, a_n] = \underbrace{\left[1, \frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right]}$

$C(X)$ and $X \times \mathbb{P}^1$ are birationally equivalent

$\implies \dim C(X) = \dim(X\times\mathbb{P}^1) = \dim X + 1$

We construct an explicit isomorph. between open subsets of $C(X)$ and $X \times \mathbb{P}^1$

$X \rightsquigarrow \underset{\neq \emptyset}{X \cap U_0}$, $U_0 : x_0 \neq 0$ (if $X \cap U_0 = \emptyset$, change coord.)

$$C(X \cap U_0) = \{(\underset{\neq 0}{a_0}, a_1, \dots, a_n) \mid [a_0, \dots, a_n] \in X\}$$

$$\underset{\shortparallel}{X_0} \times \mathbb{A}^1 \subseteq U_0 \times \mathbb{A}^1 \simeq \mathbb{A}^{n+1}$$

$$\{(\underbrace{(b_1, \dots, b_n)}_{\in X_0}, \underset{\in \mathbb{A}^1}{t})\} \longleftrightarrow \{(b_1 \dots b_n, t) \mid [1, b_1, \dots, b_n] \in X, \ t \in \mathbb{A}^1\}$$

$$C(X \cap U_0) \xrightarrow{\varphi} X_0 \times \mathbb{A}^1$$

$$(a_0, a_1, \dots a_n) \longrightarrow \left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}, a_0\right)$$

$$X_0 \times \mathbb{A}^1 \xrightarrow{\psi} C(X \cap U_0)$$

$$(b_1 \dots b_n, t) \longrightarrow [t, b_1 t, \dots, b_n t]$$

$$(a_0, \dots a_n) \longrightarrow \left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}, a_0\right) \longrightarrow (a_0, a_1, \dots, a_n)$$

$$\underset{\text{open in } X \times \mathbb{P}^1}{X_0 \times \mathbb{A}^1} \simeq \underset{\text{open in } C(X)}{C(X_0)}$$

COMPLETE VARIETIES

Def. X quasi-projective variety

$\forall$ Y locally closed in some proj. space

$X \times Y$ is locally closed in some proj. space via the Segre map

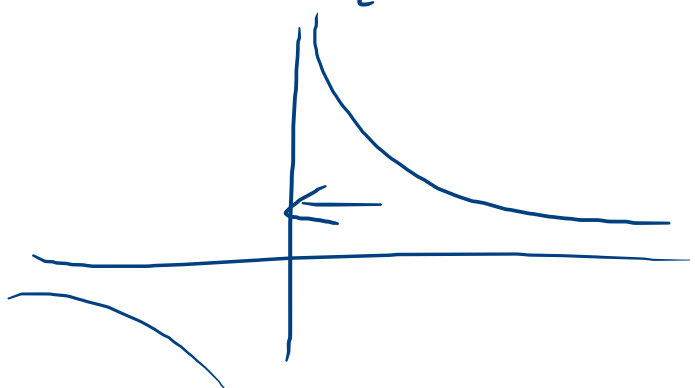
$p_2 : X \times Y \longrightarrow Y$ the projection p_2 is regular

X is complete if $\forall$ Y the proj. p_2 is a closed map: $\forall$ $W \subseteq X \times Y$ closed $\Longrightarrow$ $p_2(W)$ is closed in Y

Ex. $\mathbb{A}^1$ is not complete: for $Y = \mathbb{A}^1$

$\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1 \xrightarrow{p_2} \mathbb{A}^1$ is non closed

$Z = V(x_1 x_2 - 1)$ $\quad (x_1, x_2) \longrightarrow x_2$



$p_2(Z) = \mathbb{A}^1 - \{0\}$ not closed

Prop. 1) Ass. X is a complete variety, $f: X \longrightarrow Y$ regular $\Longrightarrow$ $f(X)$ is closed in Y and a complete variety.

2) If X is complete $\Longrightarrow$ any closed irred. subvariety of X is complete.

Pf. $X \times Y \xrightarrow{p_2} Y \supseteq f(X)$ We want to express $f(X) = p_2(Z)$, Z closed in $X \times Y$

$\Gamma_f \subseteq X \times Y$ graph of f

$\Gamma_f = \{(x,y) \mid y = f(x), x \in X\}$ $\qquad p_2(\Gamma_f) = \{y \in Y \mid p_2(x, f(x)) = y\} = f(X)$

We have to prove that Γ_f is closed in $X \times Y$

$X \times X \xrightarrow{(1_X, f)} X \times Y$:

$(x, x') \longrightarrow (x, f(x))$

$\Delta_X \subseteq X \times X$ $\qquad (1_X, f)(\Delta_X) = \Gamma_f$

diagonal

a) Δ_X is closed in $X \times X$: it is closed locally on any affine open subset; the affine open subsets form an open covering.
$\Longrightarrow \Delta_X$ is closed

b) $(1_X, f)$ is regular:
because the two comp. are regular, and then work locally in coordinates

c)

$f : X \longrightarrow Y$ $\qquad$ $X \times Y \xrightarrow{(f, 1_Y)} Y \times Y$ $\quad$ regular

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cup\!|$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \Delta_Y$ closed

$$(f, 1_Y)^{-1}(\Delta_Y) = \{ (x,y) \mid (f(x), y) \in \Delta_Y \iff y = f(x) \}$$

$\quad \|$

Γ_f closed in $X \times Y$

X complete $\Longrightarrow$ $p_2(\Gamma_f) = f(X)$ is closed

Claim $f(X)$ is complete : $\quad T$ any locally closed

$$\begin{tikzcd}
Z \subseteq f(X) \times T \arrow[r, "p_2'"] & T \\
X \times T \arrow[u, "(f, 1_T)"] \arrow[ur, "p_2" description] &
\end{tikzcd}$$

p_2 closed $\qquad$ commutative

Z closed in $f(X) \times T \Longrightarrow (f, 1_T)^{-1}(Z)$ is closed in $X \times T \Longrightarrow p_2((f, 1_T)^{-1}(Z))$ is closed

$$\| $$

$$p_2'(Z)$$

2) X complete,
$\cup\!\!\mid$
Z closed irred.

Y locally closed

$$\begin{tikzcd} C \subseteq Z\times Y \arrow[r, "p_2'"] \arrow[d, "(j,1_Y)"] & Y \\ X\times Y \arrow[ur, "p_2"'] & \end{tikzcd}$$

C closed in $Z\times Y \Longrightarrow$ C closed in $X\times Y$ because Z is closed in X

$\Longrightarrow p_2(C)$ is closed because X is complete.

<u>Cor.</u> X complete variety $\Longrightarrow \mathcal{O}(X) \simeq K$

<u>Pf.</u> $\varphi \in \mathcal{O}(X)$ $\qquad \varphi : X \longrightarrow \mathbb{A}^1 = K$ regular map

$\Longrightarrow \varphi(X)$ is closed in $\mathbb{A}^1$ and complete, $\mathbb{A}^1$ is not complete $\Longrightarrow \varphi(X) \subsetneq \mathbb{A}^1$, X irred. $\Longrightarrow$ $\varphi(X)$ is irreducible $\Longrightarrow \varphi(X)$ is one point so φ is a constant function.

Any point is a complete variety.

K alg. closed field

Cor. X affine variety, irred. Ass. X is complete

$\Longrightarrow \mathcal{O}(X) \simeq K$

$X \subseteq \mathbb{A}^n$ $\quad$ $\mathcal{O}(X) = K[X] = \dfrac{K[x_1, \dots, x_n]}{I(X)} \Longrightarrow I(X)$ is maximal

$\Longrightarrow I(X)$ is the ideal of a point $\Longrightarrow X$ is a point.

The only affine varieties which are complete are the single points.

Theorem Every irreducible projective variety is complete.

Sketch of the proof.

1) it is enough to prove that $p_2 : \mathbb{P}^n \times \mathbb{A}^m \longrightarrow \mathbb{A}^m$
is a closed map

2) characterization of closed subsets of $\mathbb{P}^n \times \mathbb{A}^m$.

$[x_0, \dots, x_n]$ homog. coordin. on $\mathbb{P}^n$

$(y_1, \dots, y_m)$ coord. on $\mathbb{A}^m$

$X \subseteq \mathbb{P}^n \times \mathbb{A}^m$ is closed $\iff$ X is the set of zeros of a family of polynomials

$F(x_0 \dots x_n; y_1 \dots y_m)$ homog. only in $x_0, \dots, x_n$

3) From a system of equations

$$(*) \left\{ F_i(x_0 \dots x_n; y_1 \dots y_m) = 0 , \quad i = 1, \dots, r \right.$$

it is possible to eliminate the homog. variables: i.e. it is possibly to find.

a system of equations only in $y_1 \dots y_m$

$$(**) \left\{ G_j(y_1 \dots y_m) = 0 \qquad j = 1, \dots, s \right.$$

such that: a point $A(a_1 \dots a_m) \in \mathbb{A}^m$ is sol. of $(**)$ iff $\exists$ $B[b_0 \dots b_m] \in \mathbb{P}^n$ s.t. $(b_0 \dots b_m, a_1 \dots a_m)$ is a sol. of $(*)$.