Chapter 10

Regular maps

10.1 Regular maps or morphisms

Let X, Y be quasi–projective varieties (or more generally locally closed sets). Let $\varphi : X \to Y$ be a map.

Definition 10.1.1. φ is a *regular map* or a *morphism* if

(i) φ is continuous for the Zariski topology;

(ii) φ preserves regular functions, i.e. for all $U \subset Y$ (U open and non–empty) and for all $f \in \mathcal{O}(U)$, then $f \circ \varphi \in \mathcal{O}(\varphi^{-1}(U))$:

$$\begin{array}{ccccc} X & \xrightarrow{\varphi} & Y & & \\ \uparrow & & \uparrow & & \\ \varphi^{-1}(U) & \xrightarrow{\varphi|} & U & \xrightarrow{f} & K \end{array}$$

Note that:
a) for all X the identity map $1_X : X \to X$ is regular;
b) for all X, Y, Z and regular maps $X \xrightarrow{\varphi} Y$, $Y \xrightarrow{\psi} Z$, the composite map $\psi \circ \varphi$ is regular.

An *isomorphism* of varieties is a regular map which possesses regular inverse, i.e. a regular map $\varphi : X \to Y$ such that there exists a regular map $\psi : Y \to X$ verifying the conditions $\psi \circ \varphi = 1_X$ and $\varphi \circ \psi = 1_Y$. In this case X and Y are said to be isomorphic, and we write: $X \simeq Y$.

If $\varphi : X \to Y$ is regular, there is a natural K–homomorphism $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, called the *comorphism associated to* φ, defined by: $f \to \varphi^*(f) := f \circ \varphi$.

The construction of the comorphism is *functorial*, which means that:

a) $1_X^* = 1_{\mathcal{O}(X)}$;
b) $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

This implies that, if $X \simeq Y$, then $\mathcal{O}(X) \simeq \mathcal{O}(Y)$. In fact, if $\varphi : X \to Y$ is an isomorphism and ψ is its inverse, then $\varphi \circ \psi = 1_Y$, so $(\varphi \circ \psi)^* = \psi^* \circ \varphi^* = (1_Y)^* = 1_{\mathcal{O}(Y)}$ and similarly $\psi \circ \varphi = 1_X$ implies $\varphi^* \circ \psi^* = 1_{\mathcal{O}(X)}$.

Example 10.1.2.

1) The homeomorphism $\varphi_i : U_i \to \mathbb{A}^n$ of Section 2.6 is an isomorphism.

2) There exist homeomorphisms which are not isomorphisms. Let $Y = V(x^3 - y^2) \subset \mathbb{A}^2$, the cuspidal cubic. We have seen (see Exercise 2, Chapter 7) that $K[X] \not\simeq K[\mathbb{A}^1]$, hence Y is not isomorphic to the affine line $\mathbb{A}^1$. Nevertheless, the map

$$\varphi : \mathbb{A}^1 \to Y \text{ such that } t \to (t^2, t^3)$$

is regular, bijective and also a homeomorphism (see Exercise 1, Lesson 7).

Its inverse $\varphi^{-1} : Y \to \mathbb{A}^1$ is defined by

$$(x, y) \to \begin{cases} \frac{y}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad (x, y) = (0, 0). \end{cases}$$

Note that φ^{-1} cannot be regular at the point $(0, 0)$.

10.2 Affine case

Next Proposition tells us how a morphism is given in practice, when the codomain is contained in an affine space.

Proposition 10.2.1. *Let $\varphi : X \to Y \subset \mathbb{A}^n$ be a map. Then φ is regular if and only if $\varphi_i := t_i \circ \varphi$ is a regular function on X, for all $i = 1, \dots, n$, where $t_1, \dots, t_n$ are the coordinate functions on Y.*

Proof. If φ is regular, then $\varphi_i = \varphi^*(t_i)$ is regular by definition.

Conversely, assume that φ_i is a regular function on X for all i. Let $Z \subset Y$ be a closed subset and we have to prove that $\varphi^{-1}(Z)$ is closed in X. Since any closed subset of $\mathbb{A}^n$ is an intersection of hypersurfaces, it is enough to consider $\varphi^{-1}(Y \cap V(F))$ with $F \in K[x_1, \dots, x_n]$:

$$\varphi^{-1}(Y \cap V(F)) = \{P \in X | F(\varphi(P)) = F(\varphi_1, \dots, \varphi_n)(P) = 0\} = V(F(\varphi_1, \dots, \varphi_n)).$$

But note that $F(\varphi_1, \ldots, \varphi_n) \in \mathcal{O}(X)$: it is the composition of F with the regular functions $\varphi_1, \ldots, \varphi_n$. Hence $\varphi^{-1}(Y \cap V(F))$ is closed, so we can conclude that φ is continuous. If $U \subset Y$ and $f \in \mathcal{O}(U)$, for any point P of U choose an open neighbourhood U_P such that $f = F_P/G_P$ on U_P. So $f \circ \varphi = F_P(\varphi_1, \ldots, \varphi_n)/G_P(\varphi_1, \ldots, \varphi_n)$ on $\varphi^{-1}(U_P)$, hence it is regular on each $\varphi^{-1}(U_P)$ and by consequence on $\varphi^{-1}(U)$. □

Remark 8. If $\varphi : X \to Y$ is a regular map and $Y \subset \mathbb{A}^n$, by Proposition 10.2.1 we can represent φ in the form $\varphi = (\varphi_1, \ldots, \varphi_n)$, where $\varphi_1, \ldots, \varphi_n \in \mathcal{O}(X)$ and $\varphi_i = \varphi^*(t_i)$. Note that $\varphi_1, \ldots, \varphi_n$ are not arbitrary in $\mathcal{O}(X)$ but such that $\operatorname{Im} \varphi \subset Y$.

Let us recall that, if Y is closed in $\mathbb{A}^n$ and K is algebraically closed, then $t_1, \ldots, t_n$ generate $\mathcal{O}(Y)$, hence $\varphi_1, \ldots, \varphi_n$ generate $\varphi^*(\mathcal{O}(Y))$ as K-algebra. This observation is the key for the following important result.

Theorem 10.2.2. *Let K be an algebraically closed field. Let X be a locally closed algebraic set and Y be an affine algebraic set. Let $Hom(X, Y)$ denote the set of regular maps from X to Y and $Hom(\mathcal{O}(Y), \mathcal{O}(X))$ denote the set of K– homomorphisms from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$.*

Then the map $Hom(X, Y) \to Hom(\mathcal{O}(Y), \mathcal{O}(X))$, such that $\varphi : X \to Y$ goes to $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, is bijective.

Proof. Let $Y \subset \mathbb{A}^n$ and let $t_1, \ldots, t_n$ be the coordinate functions on Y, so $\mathcal{O}(Y) = K[t_1, \ldots, t_n]$. Let $u : \mathcal{O}(Y) \to \mathcal{O}(X)$ be a K–homomorphism: we want to define a morphism $u^\sharp : X \to Y$ whose associated comorphism is u. By Remark 8, if $u^\sharp$ exists, its components have to be $u(t_1), \ldots, u(t_n)$. So we define

$$\begin{array}{rccc} u^\sharp : & X & \to & \mathbb{A}^n \\ & P & \to & (u(t_1)(P)), \ldots, u(t_n)(P)). \end{array}$$

This is a morphism by Proposition 10.2.1. We claim that $u^\sharp(X) \subset Y$. Let $F \in I(Y)$ and $P \in X$: then

$$\begin{aligned} F(u^\sharp(P)) = F(u(t_1)(P), \ldots, u(t_n)(P)) = \\ = F(u(t_1), \ldots, u(t_n))(P) = \\ = u(F(t_1, \ldots, t_n))(P) \text{ because } u \text{ is } K\text{-homomorphism} = \\ = u(0)(P) = \\ = 0(P) = 0. \end{aligned} \tag{10.1}$$

So $u^\sharp$ is a regular map from X to Y.

We consider now $(u^\sharp)^* : \mathcal{O}(Y) \to \mathcal{O}(X)$: it maps a function f to $f \circ u^\sharp = f(u(t_1), \ldots, u(t_n)) = u(f)$, so $(u^\sharp)^* = u$. Conversely, if $\varphi : X \to Y$ is regular, then $(\varphi^*)^\sharp$ maps P to

$$(\varphi^*(t_1)(P), \ldots, \varphi^*(t_n)(P)) = (\varphi_1(P), \ldots, \varphi_n(P)),$$

so $(\varphi^*)^\sharp = \varphi$. □

Note that, by definition, $1^\sharp_{\mathcal{O}(X)} = 1_X$, for all affine X; moreover $(v \circ u)^\sharp = u^\sharp \circ v^\sharp$ for all $u : \mathcal{O}(Z) \to \mathcal{O}(Y)$, $v : \mathcal{O}(Y) \to \mathcal{O}(X)$, K–homomorphisms of rings of regular functions of affine algebraic sets: this means that also this construction is functorial.

The construction of the comorphism associated to a regular function and the result of Theorem 10.2.2 can be rephrased using the language of categories. We will see it in Chapter 11.

If X and Y are quasi–projective varieties and $\varphi : X \to Y$ is a regular map, it is not always possible to define a comorphism $K(Y) \to K(X)$. If f is a rational function on Y with $\mathrm{dom} f = U$, it can happen that $\varphi(X) \cap \mathrm{dom} f = \emptyset$, in which case $f \circ \varphi$ does not exist. Nevertheless, if we assume that φ is **dominant**, i.e. $\overline{\varphi(X)} = Y$, then certainly $\varphi(X) \cap U \neq \emptyset$, hence $\langle \varphi^{-1}(U), f \circ \varphi \rangle \in K(X)$. We obtain a K–homomorphism, which is necessarily injective, $K(Y) \to K(X)$, also denoted by φ^*. Note that in this case, we have: $\dim X \geq \dim Y$. As above, it is possible to check that, if $X \simeq Y$, then $K(X) \simeq K(Y)$, hence $\dim X = \dim Y$. Moreover, if $P \in X$ and $Q = \varphi(P)$, then φ^* induces a map $\mathcal{O}_{Q,Y} \to \mathcal{O}_{P,X}$, such that $\varphi^* \mathcal{M}_{Q,Y} \subset \mathcal{M}_{P,X}$. Also in this case, if φ is an isomorphism, then $\mathcal{O}_{Q,Y} \simeq \mathcal{O}_{P,X}$.

10.3 Projective case

We will see now how to express in practice a regular map when the target is contained in a projective space. Let $X \subset \mathbb{P}^n$ be a quasi–projective variety and $\varphi : X \to \mathbb{P}^m$ be a map.

Proposition 10.3.1. *φ is a morphism if and only if, for any $P \in X$, there exist an open neighbourhood U_P of P and $n+1$ homogeneous polynomials $F_0, \ldots, F_m$ of the same degree in $K[x_0, x_1, \ldots, x_n]$, such that, if $Q \in U_P$, then $\varphi(Q) = [F_0(Q), \ldots, F_m(Q)]$. In particular, for any $Q \in U_P$, there exists an index i such that $F_i(Q) \neq 0$.*

Proof. "$\Rightarrow$" Let $P \in X$, $Q = \varphi(P)$ and assume that $Q \in U_0$. Then $U := \varphi^{-1}(U_0)$ is an open neighbourhood of P and we can consider the restriction $\varphi|_U : U \to U_0$, which is regular.

Possibly after restricting U, using non–homogeneous coordinates on U_0, we can assume that $\varphi|_U = (F_1/G_1, \dots, F_m/G_m)$, where (F_1, G_1), $\dots$, (F_m, G_m) are pairs of homogeneous polynomials of the same degree such that $V_P(G_i) \cap U = \emptyset$ for all index i. We can reduce the fractions F_i/G_i to a common denominator F_0, so that $\deg F_0 = \deg F_1 = \cdots = \deg F_m$ and $\varphi|_U = (F_1/F_0, \dots, F_m/F_0) = [F_0, F_1, \dots, F_m]$, with $F_0(Q) \neq 0$ for $Q \in U$.

"$\Leftarrow$" Possibly after restricting U_P, we can assume $F_i(Q) \neq 0$ for all $Q \in U_P$ and suitable i. Let $i = 0$: then $\varphi|_{U_P} : U_P \to U_0$ operates as follows:

$$\varphi|_{U_P}(Q) = (F_1(Q)/F_0(Q), \dots, F_m(Q)/F_0(Q)),$$

so it is a morphism by Proposition 10.2.1. From this remark, one deduces that also φ is a morphism. $\square$

10.4 Examples of morphisms

Example 10.4.1 (Stereographic projection)**.**

Let $X \subset \mathbb{P}^2$, $X = V_P(x_1^2 + x_2^2 - x_0^2)$, be the projective closure of the unitary circle. We define $\varphi : X \to \mathbb{P}^1$ by

$$[x_0, x_1, x_2] \to \begin{cases} [x_0 - x_2, x_1] & \text{if } (x_0 - x_2, x_1) \neq (0,0) \\ [x_1, x_0 + x_2] & \text{if } (x_1, x_0 + x_2) \neq (0,0). \end{cases}$$

φ is well–defined because, on X, $x_1^2 = (x_0 - x_2)(x_0 + x_2)$. Moreover

$$(x_1, x_0 - x_2) \neq (0,0) \Leftrightarrow [x_0, x_1, x_2] \in X \setminus \{[1,0,1]\},$$

$$(x_0 + x_2, x_1) \neq (0,0) \Leftrightarrow [x_0, x_1, x_2] \in X \setminus \{[1,0,-1]\}.$$

The map φ is the natural extension of the rational function $f : X \setminus \{[1,0,1]\} \to K$ such that $[x_0, x_1, x_2] \to x_1/(x_0 - x_2)$ (Example 9.2.6, 2). Now if we identify $\mathbb{P}^1$ with the line $V_P(x_2) \subset \mathbb{P}^2$, the North pole $N[1,0,1]$, centre of the stereographic projection, goes to the point at infinity of the line $\mathbb{P}^1$.

By geometric reasons φ is invertible and $\varphi^{-1} : \mathbb{P}^1 \to X$ takes $[\lambda, \mu]$ to $[\lambda^2 + \mu^2, 2\lambda\mu, \mu^2 - \lambda^2]$ (note the connection with the Pitagorean triples!). Indeed the line through P and $[\lambda, \mu, 0]$ has equation: $\mu x_0 - \lambda x_1 - \mu x_2 = 0$. Its intersections with X are represented by the system:

$$\begin{cases} \mu x_0 - \lambda x_1 - \mu x_2 = 0 \\ x_1^2 + x_2^2 - x_0^2 = 0 \end{cases}$$

Assuming $\mu \neq 0$ this system is equivalent to the following:

$$\begin{cases} \mu x_0 - \lambda x_1 - \mu x_2 = 0 \\ \mu^2 x_0^2 = \mu^2(x_1^2 + x_2^2) = (\lambda x_1 + \mu x_2)^2 \end{cases}$$

Therefore, either $x_1 = 0$ and $x_0 = x_2$, or

$$\begin{cases} (\mu^2 - \lambda^2)x_1 - 2\lambda\mu x_2 = 0 \\ \mu x_0 = \lambda x_1 + \mu x_2 \end{cases}$$

which gives the required expression.

Example 10.4.2. Affine transformations and affinities.

Let $A = (a_{ij})$ be a $n \times n$ matrix with entries in K, let $B = (b_1, \ldots, b_n) \in \mathbb{A}^n$ be a point. The map $\tau_A : \mathbb{A}^n \to \mathbb{A}^n$ defined by $(x_1, \ldots, x_n) \to (y_1, \ldots, y_n)$, such that

$$\{y_i = \sum_j a_{ij}x_j + b_i, i = 1, \ldots, n,$$

is a regular map called an affinity of $\mathbb{A}^n$. In matrix notation τ_A is $Y = AX + B$. If A is of rank n, then τ_A is said non–degenerate and is an isomorphism: the inverse map τ_A^{-1} is represented by $X = A^{-1}Y - A^{-1}B$. More in general, an affine transformation from $\mathbb{A}^n$ to $\mathbb{A}^m$ is a map represented in matrix form by $Y = AX + B$, where A is a $m \times n$ matrix and $B \in \mathbb{A}^m$. It is injective if and only if $\text{rk}A = n$ and surjective if and only if $\text{rk}A = m$.

The isomorphisms of an algebraic set X in itself are called **automorphisms of** X: they form a group for the usual composition of maps, denoted by $Aut\ X$. If $X = \mathbb{A}^n$, the non–degenerate affine transformations form a subgroup of $Aut\ \mathbb{A}^n$.

If $n = 1$ and the characteristic of K is 0, then $Aut\ \mathbb{A}^1$ coincides with this subgroup. In fact, let $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ be an automorphism: it is represented by a polynomial $F(x)$ such that there exists $G(x)$ satisfying the condition $G(F(t)) = t$ for all $t \in \mathbb{A}^1$, i.e. $G(F(x)) = x$ in the polynomial ring $K[x]$. Then, taking derivatives, we get $G'(F(x))F'(x) = 1$, which implies $F'(t) \neq 0$ for all $t \in K$, so $F'(x)$ is a non–zero constant. Hence, F is linear and G is linear too.

If $n \geq 2$, then $Aut\ \mathbb{A}^n$ is not completely described yet. There exist non–linear automorphisms of degree d, for all d. For example, for $n = 2$: let $\varphi : \mathbb{A}^2 \to \mathbb{A}^2$ be given by $(x, y) \to (x, y + P(x))$, where P is any polynomial of $K[x]$. Then $\varphi^{-1} : (x', y') \to (x', y' - P(x'))$. A very important and difficult open problem in Algebraic Geometry is the Jacobian conjecture, stating that, in characteristic zero, a regular map $\varphi : \mathbb{A}^n \to \mathbb{A}^n$ is an automorphism if and only if the Jacobian determinant $\mid J(\varphi) \mid$ is a non-zero constant.

Example 10.4.3. Projective transformations.

Let A be a $(n+1) \times (n+1)$–matrix with entries in K. Let $P[x_0, \dots, x_n] \in \mathbb{P}^n$: then $[a_{00}x_0 + \cdots + a_{0n}x_n, \dots, a_{n0}x_0 + \cdots + a_{nn}x_n]$ is a point of $\mathbb{P}^n$ if and only if it is different from $[0, \dots, 0]$. So A defines a regular map $\tau : \mathbb{P}^n \to \mathbb{P}^n$ if and only if $\mathrm{rk}A = n+1$. If $\mathrm{rk}A = r < n+1$, then A defines a regular map whose domain is the quasi–projective variety $\mathbb{P}^n \setminus \mathbb{P}(kerA)$. If $\mathrm{rk}A = n+1$, then τ is an isomorphism, called a projective transformation or projectivity. Note that the matrices λA, $\lambda \in K^*$, all define the same projective transformation. So $PGL(n+1, K) := GL(n+1, K)/K^*$ acts on $\mathbb{P}^n$ as the group of projective transformations.

If $X, Y \subset \mathbb{P}^n$, they are called **projectively equivalent** if there exists a projective transformation $\tau : \mathbb{P}^n \to \mathbb{P}^n$ such that $\tau(X) = Y$.

Theorem 10.4.4. *Fundamental theorem on projective transformations.*

Let two $(n+2)$–tuples of points of $\mathbb{P}^n$ in general position be fixed: $P_0, \dots, P_{n+1}$ and $Q_0, \dots, Q_{n+1}$. Then there exists one, and only one, isomorphic projective transformation τ of $\mathbb{P}^n$ in itself, such that $\tau(P_i) = Q_i$ for all index i.

Proof. Put $P_i = [v_i]$, $Q_i = [w_i]$, $i = 0, \dots, n+1$. So $\{v_0, \dots, v_n\}$ and $\{w_0, \dots, w_n\}$ are two bases of K^{n+1}, hence there exist scalars $\lambda_0, \dots, \lambda_n, \mu_0, \dots, \mu_n$ such that

$$v_{n+1} = \lambda_0 v_0 + \cdots + \lambda_n v_n, \quad w_{n+1} = \mu_0 w_0 + \cdots + \mu_n w_n,$$

where the coefficients are all different from 0, because of the general position assumption. We replace v_i with $\lambda_i v_i$ and w_i with $\mu_i w_i$ and get two new bases, so there exists a unique automorphism of K^{n+1} transforming the first basis in the second one and, by consequence, also v_{n+1} in w_{n+1}. This automorphism induces the required projective transformation on $\mathbb{P}^n$. $\square$

An immediate consequence of the above theorem is that projective subspaces of the same dimension are projectively equivalent. Also two subsets of $\mathbb{P}^n$ formed both by k points in general position are projectively equivalent if $k \leq n+2$. If $k > n+2$, this is no longer true, already in the case of four points on a projective line. The problem of describing the classes of projective equivalence of k–tuples of points of $\mathbb{P}^n$, for $k > n+2$, is one the first problems of classical Invariant Theory. The solution in the case $k = 4$, $n = 1$ is given by the notion of *cross–ratio.*

Example 10.4.5. Affine and non-affine quasi–projective varieties.

Let $X \subset \mathbb{A}^n$ be an affine variety with $I(X) = \langle G_1, \dots, G_r \rangle$, then $X_F := X \setminus V(F)$ is isomorphic to a closed subset of $\mathbb{A}^{n+1}$, i.e. to $Y = V(x_{n+1}F - 1, G_1, \dots, G_r)$. Indeed, the following regular maps are inverse each other:

- $\varphi : X_F \to Y$ such that $(x_1, \dots, x_n) \to (x_1, \dots, x_n, 1/F(x_1, \dots, x_n))$,

- $\psi : Y \to X_F$ such that $(x_1, \dots, x_n, x_{n+1}) \to (x_1, \dots, x_n)$.

Hence, X_F is a quasi–projective variety contained in $\mathbb{A}^n$, not closed in $\mathbb{A}^n$, but isomorphic to a closed subset of another affine space.

Definition 10.4.6. From now on, the term *affine variety* will denote a *locally closed subset of a projective space isomorphic to some affine closed set.*

If X is an affine variety and precisely $X \simeq Y$, with $Y \subset \mathbb{A}^n$ closed, then $\mathcal{O}(X) \simeq \mathcal{O}(Y) = K[t_1, \dots, t_n]$ is a finitely generated K–algebra. In particular, if K is algebraically closed and α is an ideal strictly contained in $\mathcal{O}(X)$, then $V(\alpha) \subset X$ is non–empty, by the relative form of the Nullstellensatz (Proposition 9.1.5). From this observation, we can deduce that the quasi–projective variety of next example is not affine.

Example 10.4.7. $\mathbb{A}^2 \setminus \{(0,0)\}$ is not affine.

Set $X = \mathbb{A}^2 \setminus \{(0,0)\}$: first of all we will prove that $\mathcal{O}(X) \simeq K[x,y] = \mathcal{O}(\mathbb{A}^2)$, i.e. any regular function on X can be extended to a regular function on the whole plane.

Indeed: let $f \in \mathcal{O}(X)$: if $P \neq Q$ are points of X, then there exist polynomials F, G, F', G' such that $f = F/G$ on a neighbourhood U_P of P and $f = F'/G'$ on a neighbourhood U_Q of Q. So $F'G = FG'$ on $U_P \cap U_Q \neq \emptyset$, which is open also in $\mathbb{A}^2$, hence dense. Therefore $F'G = FG'$ in $K[x,y]$. We can clearly assume that F and G are coprime and similarly for F' and G'. So by the unique factorization property, it follows that $F' = F$ and $G' = G$. In particular f admits a unique representation as F/G on X therefore $G(P) \neq 0$ for all $P \in X$. Hence G has no zeros on $\mathbb{A}^2$, so $G = c \in K^*$ and $f \in \mathcal{O}(X)$.

Now, the ideal $\langle x, y \rangle$ has no zeros in X and is proper: this proves that X is not affine.

We have exploited the fact that a polynomial in more than one variables has infinitely many zeros, a fact that allows to generalise the previous observation.

10.5 Open covering with affine varieties

Affine varieties are ubiquitous in view of the following Proposition.

Proposition 10.5.1. *Let $X \subset \mathbb{P}^n$ be quasi–projective. Then X admits an open covering by affine varieties.*

Proof. Let $X = X_0 \cup \dots \cup X_n$ be the open covering of X where $X_i = U_i \cap X = \{P \in X \mid P[a_0, \dots, a_n], a_i \neq 0\}$. So, fixed P, there exists an index i such that $P \in X_i$. We can assume that $P \in X_0$: X_0 is open in some affine variety Y of $\mathbb{A}^n$ (identified with U_0); set $X_0 = Y \setminus Y'$, where Y, Y' are both closed in $\mathbb{A}^n$. Since $P \notin Y'$, there exists F such that $F(P) \neq 0$ and $V(F) \supset Y'$. So $P \in Y \setminus V(F) \subset Y \setminus Y'$ and $Y \setminus V(F)$ is an affine open neighbourhood of P in $Y \setminus Y' = X_0$, that is open in X. $\square$

10.6 The Veronese maps

Let n, d be positive integers; put $N(n,d) = \binom{n+d}{d} - 1$. Note that $\binom{n+d}{d}$ is equal to the number of (monic) monomials of degree d in the variables $x_0, \dots, x_n$, that is equal to the number of $(n+1)$–tuples $(i_0, \dots, i_n)$ such that $i_0 + \dots + i_n = d$, $i_j \geq 0$. Then in $\mathbb{P}^{N(n,d)}$ we can use coordinates $\{v_{i_0 \dots i_n}\}$, where $i_0, \dots, i_n \geq 0$ and $i_0 + \dots + i_n = d$. For example: if $n = 2$, $d = 2$, then $N(2,2) = \binom{4}{2} - 1 = 5$. In $\mathbb{P}^5$ we can use coordinates $v_{200}, v_{110}, v_{101}, v_{020}, v_{011}, v_{002}$.

For all n, d we define the map $v_{n,d} : \mathbb{P}^n \to \mathbb{P}^{N(n,d)}$ such that

$$[x_0, \dots, x_n] \to [v_{d00\dots0}, v_{d-1,10\dots0}, \dots, v_{0\dots00d}]$$

where $v_{i_0 \dots i_n} = x_0^{i_0} x_1^{i_1} \dots x_n^{i_n}$: $v_{n,d}$ is clearly a morphism, its image is denoted by $V_{n,d}$ and is called *the Veronese variety* of type (n,d). It is in fact the projective variety of equations:

$$\{v_{i_0 \dots i_n} v_{j_0 \dots j_n} - v_{h_0 \dots h_n} v_{k_0 \dots k_n}, \forall i_0 + j_0 = h_0 + k_0, i_1 + j_1 = h_1 + k_1, \dots \tag{10.2}$$

We prove this statement in the particular case $n = d = 2$; the general case is similar.

First of all, it is clear that the points of $v_{n,d}(\mathbb{P}^n)$ satisfy the system (10.2). Conversely, assume that $P[v_{200}, v_{110}, \dots] \in \mathbb{P}^5$ satisfies equations (10.2), which become:

$$\begin{cases} v_{200} v_{020} = v_{110}^2 \\ v_{200} v_{002} = v_{101}^2 \\ v_{002} v_{020} = v_{011}^2 \\ v_{200} v_{011} = v_{110} v_{101} \\ v_{020} v_{101} = v_{110} v_{011} \\ v_{110} v_{002} = v_{011} v_{101} \end{cases}$$

Then, at least one of the coordinates $v_{200}, v_{020}, v_{002}$ is different from 0.

Therefore, if $v_{200} \neq 0$, then $P = v_{2,2}([v_{200}, v_{110}, v_{101}])$; if $v_{020} \neq 0$, then $P = v_{2,2}([v_{110}, v_{020}, v_{011}])$; if $v_{002} \neq 0$, then $P = v_{2,2}([v_{101}, v_{011}, v_{002}])$. Note that, if two of these three coordinates are different from 0, then the points of $\mathbb{P}^2$ found in this way have proportional coordinates, so they coincide.

We have also proved in this way that $v_{2,2}$ is an isomorphism between $\mathbb{P}^2$ and $V_{2,2}$, called the Veronese surface of $\mathbb{P}^5$. The same happens in the general case.

If $n = 1$, $v_{1,d} : \mathbb{P}^1 \to \mathbb{P}^d$ maps $[x_0, x_1]$ to $[x_0^d, x_0^{d-1}x_1, \dots, x_1^d]$: the image is called the *rational normal curve* of degree d, it is isomorphic to $\mathbb{P}^1$. If $d = 3$, we find the skew cubic (Chapter 5).

Let now $X \subset \mathbb{P}^n$ be a hypersurface of degree d: $X = V_P(F)$, with

$$F = \sum_{i_0+\cdots+i_n=d} a_{i_0\dots i_n} x_0^{i_0} \dots x_n^{i_n}.$$

Then $v_{n,d}(X) \simeq X$: it is the set of points

$$\{v_{i_0\dots i_n} \in \mathbb{P}^{N(n,d)} | \sum_{i_0+\cdots+i_n=d} a_{i_0\dots i_n} v_{i_0\dots i_n} = 0 \text{ and } [v_{i_0\dots i_n}] \in V_{n,d}\}.$$

It coincides with $V_{n,d} \cap H$, where H is a hyperplane of $\mathbb{P}^{N(n,d)}$: a hyperplane section of the Veronese variety. This is called the linearisation process, allowing to "transform" a hypersurface in a hyperplane, modulo the Veronese isomorphism.

The Veronese surface $V = V_{2,2}$ of $\mathbb{P}^5$ enjoys a lot of interesting properties. Most of them follow from its property of being covered by a 2-dimensional family of conics, which are precisely the images via $v_{2,2}$ of the lines of the plane.

To see this, we will change notation and will use as coordinates in $\mathbb{P}^5$ $w_{00}, w_{01}, w_{02}, w_{11}, w_{12}, w_{22}$, so that $v_{2,2}$ maps $[x_0, x_1, x_2]$ to the point of coordinates $w_{ij} = x_i x_j$. With this choice of coordinates, the equations of V are obtained by annihilating the 2×2 minors of the symmetric matrix:

$$M = \begin{pmatrix} w_{00} & w_{01} & w_{02} \\ w_{01} & w_{11} & w_{12} \\ w_{02} & w_{12} & w_{22} \end{pmatrix}.$$

Let ℓ be a line of $\mathbb{P}^2$ of equation $b_0x_0 + b_1x_1 + b_2x_2 = 0$. Its image is the set of points of $\mathbb{P}^5$ with coordinates $w_{ij} = x_i x_j$, such that there exists a non-zero triple $[x_0, x_1, x_2]$ with $b_0x_0 + b_1x_1 + b_2x_2 = 0$. But this last equation is equivalent to the system:

$$\begin{cases} b_0x_0^2 + b_1x_0x_1 + b_2x_0x_2 = 0 \\ b_0x_0x_1 + b_1x_1^2 + b_2x_1x_2 = 0 \\ b_0x_0x_2 + b_1x_1x_2 + b_2x_2^2 = 0 \end{cases}$$

It represents the intersection of V with the plane

$$\begin{cases} b_0w_{00} + b_1w_{01} + b_2w_{02} = 0 \\ b_0w_{01} + b_1w_{11} + b_2w_{12} = 0 \\ b_0w_{02} + b_1w_{12} + b_2w_{22} = 0 \end{cases} \tag{10.3}$$

so $v_{2,2}(\ell)$ is a plane curve. Its degree is the number of points in its intersection with a general hyperplane in $\mathbb{P}^5$: this corresponds to the intersection in $\mathbb{P}^2$ of ℓ with a conic (a hypersurface of degree 2). Therefore $v_{2,2}(\ell)$ is a conic.

So the isomorphism $v_{2,2}$ transforms the geometry of the lines in the plane in the geometry of the conics in the Veronese surface. In particular, given two distinct points on V, there is exactly one conic contained in V and passing through them.

From this observation it is easy to deduce that the *secant lines* of V, i.e. the lines meeting V at two points, are precisely the lines of the planes generated by the conics contained in V, so that the (closure of the) union of these secant lines coincides with the union of the planes of the conics of V. This union results to be the cubic hypersurface defined by the equation

$$\det M = \det \begin{pmatrix} w_{00} & w_{01} & w_{02} \\ w_{01} & w_{11} & w_{12} \\ w_{02} & w_{12} & w_{22} \end{pmatrix} = 0.$$

Indeed a point in $\mathbb{P}^5$, of coordinates $[w_{ij}]$ belongs to the plane of a conic contained in V if and only if there exists a non-zero triple $[b_0, b_1, b_2]$ which is solution of the homogeneous system (10.3).

Exercises 10.6.1. 1. Let X, Y be closed subsets of $\mathbb{A}^n$. Consider $X \times Y \subset \mathbb{A}^{2n}$ and the linear subspace, called the diagonal, $\Delta \subset \mathbb{A}^{2n}$ defined by the equations $x_i - y_i = 0$, $i = 1, \dots, n$. Prove that $(X \times Y) \cap \Delta$ is isomorphic to $X \cap Y$, constructing an explicit regular map with regular inverse.

2. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be the map defined by $f(x, y) = (x, xy)$. Check that f is regular and find the image $f(\mathbb{A}^2)$: is it open in $\mathbb{A}^2$? Dense? Closed? Locally closed? Irreducible?

3. Let $v_{1,d} : \mathbb{P}^1 \to \mathbb{P}^d$ be the d-tuple Veronese map, such that $v_{1,d}([x_0, x_1]) = [x_0^d, x_0^{d-1}x_1, \dots, x_1^d])$.

a) Check that the image of $v_{1,d}$ is C_d, the projective algebraic set defined by the 2×2 minors of the matrix

$$A = \begin{pmatrix} x_0 & x_1 & \dots & x_{d-1} \\ x_1 & x_2 & \dots & x_d \end{pmatrix}.$$

C_d is called the rational normal curve of degree d.

b) Prove that $v_{1,d} : \mathbb{P}^1 \to C_d$ is an isomorphism, by explicitly constructing its inverse morphism.

c) Prove that any $d+1$ points on C_d are linearly independent in $\mathbb{P}^d$ (Hint: Vandermonde).

Solution of Exercise 3. This exercise generalises the example of the skew cubic.

a) Let $z_0, \dots, z_d$ be coordinates in $\mathbb{P}^d$, so that the image of the Veronese map $v_{1,d}$ is given in parametric form by $z_0 = x_0^d, \dots, z_i = x_0^{d-i}x_1^i, \dots, z_d = x_1^d$. Let I be the ideal generated by the 2×2 minors of A. It is clear that the two rows of the matrix

$$\begin{pmatrix} x_0^d & x_0^{d-1}x_1 & \dots & x_0x_1^{d-1} \\ x_0^{d-1}x_1 & x_0^{d-2}x_1^2 & \dots & x_1^d \end{pmatrix}$$

are proportional for any x_0, x_1, so $v_{1,d}(\mathbb{P}^1) \subset V_P(I) = C_d$.

Conversely, let $[\bar{z}_0, \dots, \bar{z}_d] \in V_P(I)$. We observe that either $\bar{z}_0 \neq 0$ or $\bar{z}_d \neq 0$. If $\bar{z}_0 \neq 0$, then we can multiply all coordinates by $\bar{z}_0{}^{d-1}$ and we get:

$$[\bar{z}_0, \dots, \bar{z}_d] = [\bar{z}_0{}^d, \bar{z}_0{}^{d-1}\bar{z}_1, \dots, \bar{z}_0{}^{d-1}\bar{z}_i, \dots, \bar{z}_0{}^{d-1}\bar{z}_d].$$

If we can prove that $\bar{z}_0{}^{d-1}\bar{z}_i = \bar{z}_0{}^{d-i}\bar{z}_1{}^i$, then we conclude that our point is equal to $v_{1,d}([\bar{z}_0, \bar{z}_1])$. Note that $\bar{z}_0\bar{z}_k = \bar{z}_1\bar{z}_{k-1}$, for any $k = 1, \dots, d$. So $\bar{z}_0{}^{d-1}\bar{z}_i = \bar{z}_0{}^{d-2}(\bar{z}_1\bar{z}_{i-1}) = \bar{z}_0{}^{d-3}\bar{z}_1(\bar{z}_1\bar{z}_{i-2}) = \dots = \bar{z}_0{}^{d-i}\bar{z}_1{}^i$, as wanted.

If instead $\bar{z}_d \neq 0$, proceeding in a similar way we prove that $[\bar{z}_0, \dots, \bar{z}_d] = v_{1,d}([\bar{z}_{d-1}, \bar{z}_d])$.

b) The inverse map $\varphi : C_d \to \mathbb{P}^1$ operates in this way: $\varphi([z_0, \dots, z_d]) = [z_0, z_1] = [z_1, z_2] = \dots = [z_{d-1}, z_d]$. It is well defined because the columns of A are proportional, and it is regular because it is a projection.

c) Let $[z_0^{(k)}, \dots, z_d^{(k)}] = v_{1,d}([x_0^{(k)}, x_1^{(k)}])$, $k = 0, \dots d$, be $d+1$ points on C_d. Let $M = (z_i^{(j)})_{i,j=0,\dots d}$ be the matrix of their coordinates. If $x_0^{(k)} \neq 0$ for any k, we can assume $x_0^{(k)} = 1$ and

$$M = \begin{pmatrix} 1 & 1 & \dots & 1 \\ x_1^{(0)} & x_1^{(1)} & \dots & x_1^{(d)} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ x_1^{(0)}{}^d & x_1^{(1)}{}^d & \dots & x_1^{(d)}{}^d \end{pmatrix}.$$

This is a Vandermonde matrix whose determinant is different from zero because the points are distinct.

If one of the points has the first coordinate equal to zero, then it is $[0, 0, \dots, 0, 1]$, so we can assume that it is the first point, and that all the other d points have $x_0^{(k)} = 1$. Therefore

$$M = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 0 & x_1^{(1)} & \dots & x_1^{(d)} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 1 & {x_1^{(1)}}^d & \dots & {x_1^{(d)}}^d \end{pmatrix}.$$

Developing the determinant according to the first column, we find again a Vandermonde determinant, which is different from 0.

Chapter 11

The language of categories

11.1 Categories

Category theory was introduced by Samuel Eilenberg and Saunders Mac Lane in 1942-45 in their study of algebraic topology. They introduced the concepts of categories, functors, and natural transformations, with the goal of understanding the processes that preserve mathematical structures. In Algebraic Geometry it was much developed by Alexander Grothendieck, in his language of schemes.

Category theory has proven to be a powerful language for expressing some general facts and constructions that are encountered mainly in branches of algebra and geometry. Here we give an elementary introduction limiting ourselves to the simplest definitions and examples.

Definition 11.1.1. A category $\mathcal{C}$ consists of the following data:

(1) A class $ob(\mathcal{C})$ whose elements are called objects of the category;

(2) For each pair $A, B \in ob(\mathcal{C})$ of objects, a set indicated with $Hom_{\mathcal{C}}(A, B)$, or $\mathcal{C}(A, B)$, called set of morphisms or arrows from A to B. Instead of writing $f \in Hom_{\mathcal{C}}(A, B)$ it is common to use $f : A \to B$.

(3) For each triple of objects A, B, C a map of sets called composition:

$$Hom_{\mathcal{C}}(A, B) \times Hom_{\mathcal{C}}(B, C) \to Hom_{\mathcal{C}}(A, C),$$

such that

$$(f, g) \to g \circ f.$$

(4) For each object A a special element $1_A \in Hom_{\mathcal{C}}(A, A)$ called identity of A.
It is also assumed that the following axioms hold:

a) Composition is associative;

b) Identity acts as a neutral element for the composition (when it is defined).

The categories that are best known (but we will also meet others) are those in which we can interpret morphisms as particular functions between sets, their composition is the usual composition of functions, and the identity is the usual identity.

In particular we have:

(1) The category of sets, indicated with the symbol Set, in which $Hom(A, B) = Set(A, B)$ is the set of arbitrary maps from A to B.

(2) The category Grp of groups and homomorphisms between groups, Ab of abelian groups and group homomorphisms, Rng of rings and homomorphisms of rings, or Mod_R of modules on a ring R with homomorphisms of R-modules, etc.

(4) Top with objects the topological spaces and morphisms the continuous functions.

(5) The coverings of a given topological space and the covering maps.

(6) The notion of subcategory is rather natural: $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ if the class $ob(\mathcal{C}')$ is contained in $ob(\mathcal{C})$ and, for any pair of objects A, B in $\mathcal{C}'$, $Hom_{\mathcal{C}'}(A, B) \subset Hom_{\mathcal{C}}(A, B)$. The subcategory is called full if equality holds: $Hom_{\mathcal{C}'}(A, B) = Hom_{\mathcal{C}}(A, B)$.

(7) A first example of a category where morphisms cannot be thought of as simple functions is that of a *poset*. i.e. a partially ordered set P. The objects are the elements of P and

$$Hom_P(a, b) = \begin{cases} \{*\} & \text{if } a \leq b; \\ \emptyset & \text{otherwise.} \end{cases}$$

Here $\{*\}$ denotes a set with only one element denoted by $*$, also called the singleton. A particular case of a poset category is $Op(X)$, the category of the open subsets of a topological space X.

11.2 Functors

The second notion we are going to introduce formalizes the idea of transformation of categories.

Definition 11.2.1. A (covariant) functor $F : \mathcal{A} \to \mathcal{B}$ from the category $\mathcal{A}$ to the category $\mathcal{B}$ is a law that associates to every object X of $\mathcal{A}$ an object $F(X)$ of $\mathcal{B}$ and to every morphism $f : X \to Y$ in $\mathcal{A}$ a morphism $F(f) : F(X) \to F(Y)$ in $\mathcal{B}$, in such a way that

a) $F(f \circ g) = F(f) \circ F(g)$ (when the composition is defined),

b) $F(1_X) = 1_{F(X)}$.

The composition of functors can be done as in the case of functions.

Contravariant functors are defined by imposing that to every morphism $f : X \to Y$ is associated a morphism $F(f) : F(Y) \to F(X)$ so that we have $F(f \circ g) = F(g) \circ F(f)$. In other words, contravariant functors invert the arrows.

Given a category $\mathcal{C}$, we can define the opposite category $\mathcal{C}^0$, or $\mathcal{C}^{op}$, whose objects are the same as those of $\mathcal{C}$ while $Hom_{\mathcal{C}^0}(A, B) = Hom_{\mathcal{C}}(B, A)$. It is easily seen that a contravariant functor from $\mathcal{A}$ to $\mathcal{B}$ is also a covariant functor from $\mathcal{A}$ to $\mathcal{B}^0$ (or from $\mathcal{A}^0$ to $\mathcal{B}$).

Example 11.2.2. *Examples of functors.*

1. *Forgetful functors.* The law $U : Grp \to Set$ which maps a group to its underlying set and a group homomorphism to its underlying function of sets is a functor. Functors like this, which "forget" some structure, are termed forgetful functors. Another example is the functor $Rng \to Ab$ which maps a ring to its underlying additive abelian group. Morphisms in Rng (ring homomorphisms) become morphisms in Ab (abelian group homomorphisms).

2. *Free functors.* Going in the opposite direction of forgetful functors are free functors. The free functor $F : Set \to Ab$ sends every set X to the free abelian group generated by X. Functions are mapped to group homomorphisms between free abelian groups.

3. *Representable functors.* Let $\mathcal{C}$ be a category. Each object $A \in ob(\mathcal{C})$ allows to define the following functor $h^A : \mathcal{C} \to Set$. For each object $X \in ob(\mathcal{C})$, $h^A(X) := Hom_{\mathcal{C}}(A, X) \in ob(Set)$. For each morphism $f : X \to Y$ in $\mathcal{C}$, we define $h^A(f) : Hom_{\mathcal{C}}(A, X) \to Hom_{\mathcal{C}}(A, Y)$ through the composition: $h^A(g) := f \circ g$. The functor h^A is usually denoted by $h^A := Hom_{\mathcal{C}}(A, -)$ and is a covariant functor which is said to be represented by the object A of $\mathcal{C}$.

In a completely analogous way we can define the contravariant functor $h_A := Hom_{\mathcal{C}}(-, A)$.

Among the categorical ideas there is that of isomorphism, which generalizes that of bijection between sets, of isomorphism of groups, of homeomorphism between topological spaces etc.

An isomorphism f between two objects A, B of a category $\mathcal{C}$ is a morphism $f : A \to B$ such that there exists a morphism $g : B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

The following property follows easily from the axioms of category.

Proposition 11.2.3. *(1) If $f : A \to B$ is an isomorphism, the morphism $g : B \to A$ such that $g \circ f = 1_A$, $f \circ g = 1_B$ is unique (and denoted f^{-1}).*

(2) If $f : A \to B$ is an isomorphism in $\mathcal{C}$ and $F : \mathcal{C} \to \mathcal{D}$ is a functor, then also $F(f) : F(A) \to F(B)$ is an isomorphism (in $\mathcal{D}$).

11.3 Natural transformations

To complete the categorical approach it is convenient to introduce the last formal definition, the one that allows to treat the functors between two given categories $A \to B$ like the objects of a new category. To do this, we must define the morphisms between two such functors, which we will call natural transformations. We give the definition for covariant functors, the contravariant case is similar.

Definition 11.3.1. Given two functors $F, G : \mathcal{A} \to \mathcal{B}$ between two categories, a natural transformation $\varphi : F \to G$ between the two functors consists in giving, for each object $A \in ob(\mathcal{A})$ a morphism $\varphi_A : F(A) \to G(A)$ (in $\mathcal{B}$) such that, for each pair of objects $A, B \in ob(\mathcal{C})$ and for each morphism $f : A \to B$ the following diagram is commutative:

$$\begin{array}{ccc} F(A) & \xrightarrow{\varphi_A} & G(A) \\ {\scriptstyle \mathrm{F(f)}}\downarrow & & \downarrow{\scriptstyle \mathrm{G(f)}} \\ F(B) & \xrightarrow{\varphi_B} & G(B) \end{array}$$

The class of natural transformations between two functors $F, G : \mathcal{A} \to \mathcal{B}$ is denoted by $Nat(F, G)$. Often it is a set, which can therefore be taken as the set of morphisms to define the category of functors from category $\mathcal{A}$ to category $\mathcal{B}$. We will indicate with $F(\mathcal{A}, \mathcal{B})$ this category of functors. The properties of identity and composition are easy to verify.

From the general ideas, it follows the definition of natural isomorphism between two functors: it is a natural transformation that admits an inverse, and also that of **equivalence of categories.** An equivalence between the categories $\mathcal{A}, \mathcal{B}$ is a functor $F : \mathcal{A} \to \mathcal{B}$ satisying the following two conditions:

1. for any $Y \in ob(\mathcal{B})$ there exists $X \in ob(\mathcal{A})$ such that $Y \simeq F(X)$;
2. for any pair of objects A, B in $\mathcal{A}$, F gives a bijection $Hom(A, B) \xrightarrow{F} Hom(F(A), F(B))$.

We introduce a category $\mathcal{C}$ whose objects are the affine algebraic sets over a fixed algebraically closed field K and the morphisms are the regular maps. We consider also a second category $\mathcal{C}'$ with objects the K-algebras and morphisms the K-homomorphisms. Then there is a contravariant functor that operates on the objects mapping X to $\mathcal{O}(X) = K[X]$, and on the morphisms mapping φ to the associated comorphism φ^*. Note that this functor can be interpreted as the representable functor $h_{\mathbb{A}^1}$, when $\mathbb{A}^1$ is identified with K.

If we restrict the class of objects of $\mathcal{C}'$ taking only the finitely generated reduced K-algebras (a full subcategory of the previous one), then this functor becomes an equivalence of categories. Indeed the construction of the comorphism establishes a bijection between the Hom sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{C}'}(\mathcal{O}(Y), \mathcal{O}(X))$. Moreover, for any finitely generated and reduced K-algebra A, there exists an affine algebraic set X such that A is K-isomorphic to $\mathcal{O}(X)$. To see this, we choose a finite set of generators of A, such that $A = K[\xi_1, \dots, \xi_n]$. Then we can consider the surjective K-homomorphism Ψ from the polynomial ring $K[x_1, \dots, x_n]$ to A sending x_i to ξ_i for any i. In view of the fundamental theorem of homomorphism, it follows that $A \simeq K[x_1, \dots, x_n]/\ker\Psi$. The assumption that A is reduced then implies that $X := V(\ker\Psi) \subset \mathbb{A}^n$ is an affine algebraic set with $I(X) = \ker\Psi$ and $A \simeq \mathcal{O}(X)$.

We note that changing system of generators for A changes the homomorphism Ψ, and by consequence also the algebraic set X, up to isomorphism. For instance let $A = K[t]$ be a polynomial ring in one variable t: if we choose only t as system of generators, we get $X = \mathbb{A}^1$, but we can choose t, t^2, t^3, because $A = K[t, t^2, t^3]$; in this case we get the affine skew cubic in $\mathbb{A}^3$.

As a consequence of the previous discussion we have the following:

Corollary 11.3.2. *Let X, Y be affine varieties. Then $X \simeq Y$ if and only if $\mathcal{O}(X) \simeq \mathcal{O}(Y)$.*

We conclude this chapter defining an important functor. Let X be a quasi-projective algebraic variety over a field K. We consider the category $Op(X)$ of the open subsets of X, interpreted as topological space with the Zariski topology. The second category is $K - alg$, the category of K-algebras and K-homomorphisms. We define a contravariant functor $\mathcal{O}_X : Op(X) \to K - alg$ such that, for any open subset $U \subset X$, $\mathcal{O}_X(U) = \mathcal{O}(U)$, the ring of regular functions on U interpreted as quasi-projective variety. Given a morphism in $Op(X)$, this is an inclusion $U \hookrightarrow V$; this is sent by the functor $\mathcal{O}_X$ to the natural restriction map $\mathcal{O}(V) \to \mathcal{O}(U)$.

$\mathcal{O}_X$ is called the sheaf of regular functions on the variety X.

Chapter 12

Rational maps

12.1 Rational maps

Let X, Y be quasi–projective varieties over an algebraically closed field K. The idea to define rational maps is that they are to regular maps as rational functions are to regular functions.

Definition 12.1.1. The *rational maps* from X to Y are the germs of regular maps from open subsets of X to Y, i.e. they are equivalence classes of pairs (U, φ), where $U \neq \emptyset$ is open in X and $\varphi : U \to Y$ is regular. The equivalence relation is of course defined by $(U, \varphi) \sim (V, \psi)$ if and only if $\varphi|_{U \cap V} = \psi|_{U \cap V}$.

We need to prove that this is indeed an equivalence relation. The following Lemma guarantees that this is the case.

Lemma 12.1.2. *Let $\varphi, \psi : X \to Y \subset \mathbb{P}^n$ be regular maps between quasi-projective varieties. If $\varphi|_U = \psi|_U$ for $U \subset X$ open and non–empty, then $\varphi = \psi$.*

Proof. Let $P \in X$ and consider $\varphi(P)$, $\psi(P) \in Y$. There exists a hyperplane H such that $\varphi(P) \notin H$ and $\psi(P) \notin H$ (otherwise the dual projective space $\check{\mathbb{P}}^n$ would be the union of its two hyperplanes $H_{\varphi(P)}$, $H_{\psi(P)}$, defined by the conditions of containing respectively $\varphi(P)$ and $\psi(P)$).

Up to a projective transformation, we can assume that $H = V_P(x_0)$, so $\varphi(P), \psi(P) \in U_0$. Set $V = \varphi^{-1}(U_0) \cap \psi^{-1}(U_0)$: an open neighbourhood of P. Consider the restrictions of φ and ψ from V to $Y \cap U_0$: they are regular maps whose codomain is contained in $U_0 \simeq \mathbb{A}^n$. Since they coincide on $V \cap U$, their components φ_i, ψ_i, $i = 1, \dots, n$, coincide on $V \cap U$, hence on V (Corollary 9.1.4). So $\varphi_i|_V = \psi_i|_V$. In particular $\varphi(P) = \psi(P)$. $\square$

A rational map from X to Y will be denoted by $\varphi : X \dashrightarrow Y$. As for rational functions, the domain of definition of φ, dom φ, is the maximum open subset of X such that φ is regular at the points of dom φ.

The following proposition follows from the characterization of rational functions on affine varieties.

Proposition 12.1.3. *Let X, Y be affine algebraic sets, with Y closed in $\mathbb{A}^n$. Then $\varphi : X \dashrightarrow Y$ is a rational map if and only if $\varphi = (\varphi_1, \dots, \varphi_n)$, where $\varphi_1, \dots, \varphi_n \in K(X)$.*

If $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$, then a rational map $X \dashrightarrow Y$ is assigned by giving $m+1$ homogeneous polynomials of $K[x_0, x_1, \dots, x_n]$ of the same degree, $F_0, \dots, F_m$, such that *at least one* of them is not identically zero on X.

A rational map $\varphi : X \dashrightarrow Y$ is called *dominant* if the image of X via φ is dense in Y, i.e. if $\overline{\varphi(U)} = Y$, where $U = \text{dom } \varphi$.

Dominant rational maps can be composed: if $\varphi : X \dashrightarrow Y$ is dominant and $\psi : Y \dashrightarrow Z$ is any rational map, then dom $\psi \cap \text{Im}\varphi \neq \emptyset$, so we can define $\psi \circ \varphi : X \dashrightarrow Z$: it is the germ of the map $\psi \circ \varphi$, regular on $\varphi^{-1}(\text{dom } \psi \cap \text{Im}\varphi)$. We note that also the composed rational map $\psi \circ \varphi$ is dominant.

12.2 Birational maps

Definition 12.2.1. A *birational map* from X to Y is a rational map $\varphi : X \dashrightarrow Y$ such that φ is dominant and there exists $\psi : Y \dashrightarrow X$, a dominant rational map, such that $\psi \circ \varphi = 1_X$ and $\varphi \circ \psi = 1_Y$ **as rational maps**. In this case, X and Y are called *birationally equivalent* or simply *birational*.

If $\varphi : X \dashrightarrow Y$ is a dominant rational map, then we can define the comorphism $\varphi^* : K(Y) \to K(X)$ in the usual way: it is an injective K–homomorphism.

Proposition 12.2.2. *Let X, Y be quasi–projective varieties, and let $u : K(Y) \to K(X)$ be a K–homomorphism. Then there exists a rational map $\varphi : X \dashrightarrow Y$ such that $\varphi^* = u$.*

Proof. Y is covered by open affine varieties Y_α, $\alpha \in I$ (Section 10.5); note that for any index α, $K(Y) \simeq K(Y_\alpha)$ (Proposition 9.2.4) and $K(Y_\alpha) \simeq K(t_1, \dots, t_n)$, where $t_1, \dots, t_n$ can be interpreted as coordinate functions on Y_α. Choose such an open subset Y_α. Then $u(t_1), \dots, u(t_n) \in K(X)$ and there exists $U \subset X$, non–empty open subset such that $u(t_1), \dots, u(t_n)$ are all regular on U. So $u(K[t_1, \dots, t_n]) \subset \mathcal{O}(U)$ and we can consider the regular map

$u^\sharp : U \to Y_\alpha \hookrightarrow Y$. The germ of $u^\sharp$ gives a rational map $X \dashrightarrow Y$. It is possible to check that this rational map does not depend on the choice of Y_α and U. $\square$

Theorem 12.2.3. *Let X, Y be quasi–projective varieties. The following are equivalent:*

(i) X is birational to Y;

(ii) $K(X) \simeq K(Y)$;

(iii) there exist non–empty open subsets $U \subset X$ and $V \subset Y$ such that $U \simeq V$.

Proof. (i) $\Leftrightarrow$ (ii) via the construction of the comorphism φ^* associated to φ and of $u^\sharp$, associated to $u : K(Y) \to K(X)$. One checks that both constructions are functorial.

(i) $\Rightarrow$ (iii) Let $\varphi : X \dashrightarrow Y$, $\psi : Y \dashrightarrow X$ be rational maps inverse each other. Put $U' = \mathrm{dom}\,\varphi$ and $V' = \mathrm{dom}\,\psi$. By assumption, $\psi \circ \varphi$ is defined on $\varphi^{-1}(V')$ and coincides with 1_X there. Similarly, $\varphi \circ \psi$ is defined on $\psi^{-1}(U')$ and equal to 1_Y. Then φ and ψ establish an isomorphism between the corresponding sets $U := \varphi^{-1}(\psi^{-1}(U'))$ and $V := \psi^{-1}(\varphi^{-1}(V'))$.

(iii) $\Rightarrow$ (ii) $U \simeq V$ implies $K(U) \simeq K(V)$; but $K(U) \simeq K(X)$ and $K(V) \simeq K(Y)$ (Prop. 1.9, Lesson 10), so $K(X) \simeq K(Y)$ by transitivity. $\square$

Corollary 12.2.4. *If X is birational to Y, then* $\dim X = \dim Y$*.*

Corollary 12.2.5. *The projective space $\mathbb{P}^n$ and the affine space $\mathbb{A}^n$ are birationally equivalent.*

Theorem 12.2.3 can be given an interpretation in the language of categories. We can define a category $\mathcal{C}$ whose objects are the irreducible algebraic varieties over a fixed algebraically closed field K, and the morphisms are the dominant rational maps. The isomorphisms in $\mathcal{C}$ are birational maps, so two objects are isomorphic in $\mathcal{C}$ if they are birationally equivalent. We can consider also the category $\mathcal{C}'$ with objects the fields, finitely generated extensions of K, and morphisms the K-homomorphisms. Then there is a contravariant functor $\mathcal{C} \to \mathcal{C}'$ associating to a variety X its field of rational functions $K(X)$ and to a rational map $\varphi : X \dashrightarrow Y$ its comorphism φ^*. Proposition 12.2.2 and Theorem 12.2.3 say that this functor is an equivalence of categories.

There are two classification problems for algebraic varieties, up to isomorphism and up to birational equivalence. Both are central problems of Algebraic Geometry.

12.3 Examples

Example 12.3.1. a) *The cuspidal cubic $Y = V(x^3 - y^2) \subset \mathbb{A}^2$.*

We have seen in Example 10.1.2 that Y is not isomorphic to $\mathbb{A}^1$, but Y and $\mathbb{A}^1$ are birationally equivalent. Indeed, the regular map $\varphi : \mathbb{A}^1 \to Y$, $t \to (t^2, t^3)$, admits a rational inverse $\psi : Y \dashrightarrow \mathbb{A}^1$, $(x, y) \to \frac{y}{x}$. ψ is regular on $Y \setminus \{(0,0)\}$, ψ is dominant and $\psi \circ \varphi = 1_{\mathbb{A}^1}$, $\varphi \circ \psi = 1_Y$ as rational maps. In particular, $\varphi^* : K(Y) \to K(X)$ is a field isomorphism. Recall that $K[Y] = K[t_1, t_2]$, with $t_1^2 = t_2^3$, so $K(Y) = K(t_1, t_2) = K(t_2/t_1)$, because $t_1 = (t_2/t_1)^2 = t_2^2/t_1^2 = t_1^3/t_1^2$ and $t_2 = (t_2/t_1)^3 = t_2^3/t_1^3 = t_2^3/t_2^2$, so $K(Y)$ is generated by a unique transcendental element. Notice that φ and ψ establish isomorphisms between $\mathbb{A}^1 \setminus \{0\}$ and $Y \setminus \{(0,0)\}$.

b)*Any rational map from $\mathbb{P}^1$ to $\mathbb{P}^n$ is regular.*

Let $\varphi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ be a rational map: on some open $U \subset \mathbb{P}^1$,

$$\varphi([x_0, x_1]) = [F_0(x_0, x_1), \dots, F_n(x_0, x_1)],$$

with $F_0, \dots, F_n$ homogeneous of the same degree, without non–trivial common factors. Assume that $F_i(P) = 0$ for a certain index i, with $P = [a_0, a_1]$. Then $F_i \in I_h(P) = \langle a_1x_0 - a_0x_1 \rangle$, i.e. $a_1x_0 - a_0x_1$ is a factor of F_i. This remark implies that $\forall\, Q \in \mathbb{P}^1$ there exists $i \in \{0, \dots, n\}$ such that $F_i(Q) \neq 0$, because otherwise $F_0, \dots, F_n$ would have a common factor of degree 1. Hence we conclude that φ is regular.

We have obtained that any rational map from $\mathbb{P}^1$ is in fact regular.

c) *Projections.*

Let $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a rational map, that can be represented in matrix form by $Y = AX$, where A is a $(m+1) \times (n+1)$-matrix, with entries in K. Then φ is a rational map, regular on $\mathbb{P}^n \setminus \mathbb{P}(\mathrm{Ker}A)$. Put $\Lambda := \mathbb{P}(\mathrm{Ker}A)$. If $A = (a_{ij})$, this means that Λ has cartesian equations

$$\begin{cases} a_{00}x_0 + \ldots + a_{0n}x_n = 0 \\ a_{10}x_0 + \ldots + a_{1n}x_n = 0 \\ \ldots \\ a_{m0}x_0 + \ldots + a_{mn}x_n = 0. \end{cases}$$

The map φ has a geometric interpretation: it can be seen as the *projection of centre* Λ to a complementar linear space. To see how to give this interpretation, first of all we can assume that $\mathrm{rk}\ A = m + 1$, otherwise we replace $\mathbb{P}^m$ with $\mathbb{P}(\mathrm{Im}\ A)$; hence $\dim \Lambda = (n+1) - (m+1) - 1 = n - m - 1$.

Consider first the special case in which $\Lambda : x_0 = \cdots = x_m = 0$; we can identify $\mathbb{P}^m$ with the subspace of $\mathbb{P}^n$ of equations $x_{m+1} = \cdots = x_n = 0$, so Λ and $\mathbb{P}^m$ are complementar subspaces, i.e. $\Lambda \cap \mathbb{P}^m = \emptyset$ and the linear span of Λ and $\mathbb{P}^m$ is $\mathbb{P}^n$. Then, for $Q[a_0, \dots, a_n] \in \mathbb{P}^n \setminus \Lambda$, $\varphi(Q) = [a_0, \dots, a_m, 0, \dots, 0]$: it is the intersection of $\mathbb{P}^m$ with the linear span $\overline{\Lambda Q}$ of Λ and Q. In fact, $\overline{\Lambda Q}$ has equations

$$\{a_i x_j - a_j x_i = 0, i, j = 0, \dots, m \quad \text{(check!)}$$

so $\overline{\Lambda Q} \cap \mathbb{P}^m$ has coordinates $[a_0, \dots, a_m, 0, \dots, 0]$.

In the general case, if $\Lambda = V_P(L_0, \dots, L_m)$, with $L_0, \dots, L_m$ linearly independent forms, we can identify $\mathbb{P}^m$ with $V_P(L_{m+1}, \dots, L_n)$, where $L_{m+1}, \dots, L_n$ are linearly independents linear forms chosen so that $L_0, \dots, L_m$, $L_{m+1}, \dots, L_n$ is a basis of $(K^{n+1})^*$. Then $L_0, \dots, L_m$ can be interpreted as coordinate functions on $\mathbb{P}^m$.

If $m = n - 1$, then Λ is a point P and φ, often denoted by π_P, is the projection from P to a hyperplane not containing P. Also for the projection with centre Λ often the notation π_Λ is used.

d)*Rational and unirational varieties.*

A quasi–projective variety X is called *rational* if it is birational to a projective space $\mathbb{P}^n$, or equivalently to $\mathbb{A}^n$.

By Theorem 12.2.3, X is rational if and only if $K(X) \simeq K(\mathbb{P}^n) = K(x_1, \dots, x_n)$ for some n, i.e. $K(X)$ is an extension of K generated by a transcendence basis; this kind of extension is called a *purely transcendental extension of* K. In an equivalent way, X is rational if there exists a rational map $\varphi : \mathbb{P}^n \dashrightarrow X$ which is dominant and is an isomorphism if restricted to a suitable open subset $U \subset \mathbb{P}^n$. Hence X admits a *birational parameterization* by polynomials in n parameters.

A weaker notion is that of *unirational* variety: X is unirational if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$ i.e. if $K(X)$ is contained in the quotient field of a polynomial ring. Hence X can be parameterized by polynomials, but not necessarily generically one–to–one.

It is clear that, if X is rational, then it is unirational. The converse implication has been an important open problem, up to 1971, when it has been solved in the negative, for varieties of dimension ≥ 3 (Clemens–Griffiths, Iskovskih–Manin, Artin-Mumford). Nevertheless rationality and unirationality are equivalent for curves (Theorem of Lüroth, 1880, over any field) and for surfaces if char$K = 0$ (Theorem of Castelnuovo, 1894).

e)*Rational parameterization of a smooth quadric surface.*

As an example of rational variety with an explicit rational parameterization constructed geometrically, let us consider the Segre quadric in $\mathbb{P}^3$, of maximal rank: $X = V_P(x_0x_3 - x_1x_2)$, it is an irreducible hypersurface of degree 2. Let $\pi_P : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection of centre

$P[1,0,0,0]$, such that $\pi_P([y_0,y_1,y_2,y_3]) = [y_1,y_2,y_3]$. The restriction of π_P to X is a rational map $\tilde{\pi}_P : X \dashrightarrow \mathbb{P}^2$, regular on $X \setminus \{P\}$. $\tilde{\pi}_P$ has a rational inverse: indeed consider the rational map $\psi : \mathbb{P}^2 \dashrightarrow X$, $[y_1,y_2,y_3] \to [y_1y_2, y_1y_3, y_2y_3, y_3^2]$. The equation of X is satisfied by the points of $\psi(\mathbb{P}^2)$: $(y_1y_2)y_3^2 = (y_1y_3)(y_2y_3)$. ψ is regular on $\mathbb{P}^2 \setminus V_P(y_1y_2, y_3)$. Let us compose ψ and $\tilde{\pi}_P$:

$$[y_0,\dots,y_3] \in X \xrightarrow{\pi_P} [y_1,y_2,y_3] \xrightarrow{\psi} [y_1y_2, y_1y_3, y_2y_3, y_3^2];$$

$y_1y_2 = y_0y_3$ implies $\psi \circ \pi_P = 1_X$. In the opposite order:

$$[y_1,y_2,y_3] \xrightarrow{\psi} [y_1y_2, y_1y_3, y_2y_3, y_3^2] \xrightarrow{\pi_P} [y_1y_3, y_2y_3, y_3^2] = [y_1,y_2,y_3].$$

So X is birational to $\mathbb{P}^2$ hence it is a rational surface.

Note that if we consider another projection $\pi_{P'}$ whose centre P' is not on the quadric, we get a regular $2:1$ map to the plane, that is certainly not birational.

f) *A birational non–regular map from* $\mathbb{P}^2$ *to* $\mathbb{P}^2$*.*

The following rational map is called the *standard quadratic transformation*:

$$Q : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad [x_0,x_1,x_2] \to [x_1x_2, x_0x_2, x_0x_1].$$

Q is regular on $U := \mathbb{P}^2 \setminus \{A,B,C\}$, where $A[1,0,0]$, $B[0,1,0]$, $C[0,0,1]$ are the fundamental points (see Figure 1).

Let a be the line through B and C: $a = V_P(x_0)$, and similarly $b = V_P(x_1)$, $c = V_P(x_2)$. Then $Q(a) = A$, $Q(b) = B$, $Q(c) = C$. Outside these three lines Q is an isomorphism. Precisely, put $U' = \mathbb{P}^2 \setminus \{a \cup b \cup c\}$; then $Q : U' \to \mathbb{P}^2$ is regular, the image is U' and $Q^{-1} : U' \to U'$ coincides with Q. Indeed,

$$[x_0,x_1,x_2] \xrightarrow{Q} [x_1x_2, x_0x_2, x_0x_1] \xrightarrow{Q} [x_0^2x_1x_2, x_0, x_1^2x_2, x_0x_1x_2^2].$$

So $Q \circ Q = 1_{\mathbb{P}^2}$ as rational map, hence Q is birational and $Q = Q^{-1}$.

Note that another way to express Q is the following: $[x_0,x_1,x_2] \to [\frac{1}{x_0}, \frac{1}{x_1}, \frac{1}{x_2}]$.

The set of the birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a group, called the *Cremona group*. At the end of XIX century, Max Noether proved that the Cremona group is generated by $PGL(3,K)$ and by the single standard quadratic transformation Q defined above. The analogous groups for $\mathbb{P}^n$, $n \geq 3$, are much more complicated and a complete description is still unknown.

We conclude this chapter with a theorem illustrating an application of the linearisation procedure. We shall use the following notation: given a homogeneous polynomial $F \in K[x_0,x_1,\dots,x_n]$, $D(F) := \mathbb{P}^n \setminus V_P(F)$.

Theorem 12.3.2. *Let $W \subset \mathbb{P}^n$ be a closed projective variety. Let F be a homogeneous polynomial of degree d in $K[x_0, x_1, \dots, x_n]$ such that $W \nsubseteq V_P(F)$. Then $W \cap D(F)$ is an affine variety.*

Proof. The assumption $W \nsubseteq V_P(F)$ is equivalent to $W \cap D(F) \neq \emptyset$. Let us consider the d-tuple Veronese embedding $v_{n,d} : \mathbb{P}^n \to \mathbb{P}^{N(n,d)}$, with $N(n,d) = \binom{n+d}{d} - 1$, that gives the isomorphism $\mathbb{P}^n \simeq V_{n,d}$. In this isomorphism the hypersurface $V_P(F)$ corresponds to a hyperplane section $V_{n,d} \cap H$, for a suitable hyperplane H in $\mathbb{P}^{N(n,d)}$. Therefore we have $W \cap D(F) \simeq v_{n,d}(W \cap D(F)) = v_{n,d}(W) \setminus H = v_{n,d}(W) \cap (\mathbb{P}^{N(n,d)} \setminus H)$. There exists a projective isomorphism $\tau : \mathbb{P}^{N(n,d)} \to \mathbb{P}^{N(n,d)}$ such that $\tau(H) = H_0$, the fundamental hyperplane of equation $x_0 = 0$. Therefore, denoting $X := v_{n,d}(W)$, we get $X \cap (\mathbb{P}^{N(n,d)} \setminus H) \simeq \tau(X) \cap (\mathbb{P}^{N(n,d)} \setminus H_0) = \tau(X) \cap U_0$, which proves the theorem. $\square$

As a consequence of Theorem 12.3.2, we get that the open subsets of the form $W \cap D(F)$ form a topology basis for W formed by affine varieties.

Exercises 12.3.3. 1. Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^n$ be the map defined by $t \to (t, t^2, \dots, t^n)$.

a) Prove that $\varphi : \mathbb{A}^1 \to \varphi(\mathbb{A}^1)$ is an isomorphism and describe $\varphi(\mathbb{A}^1)$;

b) give a description of φ^* and φ^{-1*}.

2. Prove that the Veronese variety $V_{n,d}$ is not contained in any hyperplane of $\mathbb{P}^{N(n,d)}$.

3. Let $GL_n(K)$ be the set of invertible $n \times n$ matrices with entries in K. Prove that $GL_n(K)$ can be given the structure of an affine variety.

4. Let $\varphi : X \to Y$ be a regular map and φ^* its comorphism. Prove that the kernel of φ^* is the ideal of $\varphi(X)$ in $\mathcal{O}(Y)$. In the affine case, deduce that φ is dominant if and only if φ^* is injective.

5. Prove that $\mathcal{O}(X_F)$ is isomorphic to $\mathcal{O}(X)_f$, where X is an affine algebraic variety, F a polynomial and f the regular function on X defined by F.

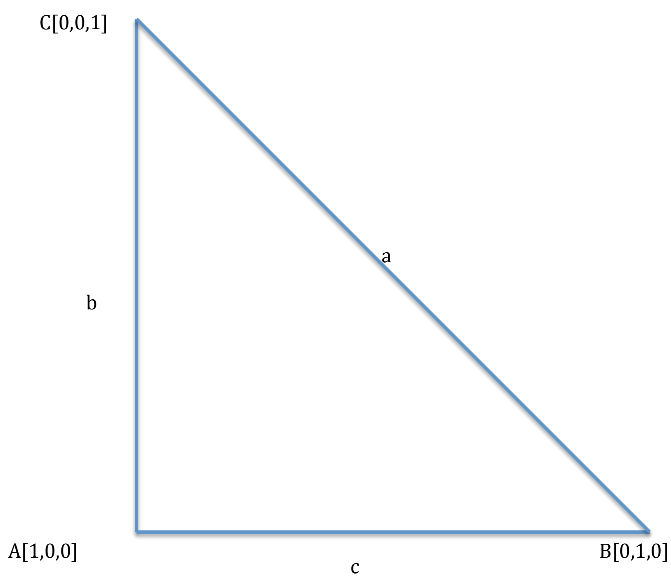

Figure 12.1:

Chapter 13

Product of quasi–projective varieties and tensors

13.1 Products

In Chapter 2, Section 2.5, we have seen how the product $\mathbb{P}^1 \times \mathbb{P}^1$ can be interpreted as a projective variety, and precisely a quadric of maximal rank, by means of the Segre map. Now we want to give a structure of algebraic variety to all products of algebraic varieties. We will seen that this can be done by generalizing the definition of the Segre map to any product of projective spaces $\mathbb{P}^n \times \mathbb{P}^m$.

Let $\mathbb{P}^n$, $\mathbb{P}^m$ be projective spaces over the same field K. The cartesian product $\mathbb{P}^n \times \mathbb{P}^m$ is simply a set: we want to define an injective map from $\mathbb{P}^n \times \mathbb{P}^m$ to a suitable projective space, so that the image is a projective variety, which will be identified with our product.

Let $N = (n+1)(m+1) - 1$ and define $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ in the following way: $\sigma([x_0, \dots, x_n], [y_0, \dots, y_m]) = [x_0y_0, x_0y_1, \dots, x_iy_j, \dots, x_ny_m]$. Using coordinates w_{ij}, $i = 0, \dots, n$, $j = 0, \dots, m$, in $\mathbb{P}^N$, σ is defined by

$$\{w_{ij} = x_iy_j,\ i = 0, \dots, n, j = 0, \dots, m.$$

It is easy to observe that σ is a well–defined map.

Let $\Sigma_{n,m}$ (or simply Σ) denote the image $\sigma(\mathbb{P}^n \times \mathbb{P}^m)$.

Proposition 13.1.1. *σ is injective and $\Sigma_{n,m}$ is a closed subset of $\mathbb{P}^N$.*

Proof. If $\sigma([x], [y]) = \sigma([x'], [y'])$, then there exists $\lambda \neq 0$ such that $x'_iy'_j = \lambda x_iy_j$ for all i, j. In particular, if $x_h \neq 0$, $y_k \neq 0$, then also $x'_h \neq 0$, $y'_k \neq 0$, and for all i $x'_i = \lambda \frac{y_k}{y'_k} x_i$, so

$[x_0,\dots,x_n]=[x'_0,\dots,x'_n]$. Similarly for the second point.
To prove the second assertion, I claim: $\Sigma_{n,m}$ is the closed set of equations:

$$\{w_{ij}w_{hk}=w_{ik}w_{hj},\ \ i,h=0,\dots,n;j,k=0\dots,m. \tag{13.1}$$

It is clear that if $[w_{ij}]\in\Sigma$, then it satisfies (13.1).

Conversely, assume that $[w_{ij}]$ satisfies (13.1) and that $w_{\alpha\beta}\neq 0$. Then

$$\begin{aligned}[w_{00},\dots,w_{ij},\dots,w_{nm}]&=[w_{00}w_{\alpha\beta},\dots,w_{ij}w_{\alpha\beta},\dots,w_{nm}w_{\alpha\beta}]=\\&=[w_{0\beta}w_{\alpha 0},\dots,w_{i\beta}w_{\alpha j},\dots,w_{n\beta}w_{\alpha m}]=\\&=\sigma([w_{0\beta},\dots,w_{n\beta}],[w_{\alpha 0},\dots,w_{\alpha m}]).\end{aligned}$$

□

σ is called the Segre map and $\Sigma_{n,m}$ the Segre variety or biprojective space. Note that Σ is covered by the affine open subsets $\Sigma^{ij}=\Sigma\cap W_{ij}$, where $W_{ij}=\mathbb{P}^N\setminus V_P(w_{ij})$. Moreover $\Sigma^{ij}=\sigma(U_i\times V_j)$, where $U_i\times V_j$ is naturally identified with $\mathbb{A}^{n+m}$.

Proposition 13.1.2. *$\sigma|_{U_i\times V_j}:U_i\times V_j=\mathbb{A}^{n+m}\to\Sigma^{ij}$ is an isomorphism of varieties.*

Proof. Assume by simplicity $i=j=0$. Choose non–homogeneous coordinates on U_0: $u_i=x_i/x_0$ and on V_0: $v_j=y_j/y_0$. So $u_1,\dots u_n,v_1,\dots,v_m$ are coordinates on $U_0\times V_0$. Take non–homogeneous coordinates also on W_{00}: $z_{ij}=w_{ij}/w_{00}$.

Using these coordinates we have:

$$\begin{aligned}\sigma|_{U_i\times V_j}:&(u_1,\dots u_n,v_1,\dots,v_m)\to(v_1,\dots,v_m,u_1,u_1v_1,\dots,u_1v_m,\dots,u_nv_m)\\&\qquad||\\&([1,u_1,\dots,u_n],[1,v_1,\dots,v_m])\end{aligned}$$

i.e. $\sigma(u_1,\dots,v_m)=(z_{01},\dots,z_{nm})$, where

$$\begin{cases}z_{i0}=u_i, & \text{if } i=1,\dots,n;\\ z_{0j}=v_j, & \text{if } j=1,\dots,m;\\ z_{ij}=u_iv_j=z_{i0}z_{0j} & \text{otherwise.}\end{cases}$$

Hence $\sigma|_{U_0\times V_0}$ is regular. Its inverse maps $(z_{01},\dots,z_{nm})$ to $(z_{10},\dots,z_{n0},z_{01},\dots,z_{0m})$, so it is also regular. □

Corollary 13.1.3. *$\mathbb{P}^n\times\mathbb{P}^m$ is irreducible and birational to $\mathbb{P}^{n+m}$.*

Proof. The first assertion follows from Exercise 5, Chapter 6, considering the covering of Σ by the open subsets Σ^{ij}. Indeed, $\Sigma^{ij}\cap\Sigma^{hk} = \sigma((U_i\times V_j)\cap(U_h\times V_k)) = \sigma((U_i\cap U_h)\times(V_j\cap V_k))$, and $U_i \cap U_h \neq \emptyset \neq V_j \cap V_k$.

For the second assertion, by Theorem 12.2.3, it is enough to note that $\Sigma_{n,m}$ and $\mathbb{P}^{n+m}$ contain isomorphic open subsets, i.e. Σ^{ij} and $\mathbb{A}^{n+m}$. $\square$

From now on, we shall identify $\mathbb{P}^n \times \mathbb{P}^m$ with $\Sigma_{n,m}$. If $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ are any quasi–projective varieties, then $X \times Y$ will be automatically identified with $\sigma(X \times Y) \subset \Sigma$.

Proposition 13.1.4. *If X and Y are projective varieties (resp. quasi–projective varieties), then $X \times Y$ is projective (resp. quasi–projective).*

Proof.

$$\begin{aligned}\sigma(X \times Y) &= \bigcup_{i,j}(\sigma(X \times Y) \cap \Sigma^{ij}) = \\ &= \bigcup_{i,j}(\sigma(X \times Y) \cap (U_i \times V_j)) = \\ &= \bigcup_{i,j}(\sigma((X \cap U_i) \times (Y \cap V_j))).\end{aligned}$$

If X and Y are projective varieties, then $X \cap U_i$ is closed in U_i and $Y \cap V_j$ is closed in V_j, so their product is closed in $U_i \times V_j$; since $\sigma|_{U_i\times V_j}$ is an isomorphism, also $\sigma(X \times Y) \cap \Sigma^{ij}$ is closed in Σ^{ij}, so $\sigma(X \times Y)$ is closed in Σ, by Lemma 9.1.3.

If X, Y are quasi–projective, the proof is similar: $X \cap U_i$ is locally closed in U_i and $Y \cap V_j$ is locally closed in V_j, so $X \cap U_i = Z \setminus Z'$, $Y \cap V_j = W \setminus W'$, with Z, Z', W, W' closed. Therefore $(Z \setminus Z') \times (W \setminus W') = Z \times W \setminus ((Z' \times W) \cup (Z \times W'))$, which is locally closed.

As for the irreducibility, see Exercise 1. $\square$

Example 13.1.5. $\mathbb{P}^1 \times \mathbb{P}^1$

The example of $\mathbb{P}^1 \times \mathbb{P}^1$, the Segre quadric, has already been studied in Section 2.5.

We recall that $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is given by the parametric equations $\{w_{ij} = x_i y_j,\ i = 0, 1,$ $j = 0, 1$. Σ has only one non–trivial equation: $w_{00}w_{11} - w_{01}w_{10}$, hence Σ is a quadric. The equation of Σ can be written as

$$\begin{vmatrix} w_{00} & w_{01} \\ w_{10} & w_{11} \end{vmatrix} = 0. \tag{13.2}$$

Σ contains two families of special closed subsets parametrised by $\mathbb{P}^1$, i.e.

$$\{\sigma(\{P\} \times \mathbb{P}^1)\}_{P\in\mathbb{P}^1} \quad \text{and} \quad \{\sigma(\mathbb{P}^1 \times \{Q\})\}_{Q\in\mathbb{P}^1}.$$

If $P = [a_0, a_1]$, then $\sigma(\{P\} \times \mathbb{P}^1)$ is given by the equations:

$$\begin{cases} w_{00} = a_0 y_0 \\ w_{01} = a_0 y_1 \\ w_{10} = a_1 y_0 \\ w_{11} = a_1 y_1 \end{cases}$$

hence it is a line. Cartesian equations of $\sigma(\{P\} \times \mathbb{P}^1)$ are:

$$\begin{cases} a_1 w_{00} - a_0 w_{10} = 0 \\ a_1 w_{01} - a_0 w_{11} = 0; \end{cases}$$

they express the proportionality of the rows of the matrix (13.2) with coefficients $[a_1, -a_0]$. Similarly, $\sigma(\mathbb{P}^1 \times \{Q\})$ is the line of equations

$$\begin{cases} a_1 w_{00} - a_0 w_{01} = 0 \\ a_1 w_{10} - a_0 w_{11} = 0. \end{cases}$$

Hence Σ contains two families of lines, called the rulings of Σ: two lines of the same ruling are clearly disjoint, while two lines of different rulings intersect at one point ($\sigma(P, Q)$). Conversely, through any point of Σ there pass two lines, one for each ruling.

Note that Σ is exactly the quadric surface of Section 12.3 e) and that the projection π_P of centre $P[1, 0, 0, 0]$ realizes an explicit birational map between $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. The two lines contained in Σ passing through P have equations $w_{10} = w_{11} = 0$ and $w_{01} = w_{11} = 0$ respectively; they are contracted to the points $E_0[1, 0, 0]$, $E_1[0, 1, 0]$ of $\mathbb{P}^2$ respectively. Conversely, the line $x_2 = 0$ in $\mathbb{P}^2$ passing through E_0, E_1 is contracted to P by π_p^{-1}.

13.2 Tensors

The product of projective spaces has a coordinate-free description in terms of tensors. Precisely, let $\mathbb{P}^n = \mathbb{P}(V)$ and $\mathbb{P}^m = \mathbb{P}(W)$. The tensor product $V \otimes W$ of the vector spaces V, W is constructed as follows: let $K(V \times W)$ be the K-vector space with basis $V \times W$ obtained as the set of formal finite linear combinations of type $\Sigma_i a_i (v_i, w_i)$ with $a_i \in K$. Let U be the vector subspace generated by all elements of the form:

$(v, w) + (v', w) - (v + v', w)$,
$(v, w) + (v, w') - (v, w + w')$,

$(\lambda v, w) - \lambda(v, w)$,

$(v, \lambda w) - \lambda(v, w)$,

with $v, v' \in V$, $w, w' \in W$, $\lambda \in K$. The tensor product is by definition the quotient $V \otimes W := K(V \times W)/U$. The class of a pair (v, w) is denoted by $v \otimes w$, and called a decomposable tensor. So $V \otimes W$ is generated by the decomposable tensors; more precisely, a general element $\omega \in V \otimes W$ is of the form $\Sigma_{i=1}^{k} v_i \otimes w_i$, with $v_i \in V$, $w_i \in W$. The minimum k such that an expression of this type exists is called the tensor rank of ω.

There is a natural bilinear map $\otimes : V \times W \to V \otimes W$, such that $(v, w) \to v \otimes w$. It enjoys the following universal property: for any K-vector space Z with a bilinear map $f : V \times W \to Z$, there exists a unique linear map $\bar{f} : V \otimes W \to Z$ such that f factorizes in the form $f = \bar{f} \circ \otimes$.

If $\dim V = n+1$, $\dim W = m+1$, and bases $\mathcal{B} = (e_0, \dots, e_n), \mathcal{B}' = (e'_0, \dots, e'_m)$ are given, then $(e_0 \otimes e'_0, \dots, e_i \otimes e'_j, \dots e_n \otimes e'_m)$ is a basis of $V \otimes W$: therefore $\dim V \otimes W = (n+1)(m+1)$.

If $v = x_0 e_0 + \dots + x_n e_n$, $w = y_0 e'_0 + \dots + y_m e'_m$, then $v \otimes w = \sum_{i,j} x_i y_j e_i \otimes e'_j$. So, passing to the projective spaces, the map $\otimes$ defines precisely the Segre map

$$\sigma : \mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes W), \quad ([v], [w]) \to [v \otimes w].$$

Indeed in coordinates we have $([x_0, \dots, x_n], [y_0, \dots, y_m]) \to [w_{00}, \dots, w_{nm}]$, with $w_{ij} = x_i y_j$. The image of $\otimes$ is the set of decomposable tensors, or rank one tensors.

The tensor product $V \otimes W$ has the same dimension, and is therefore isomorphic to the vector space of $(n+1) \times (m+1)$ matrices. The coordinates w_{ij} can be interpreted as the entries of such a $(n+1) \times (m+1)$ matrix. The equations of the Segre variety $\Sigma_{n,m}$ are the 2×2 minors of the matrix, therefore $\Sigma_{n,m}$ can be interpreted as the set of matrices of rank one.

The construction of the tensor product can be iterated, to construct $V_1 \otimes V_2 \otimes \dots \otimes V_r$. The following properties can easily be proved:

1. $V_1 \otimes (V_2 \otimes V_3) \simeq (V_1 \otimes V_2) \otimes V_3$;
2. $V \otimes W \simeq W \otimes V$;
3. $V^* \otimes W \simeq Hom(V, W)$, where $f \otimes w \to (V \to W : v \to f(v)w)$.

Also the Veronese morphism has a coordinate free description, in terms of symmetric tensors. Given a vector space V, for any $d \geq 0$ the d-th symmetric power of V, $S^d V$ or $Sym^d V$, is constructed as follows. We consider the tensor product of d copies of V: $V \otimes \dots \otimes V = V^{\otimes d}$, and we consider its subvector space U generated by all tensors of the form $v_1 \otimes \dots v_d - v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(d)}$, where $v_1, \dots, v_d$ vary in V and σ varies in the symmetric group on d elements $\mathcal{S}_d$. Then by definition $S^dV := V^{\otimes d}/U$. The equivalence class $[v_1 \otimes \cdots \otimes v_d]$ is denoted as a product $v_1 \dots v_d$. The elements of S^dV are called symmetric tensors.

There is a natural multilinear and symmetric map $V \times \cdots \times V = V^d \to S^dV$, such that $(v_1, \dots, v_d) \to v_1 \dots v_d$, which enjoys the universal property. S^dV is generated by the products $v_1 \dots v_d$.

In characteristic 0, S^dV can also be interpreted as a subspace of $V^{\otimes d}$, by considering the following map, that is an isomorphism to the image:

$$S^dV \to V^{\otimes d}, \quad v_1 \dots v_d \to \Sigma_{\sigma \in \mathcal{S}_d} \frac{1}{d!} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}.$$

For instance, in S^2V the product v_1v_2 can be identified with $\frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1)$.

If $\mathcal{B} = (e_0, \dots, e_n)$ is a basis of V, then it is easy to check that a basis of S^dV is formed by the monomials of degree d in $e_0, \dots, e_n$; therefore $\dim S^dV = \binom{n+d}{d}$.

The symmetric algebra of V is $SV := \oplus_{d \geq 0} S^dV = K \oplus V \oplus S^2V \oplus \dots$. An inner product can be naturally defined to give it the structure of a K-algebra, which results to be isomorphic to the polynomial ring in $n+1$ variables, where $n+1 = \dim V$.

If $char K = 0$ the Veronese morphism can be interpreted in the following way:

$$v_{n,d} : \mathbb{P}(V) \to \mathbb{P}(S^dV), \ [v] = [x_0e_0 + \dots x_ne_n] \to [v^d] = [(x_0e_0 + \cdots + x_ne_n)^d].$$

Moreover S^2V can be interpreted as the space of symmetric $(n+1) \times (n+1)$ matrices, and the Veronese variety $V_{n,2}$ as the subset of the symmetric matrices of rank one, because its equations express precisely the vanishing of the minors of order 2 (see Section 10.6).

Exercises 13.2.1. 1. Using Exercise 5 of Chapter 6, prove that, if $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ are irreducible projective varieties, then $X \times Y$ is irreducible.

2. Let L, M, N be the following lines in $\mathbb{P}^3$:

$$L : x_0 = x_1 = 0, M : x_2 = x_3 = 0, N : x_0 - x_2 = x_1 - x_3 = 0.$$

Let X be the union of lines meeting L, M and N: write equations for X and describe it: is it a projective variety? If yes, of what dimension and degree?

3. Let X, Y be quasi–projective varieties, identify $X \times Y$ with its image via the Segre map. Check that the two projection maps $X \times Y \xrightarrow{p_1} X$, $X \times Y \xrightarrow{p_2} Y$ are regular. (Hint: use the open covering of the Segre variety by the Σ^{ij}'s.)

Chapter 14

The dimension of an intersection

Our aim in this chapter is to investigate the dimension of the intersection of two algebraic varieties.

14.1 The theorem of the intersection

Theorem 14.1.1. *Let K be an algebraically closed field. Let $X, Y \subset \mathbb{P}^n$ be quasi–projective varieties. Assume that $X \cap Y \neq \emptyset$. Then if Z is any irreducible component of $X \cap Y$, then* $\dim Z \geq \dim X + \dim Y - n$.

To prove Theorem 14.1.1, the main ingredient will be the following theorem, known as "Krull's principal ideal theorem".

Theorem 14.1.2. *Let R be a noetherian ring, let $a \in R$ be a non-invertible element. Then, for any prime ideal $\mathcal{P} \subset R$, minimal over the ideal (a) generated by a, the height of $\mathcal{P}$ is at most 1, i.e. $ht\mathcal{P} \leq 1$. If moreover a is a non-zero divisor, then $ht\mathcal{P} = 1$.*

We postpone the proof of Theorem 14.1.2 to the end of this chapter and proceed to the proof of Theorem 14.1.1. It will be divided in three steps. Note first that, possibly passing to the closure, we can assume that X, Y are projective varieties. Then we can assume that $X \cap Y$ intersects $U_0 \simeq \mathbb{A}^n$, so, possibly after restricting X and Y to $\mathbb{A}^n$, we may work with irreducible closed subsets of the affine space. Put $r = \dim X$, $s = \dim Y$.

Step 1. Assume that $X = V(F)$ is an irreducible hypersurface, with F irreducible polynomial of $K[x_1, \dots, x_n]$. The irreducible components of $X \cap Y$ correspond, by the Nullstellensatz, to the minimal prime ideals containing $I(X \cap Y)$ in $K[x_1, \dots, x_n]$. We recall (Corollary 3.2.9) that $I(X \cap Y) = \sqrt{I(X) + I(Y)} = \sqrt{\langle I(Y), F \rangle}$. So those prime ideals are the minimal prime ideals over $\langle I(Y), F\rangle$. They correspond bijectively to the minimal prime ideals containing $\langle f\rangle$ in $\mathcal{O}(Y)$, where f is the regular function on Y defined by F. We distinguish two cases:

(i) if $Y \subset X = V(F)$, then $f = 0$ and $Y \cap X = Y$; since $s = \dim Y > r + s - n = (n-1) + s - n$, the theorem is easily true in this case;

(ii) if $Y \not\subset X$, then $f \neq 0$, moreover f is not invertible, otherwise $X \cap Y = \emptyset$: hence the minimal prime ideals over $\langle f\rangle$ in $\mathcal{O}(Y)$, which is an integral domain, have all height one by Theorem 14.1.2. So for all Z, irreducible component of $X \cap Y$, $\dim Z = \dim Y - 1 = r + s - n$ (Theorem 7.2.4).

Step 2. Assume that $I(X)$ is generated by $n - r$ polynomials (where $n - r$ is the codimension of X): $I(X) = \langle F_1, \dots, F_{n-r}\rangle$. Then we can argue by induction on $n - r$: we first intersect Y with $V(F_1)$, whose irreducible components are all hypersurfaces, and apply Step 1: all irreducible components of $Y \cap V(F_1)$ have dimension either s or $s - 1$. Then we intersect each of these components with $V(F_2)$, and so on. We conclude that every irreducible component Z has $\dim Z \geq \dim Y - (n - r) = r + s - n$.

Step 3. We use the isomorphism $\psi : X \cap Y \simeq (X \times Y) \cap \Delta_{\mathbb{A}^n}$ (see Exercise 1, Chapter 10). Note that $X \times Y$ is irreducible by Proposition 6.4.2. ψ preserves the irreducible components and their dimensions, so we consider instead of X and Y, the algebraic sets $X \times Y$ and $\Delta_{\mathbb{A}^n}$, contained in $\mathbb{A}^{2n}$. We have $\dim X \times Y = r + s$ (Proposition 7.2.7). $\Delta_{\mathbb{A}^n}$ is a linear subspace of $\mathbb{A}^{2n}$, so it satisfies the assumption of Step 2; indeed it has dimension n in $\mathbb{A}^{2n}$ and is defined by n linear equations. Hence, for all Z we have: $\dim Z \geq (r + s) + n - 2n = r + s - n$. □

The above theorem can be seen as a generalization of the Grassmann relation for linear subspaces. However, it is not an existence theorem, because it says nothing about $X \cap Y$ being non–empty. But for projective varieties, the following more precise version of the theorem holds:

Theorem 14.1.3. *Let $X, Y \subset \mathbb{P}^n$ be projective varieties of dimensions r, s. If $r + s - n \geq 0$, then $X \cap Y \neq \emptyset$.*

Proof. Let $C(X)$, $C(Y)$ be the affine cones associated to X and Y. Then $C(X) \cap C(Y)$ is certainly non–empty, because it contains the origin $O(0, 0, \dots, 0)$. Assume we know that $C(X)$ has dimension $r + 1$ and $C(Y)$ has dimension $s + 1$: then by Theorem 14.1.1 all the irreducible components Z of $C(X) \cap C(Y)$ have dimension $\geq (r + 1) + (s + 1) - (n + 1) = r + s - n + 1 \geq 1$, hence Z contains points different from O. These points give rise to points of $\mathbb{P}^n$ belonging to $X \cap Y$. The conclusion of the proof will follow from next proposition. □

Proposition 14.1.4. *Let $Y \subset \mathbb{P}^n$ be a projective variety.*

Then $\dim Y = \dim C(Y) - 1$*. If $S(Y)$ denotes the homogeneous coordinate ring, hence also* $\dim Y = \dim S(Y) - 1$.

Proof. Let $p : \mathbb{A}^{n+1} \setminus \{O\} \to \mathbb{P}^n$ be the canonical morphism. Let us recall that $C(Y) = p^{-1}(Y) \cup \{O\}$. Assume that $Y_0 := Y \cap U_0 \neq \emptyset$ and consider also $C(Y_0) = p^{-1}(Y_0) \cup \{O\}$. Then we have:

$$C(Y_0) = \{(\lambda, \lambda a_1, \dots, \lambda a_n) \mid \lambda \in K, (a_1, \dots, a_n) \in Y_0\}.$$

So we can define a birational map between $C(Y_0)$ and $Y_0 \times \mathbb{A}^1$ as follows:

$$(y_0, y_1, \dots, y_n) \in C(Y_0) \to ((y_1/y_0, \dots, y_n/y_0), y_0) \in Y_0 \times \mathbb{A}^1,$$

$$((a_1, \dots, a_n), \lambda) \in Y_0 \times \mathbb{A}^1 \to (\lambda, \lambda a_1, \dots, \lambda a_n) \in C(Y_0).$$

Therefore $\dim C(Y_0) = \dim(Y_0 \times \mathbb{A}^1) = \dim Y_0 + 1$. To conclude, it is enough to remark that $\dim Y = \dim Y_0$ and $\dim C(Y) = \dim C(Y_0) = \dim S(Y)$. $\square$

We observe that also $C(Y)$ and $Y \times \mathbb{P}^1$ are birationally equivalent.

Corollary 14.1.5. *1. If $X, Y \subset \mathbb{P}^2$ are projective curves over an algebraically closed field, then $X \cap Y \neq \emptyset$.*
2. $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$.

Proof. 1. is a straightforward application of Theorem 14.1.3.

To prove 2., assume by contradiction that $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ is an isomorphism. Let L, L' be two skew lines in $\mathbb{P}^1 \times \mathbb{P}^1$; since φ is an isomorphism, then $\varphi(L)$, $\varphi(L')$ are rational disjoint curves in $\mathbb{P}^2$, but this contradicts 1. $\square$

If $X, Y \subset \mathbb{P}^n$ are varieties of dimensions r, s, then $r+s-n$ is called the *expected dimension* of $X \cap Y$. If all irreducible components Z of $X \cap Y$ have the expected dimension, then we say that the intersection $X \cap Y$ is *proper* or that *X and Y intersect properly.*

For example, two plane projective curves X, Y intersect properly if they don't have any common irreducible component. In this case, it is possible to predict the number of points of intersections. Precisely, it is possible to associate to every point $P \in X \cap Y$ a number $i(P; X, Y)$, called the *multiplicity of intersection of X and Y at P*, in such a way that

$$\sum_{P \in X \cap Y} i(P; X, Y) = dd',$$

where d is the degree of X and d' is the degree of Y. This result is the Theorem of Bézout, and is the first result of the branch of algebraic geometry called Intersection Theory. For a proof of the Theorem of Bézout, see for instance the classical [W], or [F].

14.2 Complete intersections

Let X be a closed subvariety of $\mathbb{P}^n$ (resp. of $\mathbb{A}^n$) of codimension r. X is called a *complete intersection* if $I_h(X)$ (resp. $I(X)$) is generated by r polynomials, the minimum possible number.

Hence, if X is a complete intersection of codimension r, then X is certainly the intersection of r hypersurfaces. Conversely, if X is intersection of r hypersurfaces, then, by Theorem 14.1.1, using induction, we deduce that $\dim X \geq n-r$; even assuming equality, we cannot conclude that X is a complete intersection, but simply that $I(X)$ is the radical of an ideal generated by r polynomials.

Example 14.2.1. *The skew cubic (again).*

Let $X \subset \mathbb{P}^3$ be the skew cubic. The homogeneous ideal of X is generated by the three polynomials F_1, F_2, F_3, the 2×2–minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix},$$

which are linearly independent polynomials of degree 2. Note that $I_h(X)$ does not contain any linear polynomial, because X is not contained in any hyperplane, and that the homogeneous component of minimal degree 2 of $I_h(X)$ is a vector space of dimension 3. Hence $I_h(X)$ cannot be generated by two polynomials, i.e. X is not a complete intersection.

Nevertheless, X is the intersection of the surfaces $V_P(F)$, $V_P(G)$, where

$$F = F_1 = \begin{vmatrix} x_0 & x_1 \\ x_1 & x_2 \end{vmatrix} \text{ and } G = \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_0 \end{vmatrix}.$$

Indeed, clearly $F, G \in I_h(X)$ so $X \subset V_P(F) \cap V_P(G)$. Conversely, observe that

$$G = x_0F - x_3(x_0x_3 - x_1x_2) + x_2(x_1x_3 - x_2^2) = x_0F_1 - x_3F_2 + x_2F_3.$$

If $P[x_0, \ldots, x_3] \in V_P(F) \cap V_P(G)$, then P is a zero also of $G - x_0F = x_0x_3^2 - 2x_1x_2x_3 + x_2^3$, and therefore also of

$$x_2(x_0x_3^2 - 2x_1x_2x_3 + x_2^3) = x_1^2x_3^2 - 2x_1x_2^2x_3 + x_2^4 = (x_1x_3 - x_2^2)^2 = F_3^2,$$

because $x_0x_2 = x_1^2$. Hence P is a zero also of $F_3 = x_1x_3 - x_2^2$. So P annihilates $G - x_0F - x_2F_3 = x_3(x_0x_3 - x_1x_2) = x_3F_2$. If P satisfies the equation $x_3 = 0$, then it satisfies also $x_2 = 0$ and $x_1 = 0$, therefore $P = [1, 0, 0, 0] \in X$. If $x_3 \neq 0$, then $P \in V_P(F_1, F_2, F_3) = X$.

The geometric description of this phenomenon is that the skew cubic X is the set-theoretic intersection of a quadric and a cubic, which are tangent along X, so their intersection is X "counted with multiplicity 2".

This example motivates the following definition: X is a *set–theoretic complete intersection* if $\operatorname{codim} X = r$ and the ideal of X is the radical of an ideal generated by r polynomials. It is an open problem if all irreducible curves of $\mathbb{P}^3$ are set–theoretic complete intersections. For more details, see [K].

14.3 Krull's principal ideal theorem

We conclude this chapter with the proof of Krull's principal ideal Theorem 14.1.2.

Proof. Let $\mathcal{P}$ be a prime ideal, minimal among those containing (a), let $R_{\mathcal{P}}$ be the localization. Then $ht\mathcal{P} = \dim R_{\mathcal{P}}$, because of the bijection between prime ideals of $R_{\mathcal{P}}$ and prime ideals of R contained in $\mathcal{P}$. Moreover $\mathcal{P}R_{\mathcal{P}}$ is a minimal prime ideal over $aR_{\mathcal{P}}$, the ideal generated by a in $R_{\mathcal{P}}$. So, we can replace the ring R with its localization $R_{\mathcal{P}}$, or, in other words, we can assume that R is a local ring and that its maximal ideal $\mathcal{M}$ is minimal over (a).

It is enough to prove that, for any prime ideal $\mathcal{Q}$ of R, with $\mathcal{Q} \neq \mathcal{M}$, we have $ht\mathcal{Q} = 0$. Indeed this will imply $ht\mathcal{M} \leq 1$. Let $j : R \to R_{\mathcal{Q}}$ be the natural homomorphism. For any integer i, $i \geq 1$, we consider $\mathcal{Q}^i$, and its saturation with $\mathcal{Q}$: $\mathcal{Q}^{(i)} := j^{-1}(\mathcal{Q}^i R_{\mathcal{Q}})$, called the i-th symbolic power of $\mathcal{Q}$. It is $\mathcal{Q}$-primary. We have $\mathcal{Q}^i \subset \mathcal{Q}^{(i)}$ and

$$\mathcal{Q} = \mathcal{Q}^{(1)} \supseteq \mathcal{Q}^{(2)} \supseteq \cdots \supseteq \mathcal{Q}^{(i)} \supseteq \ldots .$$

We also have

$$(a) + \mathcal{Q} \supseteq (a) + \mathcal{Q}^{(2)} \supseteq \cdots \supseteq (a) + \mathcal{Q}^{(i)} \supseteq \ldots . \tag{14.1}$$

We observe that in $R/(a)$ there is only one prime ideal, $\mathcal{M}/(a)$, because R is local and $\mathcal{M}$ is minimal over (a), therefore $R/(a)$ has dimension 0; since it is noetherian of dimension 0, $R/(a)$ is artinian, and we can conclude that the chain of ideals (14.1) is stationary, so there exists an integer n such that $(a) + \mathcal{Q}^{(n)} = (a) + \mathcal{Q}^{(n+1)}$.

Let $q \in \mathcal{Q}^{(n)}$: so $q \in (a) + \mathcal{Q}^{(n+1)}$, and it can be written in the form $q = ra + q'$, where $r \in R$, $q' \in \mathcal{Q}^{(n+1)} \subset \mathcal{Q}^{(n)}$. Therefore $ra = q - q' \in \mathcal{Q}^{(n)}$; but $a \notin \mathcal{Q}$ (because $\mathcal{M}$ is minimal over (a)), and $\mathcal{Q}^{(n)}$ is $\mathcal{Q}$-primary, so $r \in \mathcal{Q}^{(n)}$. We conclude that $\mathcal{Q}^{(n)} = a\mathcal{Q}^{(n)} + \mathcal{Q}^{(n+1)}$.

We can apply now Nakayama's lemma (Theorem 14.3.1 below), and get $\mathcal{Q}^{(n)} = \mathcal{Q}^{(n+1)}$. Therefore $\mathcal{Q}^n R_{\mathcal{Q}} = \mathcal{Q}^{n+1} R_{\mathcal{Q}}$. We apply Nakayama's lemma again, and we conclude that

$\mathcal{Q}^n R_{\mathcal{Q}} = (0)$. So every element of the maximal ideal $\mathcal{Q}R_{\mathcal{Q}}$ of $R_{\mathcal{Q}}$ is nilpotent, which implies that $ht\mathcal{Q}R_{\mathcal{Q}} = 0$.

We recall here the statement of Nakayama's lemma.

Theorem 14.3.1. *Let $I \subset R$ be an ideal contained in the Jacobson radical of R (the intersection of the maximal ideals). Let M be a finitely generated R-module, let $N \subset M$ be a submodule.*

If $M = N + IM$, then $M = N$.

We have applied Nakayama's lemma the first time in the situation where R is a local ring and $I = (a) \subset \mathcal{M}$, which is the Jacobson radical of R. The R-module M is $\mathcal{Q}^{(n)}$ and its submodule N is $\mathcal{Q}^{(n+1)}$. The second time, we are instead in the situation where the ring is $R_{\mathcal{Q}}$, $I = \mathcal{Q}R_{\mathcal{Q}}$, the module M is $\mathcal{Q}^n R_{\mathcal{Q}}$ and N is (0).

To conclude the proof of the theorem, we observe that the second assertion follows from the first one, because if $\mathcal{P}$ is a prime ideal of height zero, all its elements are zero-divisors. Indeed, let $r \in \mathcal{P}$, $r \neq 0$; we can find an element $t \notin \mathcal{P}$ belonging to the intersection $\cap_i \mathcal{P}_i$ of the prime ideals of height zero different from $\mathcal{P}$ (there is a finite number of such ideals because R is noetherian). Otherwise $\mathcal{P} \subset \cap_i \mathcal{P}_i$, but this would imply $\mathcal{P} \subset \mathcal{P}_i$ for some i. Now observe that rt belongs to the intersection of all minimal prime ideals of R, so rt is nilpotent: there exists $\alpha \geq 0$ such that $(rt)^\alpha = 0$. Since $t \notin \mathcal{P}$, it is not nilpotent, so $t^\alpha \neq 0$. Hence there is a minimum $\beta \geq 0$ such that $r^\beta t^\alpha \neq 0$ but $r^{\beta+1} t^\alpha = r(r^\beta t^\alpha) = 0$. This proves that r is a zero-divisor. $\square$

Exercises 14.3.2. 1. Let $X \subset \mathbb{P}^2$ be the union of three points not lying on a line. Prove that the homogeneous ideal of X cannot be generated by two polynomials.

Chapter 15

Complete varieties

We work over an algebraically closed field K.

In this chapter, we will prove that the algebra of regular functions $\mathcal{O}(X)$ of an irreducible projective variety X is the base field K, i.e. that the only regular functions on X are the constants. We will obtain this theorem as a consequence of the theorem of completeness of projective varieties. The property of a variety to be complete can be seen as an analogue of compactness in the context of algebraic geometry.

15.1 Complete varieties

Definition 15.1.1. Let X be a quasi–projective variety. X is *complete* if, for any quasi–projective variety Y, the natural projection on the second factor $p_2 : X \times Y \to Y$ is a closed map.

Note that both projections p_1, p_2 are morphisms: see Exercise 3, Chapter 14.

We recall that a topological space X is compact if and only if the above projection map is closed with respect to the product topology. Here the product variety $X \times Y$ does not carry the product topology but the Zariski topology, that is in general strictly finer (Proposition 2.4.1).

Example 15.1.2. *The affine line $\mathbb{A}^1$ is not complete: let $X = Y = \mathbb{A}^1$, $p_2 : \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2 \to \mathbb{A}^1$ is the map such that $(x_1, x_2) \to x_2$. Then $Z := V(x_1x_2 - 1)$ is closed in $\mathbb{A}^2$ but $p_2(Z) = \mathbb{A}^1 \setminus \{O\}$ is not closed.*

Proposition 15.1.3. *(i) If $f : X \to Y$ is a regular map and X is complete, then $f(X)$ is a closed complete subvariety of Y.*

(ii) If X is complete, then all closed subvarieties of X are complete.

Proof. (i) Let $\Gamma_f \subset X \times Y$ be the graph of f: $\Gamma_f = \{(x, f(x)) \mid x \in X\}$. It is clear that $f(X) = p_2(\Gamma_f)$, so to prove that $f(X)$ is closed it is enough to check that Γ_f is closed in $X \times Y$. Let us consider the diagonal of Y: $\Delta_Y = \{(y, y) \mid y \in Y\} \subset Y \times Y$. If $Y \subset \mathbb{P}^n$, then $\Delta_Y = \Delta_{\mathbb{P}^n} \cap (Y \times Y)$, so it is closed in $Y \times Y$, because $\Delta_{\mathbb{P}^n}$ is the closed subset defined in $\Sigma_{n,n}$ by the equations $w_{ij} - w_{ji} = 0$, $i, j = 0, \dots, n$. There is a natural map $f \times 1_Y : X \times Y \to Y \times Y$, $(x, y) \to (f(x), y)$, such that $(f \times 1_Y)^{-1}(\Delta_Y) = \Gamma_f$. It is easy to see that $f \times 1_Y$ is regular, so Γ_f is closed, so also $f(X)$ is closed.

Let now Z be any variety and consider $p_2 : f(X) \times Z \to Z$ and the regular map $f \times 1_Z : X \times Z \to f(X) \times Z$. There is a commutative diagram:

$$\begin{array}{ccc} X \times Z & \xrightarrow{p_2'} & Z \\ \downarrow {\scriptstyle f \times 1_Z} & \nearrow & {\scriptstyle p_2} \\ f(X) \times Z & & \end{array}$$

If $T \subset f(X) \times Z$, then $(f \times 1_Z)^{-1}(T)$ is closed and $p_2(T) = p_2'((f \times 1_Z)^{-1}(T))$ is closed because X is complete. We conclude that $f(X)$ is complete.

(ii) Let $T \subset X$ be a closed subvariety and Y be any variety. We have to prove that $p_2 : T \times Y \to Y$ is closed. If $Z \subset T \times Y$ is closed, then Z is closed also in $X \times Y$, hence $p_2(Z)$ is closed because X is complete. ☐

Corollary 15.1.4. *1. If X is a complete variety, then $\mathcal{O}(X) \simeq K$.*

2. If X is an affine complete irreducible variety, then X is a point.

Proof. 1. If $f \in \mathcal{O}(X)$, f can be interpreted as a regular map $f : X \to \mathbb{A}^1$. By Proposition 15.1.3, (i), $f(X)$ is a closed complete subvariety of $\mathbb{A}^1$, which is not complete. Hence $f(X)$ has dimension < 1 and is irreducible, hence it is a point, so $f \in K$.

2. By part 1., $\mathcal{O}(X) \simeq K$. But $\mathcal{O}(X) \simeq K[x_1, \dots, x_n]/I(X)$, hence $I(X)$ is maximal. By the Nullstellensatz, X is a point. ☐

15.2 Completeness of projective varieties

Before stating Theorem 15.2.2 of completeness of projective varieties, we give a characterization of the closed subsets of a biprojective space $\mathbb{P}^n \times \mathbb{P}^m$, that will be needed in its proof. It is expressed in terms of equations in two series of variables, corresponding to the homogeneous coordinates $[x_0, \dots, x_n]$ on $\mathbb{P}^n$ and $[y_0, \dots, y_m]$ on $\mathbb{P}^m$.

Let $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ be the Segre map. A closed subvariety X in $\mathbb{P}^N$ is defined by finitely many equations $F_k(w_{00}, \ldots, w_{nm})$, where the F_k are homogeneous polynomials in the w_{ij}. On the subvariety $X \cap \Sigma$, where Σ is the Segre variety, we have $w_{ij} = x_i y_j$, so we can make this substitution in F_k and get equations $G_k(x_0, \ldots, x_n; y_0, \ldots, y_m) = 0$, where $G_k = F_k(x_0 y_0, \ldots, x_n y_m)$: they are equations characterizing the subset $\sigma^{-1}(X)$. Note that each G_k is homogeneous in each set of variables x_i and y_j, and of the same degree in both.

Conversely, it is easy to see that a polynomial with this property of bihomogeneity can always be written as a polynomial in the products $x_i y_j$, and the possible ambiguity depending on the choice disappears in view of the equations of the Segre variety. So it describes a subset of $\mathbb{P}^n \times \mathbb{P}^m$ whose image in σ is closed. However, equations that are bihomogeneous in x_i and y_j always define an algebraic closed subvariety of $\mathbb{P}^n \times \mathbb{P}^m$ even if the degrees of homogeneity in the two sets of variables are different. Indeed if $G(x_0, \ldots, x_n; y_0, \ldots, y_m)$ has degree r in x_i and s in y_j, and for instance $r > s$, then the equation $G = 0$ is equivalent to the system of equations $y_i^{r-s} G = 0$, $i = 0, \ldots, m$, and these define an algebraic variety.

We will need the answer to the analogous question also for the product $\mathbb{P}^n \times \mathbb{A}^m$. Let us assume that $\mathbb{A}^m = U_0 \subset \mathbb{P}^m$, defined by $y_0 \neq 0$. If we have a closed subset of $\mathbb{P}^n \times \mathbb{P}^m$ defined by equations $G_k(x_0, \ldots, x_n; y_0, \ldots, y_m) = 0$, with G_k homogeneous of degree r_k in y_j, dividing by $y_0^{r_k}$ and setting $v_j = y_j / y_0$, we get equations $g_k(x_0, \ldots, x_n; v_1, \ldots, v_m) = 0$ that are homogeneous in the x_i and in general non-homogeneous in the v_j.

These observations can be collected in the following result.

Theorem 15.2.1. *A subset $X \subset \mathbb{P}^n \times \mathbb{P}^m$ is a closed algebraic subvariety if and only if it is defined by a system of equations $G_k(x_0, \ldots, x_n; y_0 \ldots, y_m) = 0$, homogeneous separately in each set of variables. Every closed algebraic subvariety of $\mathbb{P}^n \times \mathbb{A}^m$ is defined by a system of equations $g_k(x_0, \ldots, x_n; v_1, \ldots, v_m) = 0$ that are homogeneous in $x_0, \ldots, x_n$.*

Theorem 15.2.2. *Let $X \subset \mathbb{P}^n$ be a projective irreducible variety over an algebraically closed field K. Then X is complete.*

Proof. (see [S], Theorem 3, Ch.1, §5)

1. It is enough to prove that $p_2 : \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ is closed, for any positive n, m. This can be observed by using the local character of closedness and the existence of an affine open covering of any quasi–projective varieties.

Indeed, let us assume first that $p_2 : \mathbb{P}^n \times Y \to Y$ is a closed map for any quasi-projective variety Y. We observe that $X \times Y$ is closed in $\mathbb{P}^n \times Y$, because X is closed in $\mathbb{P}^n$. So, if $Z \subset X \times Y$ is closed, it is also closed in $\mathbb{P}^n \times Y$, which implies that $p_2(Z)$ is closed in Y. So we can replace X with $\mathbb{P}^n$.

Secondly, since being closed is a local property, it is enough to cover Y by affine open subsets U_i, and prove the theorem for each of them. Hence we can assume that Y is an affine variety. Finally, if $Y \subset \mathbb{A}^m$ is closed, then $\mathbb{P}^n \times Y$ is closed in $\mathbb{P}^n \times \mathbb{A}^m$, so it is enough to prove the theorem in the particular case $X = \mathbb{P}^n$ and $Y = \mathbb{A}^m$.

2. If $x_0, \dots, x_n$ are homogeneous coordinates on $\mathbb{P}^n$ and $y_1, \dots, y_m$ are non-homogeneous coordinates on $\mathbb{A}^m$, then any closed subvariety of $\mathbb{P}^n \times \mathbb{A}^m$ can be characterised as the set of common zeros of a set of polynomials in the variables $x_0, \dots, x_n, y_1, \dots, y_m$, homogeneous in the first group of variables $x_0, \dots, x_n$ (Theorem 15.2.1).

3. Let $Z \subset \mathbb{P}^n \times \mathbb{A}^m$ be closed. Then Z is the set of solutions of a system of equations

$$\{G_i(x_0, \dots, x_n; y_1, \dots, y_m) = 0, i = 1, \dots, t,$$

where G_i is homogeneous in the x's. A point $P(\overline{y}_1, \dots, \overline{y}_m)$ is in $p_2(Z)$ if and only if the system

$$\{G_i(x_0, \dots, x_n; \overline{y}_1, \dots, \overline{y}_m) = 0, i = 1, \dots, t,$$

has a solution in $\mathbb{P}^n$, i.e. if the ideal of $K[x_0, \dots, x_n]$ generated by $G_1(x; \overline{y})$,..., $G_t(x; \overline{y})$ has at least one zero in $\mathbb{P}^n$. Hence

$$\begin{aligned} p_2(Z) &= \{(\overline{y}_1, \dots, \overline{y}_m) \in \mathbb{A}^m | \ \forall\ d \geq 1\ \langle G_1(x; \overline{y}), \dots, G_t(x; \overline{y})\rangle \not\supset K[x_0, \dots, x_n]_d\} \\ &= \textstyle\bigcap_{d \geq 1} \{(\overline{y}_1, \dots, \overline{y}_m) | \ \langle G_1(x; \overline{y}), \dots, G_t(x; \overline{y})\rangle \not\supset K[x_0, \dots, x_n]_d\} = \cap_{d \geq 1} T_d, \end{aligned} \tag{15.1}$$

where $T_d = \{(\overline{y}_1, \dots, \overline{y}_m) | \ \langle G_1(x; \overline{y}), \dots, G_t(x; \overline{y})\rangle \not\supset K[x_0, \dots, x_n]_d\}$. To conclude the proof of the theorem it is enough to prove that T_d is closed in $\mathbb{A}^m$ for any $d \geq 1$.

Let $\{M_\alpha\}_{\alpha = 1, \dots, \binom{n+d}{d}}$ be the set of the monomials of degree d in $K[x_0, \dots, x_n]$; let $d_i = \deg G_i(x; \overline{y})$, let $\{N_i^\beta\}_\beta$ be the set of the monomials of degree $d - d_i$.

Note that $P(\overline{y}_1, \dots, \overline{y}_m) \notin T_d$ if and only if $M_\alpha = \sum_i G_i(x; \overline{y}) F_{i,\alpha}(x_0, \dots, x_n)$, for all α and for suitable polynomials $F_{i,\alpha}$ homogeneous of degree $d - d_i$. So $P \notin T_d$ if and only if, for all index α, M_α is a linear combination of the polynomials $\{G_i(x; \overline{y}) N_i^\beta\}$, i.e. the matrix A of the coordinates of the polynomials $G_i(x; \overline{y}) N_i^\beta$ with respect to the basis $\{M_\alpha\}$ has maximal rank $\binom{n+d}{d}$. So T_d is the set of zeros of the minors of a fixed order of the matrix A, hence it is closed. □

Corollary 15.2.3. *Let X be a projective variety. Then $\mathcal{O}(X) \simeq K$.*

Corollary 15.2.4. *Let X be a projective variety, let $\varphi : X \to Y \subset \mathbb{P}^n$ be any regular map. Then $\varphi(X)$ is a projective variety. In particular, if $X \simeq Y$, then Y is projective.*

Corollary 15.2.4 says that the notion of projective variety, differently from that of affine variety, is invariant by isomorphism, i.e. quasi-projective varieties that are isomorphic to projective varieties are already projective.

In algebraic terms, Theorem 15.2.2 can be seen as a result in Elimination Theory. Indeed it can be reformulated by saying that, given a system of algebraic equations in two sets of variables, $x_0, \dots, x_n$ and $y_1, \dots, y_m$, homogeneous in the first ones, it is possible to find another system of algebraic equations only in $y_1, \dots, y_m$, such that $\bar{y}_1, \dots, \bar{y}_m$ is a solution of the second system if and only if there exist $\bar{x}_0, \dots, \bar{x}_n$, that, together with $\bar{y}_1, \dots, \bar{y}_m$, are a solution of the first system. In other words, it is possible to eliminate a set of homogeneous variables from any system of algebraic equations.

Example 15.2.5. Let $S = K[x_0, \dots, x_n]$. Let $d \geq 1$ be an integer number and consider S_d, the vector space of homogeneous polynomials of degree d. As an application of Theorem 15.2.2, we shall prove that the set of (proportionality classes of) reducible polynomials is a projective algebraic set in $\mathbb{P}(S_d)$.

We denote by $X \subset \mathbb{P}(S_d)$ the set of reducible polynomials. For any integer k, $0 < k < d$, let $X_k \subseteq X$ be the set of polynomials of the form F_1F_2 with $\deg F_1 = k, \deg F_2 = d-k$. Then $X = \bigcup_{k=1}^{d-1} X_k$. Let $f_k : \mathbb{P}(S_k) \times \mathbb{P}(S_{d-k}) \to \mathbb{P}(S_d)$ be the multiplication of polynomials, i.e. $f_k([F_1],[F_2]) = [F_1F_2]$. f_k is clearly a regular map, and its image is $X_k = X_{d-k}$. Since the domain is a projective variety, and precisely a Segre variety, it follows from Theorem 15.2.2 that also X_k is projective.

In the special case $d = 2$, the quadratic polynomials, the equations of $X = X_1$ are the minors of order 3 of the matrix associated to the quadric.