May, 6th Fourier transform of L^2 functions.

def. $S \in \mathcal{S}'(\mathbb{R}^d)$, $\widehat{S}(f) = S(\hat{f})$.

rem. if $f \in L^2(\mathbb{R}^d) \Rightarrow T_f \in \mathcal{S}'(\mathbb{R}^d)$ so $\boxed{\widehat{T_f} \in \mathcal{S}'}$

$$\left(T_f(g) = \int_{\mathbb{R}^d} fg \right)$$

Lemma let $f, g \in \mathcal{S}(\mathbb{R}^d)$

then i) $\int_{\mathbb{R}^d} f\hat{g} = \int_{\mathbb{R}} \hat{f} g$

ii) $\int_{\mathbb{R}} f \bar{g} = (2\pi)^{-n} \int \hat{f}\,\overline{\hat{g}}$

$\boxed{\text{iii)}\ \widehat{fg} = (2\pi)^{-n} \hat{f} * \hat{g}}$

iv) $\widehat{f*g} = \hat{f} \cdot \hat{g}$.

proof. i) $(x, \eta) \mapsto f(x) g(\eta) \in L^1(\mathbb{R}^{2n})$

$$\int_{\mathbb{R}^d} f(x) \hat{g}(x)\, dx = \underbrace{\int_{\mathbb{R}^d} f(x) \left(\int_{\mathbb{R}} e^{-ix\eta} g(\eta)\, d\eta \right)}_{\text{no problem to exchange the order of integration}} dx$$

$$= \int g(\eta) \left(\int e^{-ix\eta} f(x)\, dx \right) d\eta = \int_{\mathbb{R}^d} \hat{f}(\eta) g(\eta)\, d\eta$$

conjugation in $\mathbb{C}$

ii) $$\int_{\mathbb{R}^d} f(x) \overline{g(x)}\, dx = \int_{\mathbb{R}^d} f(x) \cdot \overline{\frac{1}{(2\pi)^n} \int_{\mathbb{R}^d} e^{ix\xi} \hat{g}(\xi)\, d\xi} \, dx$$

$$= \int_{\mathbb{R}^d} f(x) \frac{1}{(2\pi)^n} \int e^{-ix\xi}\, \overline{\hat{g}}(\xi)\, d\xi\, dx$$

$$= \int_{\mathbb{R}^d} \overline{\hat{g}}(\xi) \cdot \frac{1}{(2\pi)^n} \underbrace{\left(\int e^{-ix\xi} f(x)\, dx \right)}_{\hat{f}(\xi)} d\xi$$

iii) $$\int_{\mathbb{R}^d} e^{-ix\xi} f(x) g(x)\, dx = \int_{\mathbb{R}^d} e^{-ix\xi} f(x) \left(\frac{1}{(2\pi)^n} \int e^{ix\eta} \hat{g}(\eta)\, d\eta \right) dx$$

$$= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^d} \hat{g}(\eta) \left(\int e^{-ix(\xi - \eta)} f(x)\, dx \right) d\eta$$

$$= \frac{1}{(2\pi)^n} \int \hat{g}(\eta) \hat{f}(\xi - \eta)\, d\eta = \frac{1}{(2\pi)^n} \hat{g} * \hat{f}\,(\xi)$$

Theorem (PLANCHEREL)

Let $f \in L^2(\mathbb{R}^d)$. There exists a unique $g \in L^2(\mathbb{R}^d)$ s.t. $\widehat{T_f} = T_g$

g will be called the Fourier transform of f ($\mathcal{F}(f)$ or $\hat{f}$)

Moreover $\|g\|_{L^2} = (2\pi)^{d/2} \|f\|_{L^2}$

$$\|g\|_{L^2} = \left(\|\mathcal{F}(f)\|_{L^2}\right) = \|\hat{f}\|_{L^2}$$

~~proof~~. We know that $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$ (actually $C_0^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$)

so if $f \in L^2(\mathbb{R}^d)$ $\exists (\varphi_n)_n$ in $\mathcal{D}(\mathbb{R}^d)$ ($\subseteq \mathcal{S}(\mathbb{R}^d)$) such that $\lim_n \|\varphi_n - f\|_{L^2} = 0$

the sequence $(\varphi_n)_n$ is a Cauchy sequence in $L^2(\mathbb{R}^d)$

I consider $(\hat{\varphi}_n)_n$ (mind that $\hat{\varphi}_n \in \mathcal{S}(\mathbb{R}^d)$)

and $\|\hat{\varphi}_n - \hat{\varphi}_m\|_{L^2} = \underbrace{\|\widehat{\varphi_n - \varphi_m}\|_{L^2} = (2\pi)^d \|\varphi_n - \varphi_m\|_{L^2}}_{\text{use point ii) in previous lemma}}$

$$\left(\int f\bar{g} \overset{?}{=} \int \hat{f}\bar{\hat{g}} \Rightarrow \int f\bar{f} =? \int \hat{f}\bar{\hat{f}} \right)$$

$$\|f\|_{L^2}^2 =? \|\hat{f}\|_{L^2}^2$$

$(\hat{\varphi}_n)_n$ is Cauchy in $L^2 \Rightarrow \exists g \in L^2$ st $\lim_n \hat{\varphi}_n = g$ in L^2.

I prove that $T_g = \widehat{T_f}$

remark that if $\varphi_n \to f$ in L^2 then $\varphi_n \rightharpoonup f$ (weak convergence)

this means that $T_{\varphi_n}(\psi) = \int \varphi_n \psi \longrightarrow \int f \psi = T_f \psi$

so that $T_g(\psi) = \lim_n \int \hat{\varphi}_n \psi \overset{\text{lemma}}{=} \lim_n \int \varphi_n \hat{\psi} = T_f(\hat{\psi}) = \widehat{T_f}(\psi)$

So g is unique since $T_g(\psi) = \widehat{T_f}(\psi)$

if $T_{g_1}(\psi) = T_{g_2}(\psi) \Rightarrow \underbrace{\int (g_1 - g_2)\psi = 0 \quad \forall \psi \in \mathcal{S}}$

$\Downarrow$

$g_1 = g_2$ a.e.

QED

Ex. Let $f, g \in L^2(\mathbb{R}^d)$

then $\int_{\mathbb{R}^d} f(x)\hat{g}(x)\,dx = \int_{\mathbb{R}^d} \hat{f}(x) g(x)\,dx$?

solution

idea: $\int f(x) \left(\underbrace{\int e^{-ix\xi} g(\xi)\,d\xi}_{\text{? no sense !!}}\right) dx$ and exchange the order of integration

NO

OK only if $g \in L^1$

then through approximation

$(\varphi_n)_n$ in $\mathcal{S}(\mathbb{R}^d)$, $(\psi_n)_n$ in $\mathcal{S}(\mathbb{R}^d)$

$\varphi_n \to f$ in L^2, $\psi_n \to g$ in $L^2 \iff \hat{\psi}_n \to \hat{g}$ in L^2

$\iff \hat{\varphi}_n \to \hat{f}$ in L_2

and $\int \varphi_n \hat{\psi}_n = \int \hat{\varphi}_n \psi_n$

$\downarrow_n \int f\hat{g} \qquad \downarrow_n \int \hat{f} g$

Fourier-Laplace transform of a distribution with compact support.

def. Let $S \in \mathcal{E}'(\mathbb{R}^d)$ (dist. with compact support)

Fix $\xi \in \mathbb{R}^d$ consider $x \longmapsto \psi_\xi(x) = e^{-ix\cdot\xi}$

$$= \underbrace{\cos(x\cdot\xi) - i\sin(x\cdot\xi)}$$

it is a $\mathcal{E}^\infty(\mathbb{R}^d)$ as function of x

I define the function

$$\xi \longmapsto S(\psi_\xi)$$

I denote this function as the Fourier-Laplace transf. of S

$$\widehat{S}^{FL}$$

Rem. let $f \in \mathcal{E}_0(\mathbb{R}^d)$ (f cont. with compact support)

let T_f be the distribution associated to f

then $T_f \in \mathcal{E}'(\mathbb{R}^d)$

$$\widehat{T_f}^{FL} = \widehat{f}$$

F-L. trans, — traditional Fourier transform as an L^1 function

In fact $\widehat{T_f}^{FL}(\xi) = T_f(\psi_\xi) = \int_{\mathbb{R}^d} f(x)\,\psi_\xi(x)\,dx = \underbrace{\int f(x)\,e^{-i(x\cdot\xi)}\,dx}_{=\widehat{f}(\xi)}$

moreover $\xi \longmapsto \widehat{T_f}^{FL}(\xi)$ is a $\mathcal{E}^\infty$ function (next theorem)

$\Rightarrow$ it is a L^1_{loc} function

so $T_{\widehat{T_f}^{FL}} = \widehat{T_f}$

distribution associated to the F-L trans of f

$$T_{\widehat{T_f}^{FL}}(\varphi) = \int \widehat{T_f}^{FL}(\xi)\,\varphi(\xi) = \int \widehat{f}(\xi)\,\varphi(\xi) = \int f(\xi)\,\widehat{\varphi}(\xi) = T_f(\widehat{\varphi}) = \widehat{T_f}(\varphi)$$

I can exchange since

$(x,\xi) \longmapsto f(x)\varphi(\xi)$ is in $L^1(\mathbb{R}^{2d})$

Th. Let $S \in \mathcal{E}'(\mathbb{R}^d)$

then $\widehat{S}^{FL}$ is a $\mathcal{C}^\infty$ function.

proof (idea)

I prove that $\widehat{S}^{FL}$ is continuous

let $\bar{\xi} \in \mathbb{R}^d$ with $\xi_n \xrightarrow{n} \bar{\xi}$

$\widehat{S}^{FL}(\xi_n) = S(\psi_{\xi_n})$ where $\psi_{\xi_n} : x \longmapsto e^{-i x \cdot \xi_n}$

$x \longmapsto \cos(x \cdot \xi_n) - i \sin(x \cdot \xi_n)$

we remark that $\lim_n \underbrace{\cos(x \cdot \xi_n) - i \sin(x \cdot \xi_n)}_{f_n(x)} = \underbrace{\cos(x \cdot \bar{\xi}) - i \sin(x \cdot \bar{\xi})}_{\bar{f}(x)}$

$\therefore f_n \to \bar{f}$ uniformly on compact sets

$|\cos(x \cdot \xi_n) - \cos(x \cdot \bar{\xi})| \leq |x| \, |\xi_n - \bar{\xi}|$

similarly $\partial^\alpha f_n \to \partial^\alpha \bar{f}$ uniformly on compact sets

so $\psi_{\xi_n} \longrightarrow \psi_{\bar{\xi}}$ in the sense of $\mathcal{E}(\mathbb{R}^d)$

then $S(\psi_n) \longrightarrow S(\psi_{\bar{\xi}})$

similarly for the derivatives of $\widehat{S}^{FL}$.

def. Let $f \in \mathcal{E}^\infty(\Omega)$ (Ω open set in $\mathbb{R}^d$)

f is analytic if $\forall x_0 \in \Omega$, $\exists r > 0$

s.t. $\forall x \in B(x_0, r)$

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} \frac{\partial^\alpha f(x_0)}{\alpha!}(x - x_0)^\alpha$$

if $f \in \mathcal{E}^\infty(\mathbb{R}^d)$

f is entire analytic if $\forall x \in \mathbb{R}^d$ (can be extended to $\mathbb{C}^n$)

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} \frac{\partial^\alpha f(0)}{\alpha!} x^\alpha \qquad z^\alpha$$

Lemma. Let $f \in \mathcal{E}^\infty(\mathbb{R}^d)$ suppose that for all $R > 0$ $\exists\, C_R, M_\alpha$ s.t.

$$\sup_{|x| \le R} |\partial^\alpha f(x)| \le C_R M_R^{|\alpha|}$$

Then f is entire analytic (book of Cartan)

Lemma (Weierstrass th. on convergence of analytic functions) Let $(f_n)_n$ be a sequence of entire analytic function on $\mathbb{R}^d$. Let f be a function on $\mathbb{R}^d$.

Suppose $f_n \xrightarrow{n} f$ uniformly on compact sets

then f is entire analytic (book of Cartan)

Lemma. Let $S \in \mathcal{E}'(\mathbb{R}^d)$ and let $(\rho_n)_n$ be a family of mollifiers

then $T_{S*\rho_n} \longrightarrow S$ (in the sense of $\mathcal{E}'$ weak topology)

proof $S*\rho_n$ is $\mathcal{E}^\infty_0(\mathbb{R}^d)$

$$S*\rho_n(x) = S(\psi_{n,x}) \qquad \psi_{n,x}(y) = \rho_n(x-y)$$

$$T_{S*\rho_n}(f) = \int S(\psi_{n,x}) f(x)\,dx$$

$$= \lim_{\varepsilon \to 0} \underbrace{\varepsilon^n \sum_{\nu \in \mathbb{Z}^n} S(\psi_{n,\varepsilon\nu}) f(\varepsilon\nu)}_{\text{this is not a series, is a sum}}$$

$$= \lim_{\varepsilon \to 0} S\Big(\underbrace{\varepsilon^n \sum_{\nu \in \mathbb{Z}^n} \psi_{n,\varepsilon\nu}(y) f(\varepsilon\nu)}_{\text{this is a } \mathcal{E}^\infty \text{ function in } y}\Big)$$

$$= S\Big(\underbrace{\lim_{\varepsilon \to 0} \varepsilon^n \sum_{\nu \in \mathbb{Z}^n} \overbrace{\psi_{n,\varepsilon\nu}(y)}^{\rho_n(\varepsilon\nu - y)} f(\varepsilon\nu)}_{\text{this is an integral}}\Big)$$

$$\int \rho_n(x-y) f(x)\,dx$$

$$\boxed{T_{S*\rho_n}(f) = S(\check{\rho}_n * f)}$$

remark that $\check{\rho}_n * f \longrightarrow f$ in the sense of $\mathcal{E}$

$$\lim_n T_{S*\rho_n}(f) = S(f) \qquad \check{\rho}(x) = \rho(-x)$$

Th. Let $T \in \mathcal{E}'(\mathbb{R}^d)$

Then $\boxed{\widehat{T}^{FL} \text{ is an entire analytic}}$ function

Moreover the distribution associated to $\widehat{T}^{FL}$ is the Fourier transform of T as tempered distribution.

proof. we prove first that if $f \in \mathcal{C}_0(\mathbb{R}^d)$ then $\widehat{f} = \widehat{T_f}^{FL}$ is an entire analytic function

recall that $$\left|\partial_\xi^\alpha \widehat{f}(\xi)\right| \leq \int_{\mathbb{R}^d} |x^\alpha f(x)|\, dx$$

but supp $f \subseteq B(0,R)$ for some R

so $$\leq \int_{|x|\leq R} |x|^\alpha |f(x)|\, dx \leq R^{|\alpha|} \cdot \|f\|_{L^1}$$

$$\sup_{\xi \in \mathbb{R}^d} |\partial_\xi^\alpha \widehat{f}(\xi)| \leq C \cdot R^{|\alpha|}$$

Analyticity from lemma 1

general case Let $S \in \mathcal{E}'(\mathbb{R}^d)$

Let (ρ_n) mollifier

we know (lemma 3) $T_{S*\rho_n} \to S$ in $\mathcal{E}'$

so that also in $\mathcal{S}'$

so $\widehat{T_{S*\rho_n}} \to \widehat{S}$ in $\mathcal{S}'$

$\|$

$T_{\widehat{S*\rho_n}}$

let's compute $\widehat{S*\rho_n}(\xi) = S*\rho_n(\psi_\xi)$ $\quad \psi_\xi(x) = e^{-ix\cdot\xi}$

$= ((S*\rho_n)*\check{\psi}_\xi)(0)$

$= (S*(\rho_n*\check{\psi}_\xi))(0)$

$= S((\rho_n*\check{\psi}_\xi)^\vee)$ $\quad (\rho_n*\check{\psi}_\xi)^\vee = \check{\rho}_n*\psi_\xi$

$= S(\check{\rho}_n*\psi_\xi)$

$\check{\rho}_n*\psi_\xi(x) = \int \check{\rho}_n(x-y)\, e^{-iy\xi}\, dy = \int \check{\rho}_n(y)\, e^{-i(x-y)\cdot\xi}\, dy$

$= e^{-ix\cdot\xi} \underbrace{\int e^{iy\xi}\, \check{\rho}_n(y)\, dy}_{\widehat{\rho}_n(\xi)}$ $\quad y = -z$

$=$

$= \widehat{\rho}_n(\xi)\cdot e^{-ix\cdot\xi}$

$= S(\widehat{\rho}_n(\xi)\cdot e^{-ix\cdot\xi})$

$\widehat{S*\rho_n}(\xi) = \widehat{\rho}_n(\xi)\cdot \underbrace{S(e^{-ix\cdot\xi})}_{\widehat{S}^{FL}(\xi)}$

$$\boxed{\widehat{S*\rho_n}(\xi) = \widehat{\rho}_n(\xi)\cdot \widehat{S}^{FL}(\xi)}$$

$\mathcal{E}_0$

analytic

ρ_n mollifier

$\widehat{\rho}_n(\xi) \xrightarrow{n} 1$ uniformly on compact sets

so $\widehat{S*\rho_n}(\xi) \longrightarrow \widehat{S}^{FL}(\xi)$ uniformly on compact sets (Weierstrass)

the limit is entire analytic.

if $S \in \mathcal{E}'(\mathbb{R}^d)$ then $\widehat{S}^{FL}$ is entire analytic

¿ take $\bar{U}$ entire analytic

is $\bar{U}$ the F-L transf of a distribution?

Paley-Wiener thm.