May 10th Sobolev Spaces (book of Brezis ch. 8-9)
case of 1D.

definition Let I open interval in $\mathbb{R}$, $p \in [1, +\infty]$.

We define
$$W^{1,p}(I) = \{ u \in L^p(I) \text{ s.t. } \exists g \in L^p(I) \text{ s.t. } \forall \varphi \in \mathcal{C}^1_0(I), \ \underbrace{\int_I u\varphi' = -\int_I g\varphi}_{\circledast} \}$$

in principle g is not unique

Sobolev space of indexes 1 and p

Remarks 1) It is sufficient that $\circledast$ holds for $\varphi \in \mathcal{C}^\infty_0(I)$
(it is sufficient to make the test only on $\mathcal{D}(I)$)

in fact let $\psi \in \mathcal{C}^1_0(I)$

consider $(\rho_n)_n$ mollifier and consider $\rho_n * \psi$

if $n \geq \bar{n}$, $\rho_n * \psi \in \mathcal{C}^\infty_0(I)$

and we know that
$$\left.\begin{array}{l} \rho_n * \psi \longrightarrow \psi \\ (\rho_n * \psi)' = \rho_n * \psi' \longrightarrow \psi' \end{array}\right\} \text{uniformly}$$
$\mathcal{C}^0_0(I)$

$\Downarrow$

also in $L^{p'}$ for all $p' \in [1, +\infty]$

then
$$\int_I u(\rho_n * \psi)' \longrightarrow \int u\psi'$$
$$\shortparallel$$
$$-\int g(\rho_n * \psi) \longrightarrow -\int g\psi$$

equality here implies $\longrightarrow$ equality here.

remark 2 the function g is unique

in fact: we know that if
$$-\int_I g_1\varphi = \int_I u\varphi' = -\int_I g_2\varphi \qquad \forall \varphi \in \mathcal{C}^\infty_0(I)$$
then $\int_I (g_1 - g_2)\varphi = 0 \quad \forall \varphi \in \mathcal{C}^\infty_0(I)$

$\Rightarrow g_1 - g_2 = 0$ a.e.

remark 3. T_g is the derivative in the sense of distribution of the distribution T_u

So
$$W^{1,p}(I) = \{ u \in L^p : T_u' = T_g, \ \exists g \in L^p(I) \}$$

we denote $g = u'$ $\leftarrow$ we call u' the "weak derivative" of u

in the case of $p = \infty$

def. We set
$$\|u\|_{W^{1,p}(I)} = \|u\|_{L^p(I)} + \|u'\|_{L^p(I)}$$

or equivalently
$$\left(\|u\|^p_{L^p(I)} + \|u'\|^p_{L^p(I)} \right)^{\frac{1}{p}}$$

$p = 2$

Theorem $W^{1,p}(I)$ is a Banach space

if $1 \le p < +\infty$ it is a separable space

if $1 < p < +\infty$ it is a reflexive space

if $p = 2$ $W^{1,2}(I) = H^1(I)$ is an Hilbert space

with

$$(u,v)_{H^1} = \int_I u\bar{v} + \int_I u'\overline{v'}$$

proof. Consider

$$\Phi : W^{1,p}(I) \longrightarrow L^p(I) \times L^p(I)$$

$$u \longmapsto (u, u') = \Phi(u)$$

Φ is an isometry

to conclude the proof it is sufficient to prove that $\Phi(W^{1,p}(I))$ is a closed subspace

let $(u,v) \in \overline{\Phi(W^{1,p}(I))}$

$\exists\, (u_n)_n$ in $W^{1,p}(I)$ st. $\Phi(u_n) \longrightarrow (u,v)$

i.e. $u_n \longrightarrow u$ in $L^p(I)$

$u_n' \longrightarrow v$ in $L^p(I)$

$u_n \to u$ strongly $\Rightarrow$ $u_n \rightharpoonup u$ (weakly)

$u_n' \to v$ " " $u_n' \rightharpoonup v$ (weakly)

so that

$$\int_I u_n \varphi' \longrightarrow \int_I u\varphi' \qquad \forall \varphi \in C_0^\infty(I)$$

$$\| $$

$$-\int u_n'\varphi \longrightarrow -\int v\varphi$$

so that $\int u\varphi' = -\int v\varphi, \ \forall \varphi \in C_0^\infty(I)$

this means that $u \in W^{1,p}(I)$ and $u' = v$

so that $(u,v) \in \Phi(W^{1,p}(I))$.

QED

remark: we have proved that

if $(u_n)_n$ in $W^{1,p}(I)$

and we know that $\left.\begin{array}{l} u_n \to u \\ u_n' \to v \end{array}\right\}$ in L^p

then $u \in W^{1,p}$ and $v = u'$.

Ex. suppose $1 < p \leq +\infty$. suppose $(u_n)_n \in W^{1,p}(I)$

suppose that $u_n \to u$ in $L^p(I)$

suppose that $\exists C > 0 : \forall n, \|u_n'\|_{L^p(I)} \leq C$

then $u \in W^{1,p}(I)$

Hint. consider p' s.t. $\frac{1}{p} + \frac{1}{p'} = 1$

$p' \in [1, +\infty[$

We can think at $L^p(I)$ as the dual space of $L^{p'}(I)$

and $(u_n')_n$ is bounded in $L^p(I)$
strongly

so we can extract a weakly $*$ converging subsequence to $w \in L^p(I)$

$\forall \psi \in L^{p'}(I), \quad \int u_{n_k}' \psi \longrightarrow \int w \psi$

$$\Downarrow$$

$$\int u_{n_k}' \psi \longrightarrow \int w \psi \quad \forall \psi \in C_c^\infty(I)$$

$$\|$$

$$-\int u_{n_k} \psi' \longrightarrow -\int u \psi'$$

so $u \in L^p(I)$ and $\int u \psi' = -\int w \psi \quad \forall \psi$.

Continuous representative of a function in $W^{1,p}(I)$.

<u>Theorem</u>. Let $u \in W^{1,p}(I)$ $\quad$ I open interval in $\mathbb{R}$, $p \in [1,+\infty]$

then there exists $\bar{u} \in \mathcal{C}(\bar{I})$ s.t.

$\bar{u} = u$ a.e. in I and

for all $x, y \in I$, $\quad \bar{u}(y) - \bar{u}(x) = \int_x^y u'(t)\,dt$

(corollary: $W^{1,p}(I) \subseteq AC(I)$)
if $|I| < +\infty$
and $W^{1,1}(I) = AC(I)$)

proof (alternative to Brezis' one)

<u>Lemma</u> Let T be a distribution on $I \subseteq \mathbb{R}$ open interval

suppose that $T' = 0$. Then $\exists c \in \mathbb{R}$ s.t. $T = T_c$

<u>proof</u>. Let $\chi \in \mathcal{C}_0^\infty(I)$ s.t. $\int_I \chi(t)\,dt = 1$

take $\varphi \in \mathcal{C}_0^\infty(I)$

and consider $\psi(x) = \int_{-\infty}^{x} \varphi(t)\,dt - \left(\int_{-\infty}^{x} \chi(t)\,dt\right) \underbrace{\int_{-\infty}^{+\infty} \varphi(s)\,ds}_{\text{this a constant}}$

¿ $\psi \in \mathcal{C}_0^\infty(I)$? $\quad$ <u>yes</u>

$\psi \in \mathcal{C}^\infty$ and it "starts from 0 and goes back to 0"

and $\quad \psi'(x) = \varphi(x) - \chi(x) \cdot \int_I \varphi(s)\,ds$

constant

so $\quad T(\psi') = T(\varphi) - \int_I \varphi(s)\,ds \; T(\chi)$

$\qquad \| $

$\quad -T'(\psi)$

$\qquad \|$

$\qquad 0$

conclusion: $\forall \varphi \in \mathcal{D}(I)$

$$T(\varphi) = \underbrace{T(\chi)}_{c} \cdot \underbrace{\int_I \varphi(s)\,ds}_{T_1(\varphi)}$$

$$= T_c(\varphi)$$

QED.

proof. Notice that $L^p(I) \subseteq L^1_{loc}(I)$

fix $x_0 \in I$ and denote by

$$w(x) = \int_{x_0}^{x} \underbrace{u'(t)}_{\in L^1([x_0,x])}\,dt$$

we deduce that $w \in AC(\tilde{I})$

for all $\tilde{I}$ bounded, $\underset{\underset{x_0}{\cup}}{\tilde{I}} \subseteq I$.

moreover $\underbrace{w'(x)}_{\text{the classical derivative}} = u'(x)$ a.e.

use the integration by parts for AC functions

we have that $\int_I w \cdot \varphi' = -\int_I u' \varphi \quad \forall \varphi \in C_0^\infty(I)$

As a consequence, since $u \in W^{1,p}(I)$

we have $\int_I w \varphi' = \int_I u \varphi' \qquad \forall \varphi \in C_0^\infty(I)$

From the lemma $\exists c$ constant s.t

$$T_w - T_u = T_c$$

we define $\bar{u}(x) = w(x) + c$ then $\bar{u}$ is continuous and it is equal to u a.e.

QED

Characterization of $W^{1,p}(I)$ for $1<p\le+\infty$.

Theorem Let I open interval in $\mathbb{R}$, let $p\in]1,+\infty]$.
Let $u\in L^p(I)$.
The following are equivalent
i) $u\in W^{1,p}(I)$

ii) $\exists C>0: \quad \forall \varphi\in \mathcal{C}_0^\infty(I), \quad \left|\int_I u\varphi'\right|\le C\|\varphi\|_{L^{p'}}$

(where $\frac{1}{p}+\frac{1}{p'}=1$)

iii) $\exists C>0: \quad \forall \omega\subset\subset I \quad \forall h\in\mathbb{R}: \quad |h|\le \operatorname{dist}(\omega,\partial I)$

$$\|\tau_h u-u\|_{L^p(\omega)}\le C|h|$$

$\tau_h u(x)=u(x-h)$ ← translation of u

proof. i) => ii)

$u\in W^{1,p}(I) \implies \int u\varphi'=-\int u'\varphi$

~~so~~ that $\left|\int u\varphi'\right|=\left|\int u'\varphi\right|\le \|u'\|_{L^p}\|\varphi\|_{L^{p'}}$

so condition ii) is valid with $C=\|u'\|_{L^p}$

conversely ii) => i)

we know that $\exists C>0: \quad \underbrace{\left|\int u\varphi'\right|\le C\|\varphi\|_{L^{p'}}}_{\circledast}$

Consider $\mathcal{C}_0^\infty(I)$ as a subspace of $L^{p'}(I)$

consider $\Phi: \mathcal{C}_0^\infty(I)\longrightarrow \mathbb{C}$ (this is linear)

$$\varphi\longmapsto \int u\varphi'$$

$\circledast$ says that Φ is continuous (w.r.t. the topology of $L^{p'}$)

I use Hahn-Banach, $\tilde\Phi$ is the extension to $L^{p'}(I)$

$1\le p'<+\infty$

$$\tilde\Phi\in (L^{p'})'=L^p$$

$$\exists g\in L^p(I): \quad \tilde\Phi(\varphi)=\int_I g\varphi$$

combining $\exists g\in L^p(I)$ s.t. $\int u\varphi'=\int g\varphi$

$\implies u\in W^{1,p}(I)$ and $u'=-g$

I prove i) => iii)

let $u\in W^{1,p}(I)$

i) $\Rightarrow$ iii) let $u \in W^{1,p}(I)$

let $\omega \subset\subset I$ let h s.t. $|h| \le \operatorname{dist}(\omega, \partial I)$

consider $u(x) - u(x-h) = \int_{x-h}^{x} u'(t)\,dt = h \int_0^1 u'(x-sh)\,ds$

$t = x - sh \qquad t = x-h \iff s = 1$

$dt = -h\,ds \qquad t = x \iff s = 0$

$$|u(x-h) - u(x)| \le |h| \int_0^1 |u'(x-sh)|\,ds$$

$1 \in L^{p'}$, $|u'(x-sh)| \in L^p$ Hölder

$$|u(x-h) - u(x)|^p \le |h|^p \cdot \int_0^1 |u'(x-sh)|^p\,ds$$

$$\int_\omega |u(x-h) - u(x)|^p\,dx \le |h|^p \underbrace{\int_\omega \Big(\int_0^1}_{\text{exchange}} |u'(x-sh)|^p\,ds \Big)\,dx$$

$$\|\tau_h u - u\|^p_{L^p(\omega)} \le |h|^p \int_0^1 \underbrace{\Big(\int_\omega |u'(x-sh)|^p\,dx \Big)}_{\|u'\|^p_{L^p(I)}}\,ds$$

Finally $\|\tau_h u - u\|_{L^p(\omega)} \le |h| \underbrace{\|u'\|^p_{L^p(I)}}_{C} \Rightarrow$ iii)

Finally iii) $\Rightarrow$ ii) $\exists C: \forall \varphi \in C_c^\infty(I) \ \left|\int u \varphi'\right| \le C \|\varphi\|_{L^{p'}}$

take $\varphi \in C_c^\infty(I)$ suppose $\omega \subset\subset I$ s.t. $\operatorname{supp}(\varphi) \subseteq \omega$ take h s.t. $|h| \le \operatorname{dist}(\omega, \partial I)$

$$\underbrace{\int_\omega (u(x-h) - u(x))\varphi(x)\,dx}_{\text{is in condition iii)}} = \int_I u(x-h)\varphi(x)\,dx - \int_I u(x)\varphi(x)$$

change $x - h = z$

$$= \int_I u(z)\varphi(z+h)\,dz - \int_I u(x)\varphi(x)\,dx$$

$$= \int_I u(x)\big(\varphi(x+h) - \varphi(x)\big)\,dx$$

$$\left|\int_I u(x)\big(\varphi(x+h) - \varphi(x)\big)\,dx\right| = \left|\int_\omega (u(x-h) - u(x))\varphi(x)\,dx\right|$$

Hölder

$$\le \|\tau_h u - u\|_{L^p(\omega)} \cdot \|\varphi\|_{L^{p'}(I)}$$

use iii)

$$\le C\,|h| \cdot \|\varphi\|_{L^{p'}(I)}$$

Finally
$$\left|\int u(x)\left(\frac{\varphi(x+h) - \varphi(x)}{h}\right)dx\right| \le C \|\varphi\|_{L^{p'}(I)}$$

pass to the limit in $h \to 0$

you obtain $\left|\int u(x)\varphi'(x)\right| \le C \|\varphi\|_{L^{p'}(I)}$ ii)

rem. $p = 1$

it is possible to prove $u \in BV(I)$

i) $\Rightarrow$ ii) $\iff$ iii)