May 13th. Sobolev spaces.

Ex. 1. Let $f, g \in L^2(\mathbb{R})$ then $\int f\hat{g} = \int \hat{f} g$ ←

and

$\int f\bar{g} = \underbrace{\ }_{(2\pi)^{-1}}\int \hat{f}\,\overline{\hat{g}}$ ←

we saw this for $f, g \in \mathcal{S}(\mathbb{R})$

Hint. approximation of L^2 via $\mathcal{S}$

Ex. 2. Let $u \in W^{1,p}(\mathbb{R})$ let $f \in \mathcal{S}(\mathbb{R})$

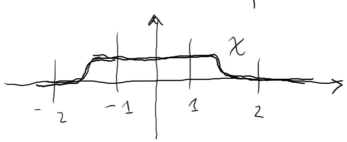
then $\int_{\mathbb{R}} u f' = -\int_{\mathbb{R}} u' f$ ← $f \in \mathcal{S}$

(mind that $u \in W^{1,p}(\mathbb{R}) \iff$ $u \in L^p$, $\exists g \in L^p$ s.t. $\int u \varphi' = -\int g \varphi$ with $\varphi \in C_0^\infty$

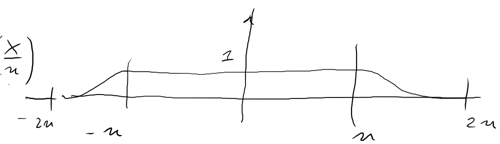
Solution: let $f \in \mathcal{S}(\mathbb{R})$ let $\chi \in C_0^\infty(\mathbb{R})$ $\chi(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| \geq 2 \end{cases}$

$0 \leq \chi \leq 1$

truncation function



consider $\chi_n(x) = \chi\left(\frac{x}{n}\right)$



$f_n(x) = \chi_n(x) f(x) \in C_0^\infty$

so $\int u(x)\left(f_n(x)\right)' dx = -\int u'(x) f_n(x)\, dx$

$= \underbrace{\int u(x)\left(\frac{1}{n}\chi'\left(\frac{x}{n}\right) f(x) + \chi_n(x) f'(x)\right) dx}_{\downarrow n} = \underbrace{-\int u'(x) f_n(x)\, dx}_{\downarrow n}$

bounded

$\frac{1}{n}$

dominated convergence →

← Lebesgue's thm

$\int u(x) f'(x)\, dx$ $\qquad$ $-\int u'(x) f(x)\, dx$

Theorem let $u \in L^2(\mathbb{R})$

$u \in W^{1,2}(\mathbb{R}) = H^1(\mathbb{R}) \iff (1+|\xi|^2)^{\frac{1}{2}} \hat{u}(\xi) \in L^2(\mathbb{R})$

and $\|u\|_{H^1} = \frac{1}{(2\pi)^{\frac{1}{2}}} \left(\int (1+|\xi|^2) |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}$

$\mathbb{R}$

(Rem. let $s \in \mathbb{R}$

$H^s(\mathbb{R}) = \{ u \in \mathcal{S}'(\mathbb{R}) \text{ s.t. } (1+|\xi|^2)^{s/2} \hat{u}(\xi) \in L^2(\mathbb{R}) \}$)

proof. let $u \in H^1(\mathbb{R})$

we have $u \in L^2(\mathbb{R})$, $u' \in L^2(\mathbb{R})$, $\hat{u} \in L^2(\mathbb{R})$, $\widehat{u'} \in L^2(\mathbb{R})$

$\psi \in \mathcal{S}$

$$\int_{\mathbb{R}} \widehat{u'}(\xi) \psi(\xi) = \int_{\mathbb{R}} u'(x) \underset{\mathcal{S}}{\underset{\Uparrow}{\hat{\psi}(\xi)}} dx = - \int u(x) (\hat{\psi})' dx$$

$$= - \int u(x) i \widehat{\xi \psi(\xi)} dx = -i \int \hat{u}(\xi) \xi \psi(\xi) d\xi$$

conclusion

$$-i \hat{u}(\xi) \xi = \underset{L^2}{\underset{\Uparrow}{\widehat{u'}(\xi)}} \quad a.e.$$

$$\Rightarrow \hat{u}(\xi) \cdot \xi \in L^2$$

$$\Downarrow$$

$$(1+|\xi|^2)^{\frac{1}{2}} \hat{u}(\xi) \in L^2$$

conversely suppose $\left.\begin{array}{l} \hat{u}(\xi) \in L^2 \\ \xi \hat{u}(\xi) \in L^2 \end{array}\right\} \iff (1+|\xi|^2)^{\frac{1}{2}} \hat{u}(\xi) \in L^2$

Let

take $\varphi \in C_0^\infty(\mathbb{R})$

$$\left| \int u(x) \varphi'(x) dx \right| = \left| \frac{1}{2\pi} \int \hat{u}(\xi) \widehat{\varphi'}(\xi) d\xi \right|$$

$$\leq \frac{1}{2\pi} \left| \int \underbrace{\hat{u}(\xi) \xi}_{\in L^2} \hat{\varphi}(\xi) d\xi \right|$$

$$\leq C \|\hat{u}(\xi) \cdot \xi\|_{L^2} \|\varphi\|_{L^2}$$

at the end

$$\left| \int u \varphi' \right| \leq \tilde{C} \|\varphi\|_{L^2}$$

$$\Updownarrow$$

$$u \in W^{1,2}(\mathbb{R})$$

QED

The extension operator

Rem. take $I \subseteq \mathbb{R}$ (interval), take $u \in W^{1,p}(I)$,

¿ $\exists\, \tilde{u} \in W^{1,p}(\mathbb{R}) : \tilde{u}|_I = u$?

Th. Let $I \subseteq \mathbb{R}$, Let $p \in [1, +\infty]$

$\exists\, P : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$ s.t.

i) $\forall u \in W^{1,p}(I)$, $Pu|_I = u$

ii) $\exists\, C_0 > 0 : \forall u \in W^{1,p}(I)$ $\quad \|Pu\|_{L^p(\mathbb{R})} \le C_0 \|u\|_{L^p(I)}$

iii) $\exists\, C_1 > 0 : \forall u \in W^{1,p}(I)$ $\quad \|Pu\|_{W^{1,p}(\mathbb{R})} \le C_1 \|u\|_{W^{1,p}(I)}$

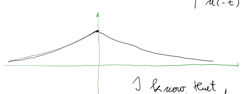
where C_0 and C_1 depend only on $|I|$ (lenght of I)

proof. step 1. suppose $I =]0, +\infty[$

$u \in W^{1,p}(]0, +\infty[)$ from theorem on continuous representative

we know that $u \in \mathcal{C}([0, +\infty[)$

I define $Pu(t) := \begin{cases} u(t) & \text{if } t \ge 0 \\ u(-t) & \text{if } t < 0 \end{cases}$ (reflection)



I know that $u(t) = u(0) + \int_0^t u'(s)\, ds$ if $t \ge 0$

from the theory of AC functions

so that I can verify that

$$Pu(t) = \begin{cases} u(0) + \int_0^t u'(s) ds \\ u(0) + \int_0^t -u'(-s)\, ds \end{cases} \quad \text{and} \quad (Pu)'(t) \begin{cases} u'(t) & \text{if } t \ge 0 \\ -u'(-t) & \text{if } t \le 0 \end{cases}$$

we have that $Pu \in W^{1,p}(\mathbb{R})$ (because $Pu \in L^p$, $(Pu)' \in L^p$)

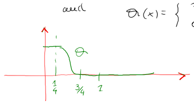
(here we use the integration by parts in AC functions.

and also $\|Pu\|_{L^p(\mathbb{R})} \le 2 \|u\|_{L^p(]0,+\infty[)}$, $\|Pu\|_{W^{1,p}(\mathbb{R})} \le 2 \|u\|_{W^{1,p}(]0,+\infty[)}$

2^{nd} step $\quad I =]0, 1[$

lemma let $\theta \in \mathcal{C}^\infty([0, +\infty[)$ s.t. $0 \le \theta \le 1$

and $\theta(x) = \begin{cases} 1 & \text{if } x \le \frac{1}{4} \\ 0 & \text{if } x \ge \frac{3}{4} \end{cases}$



Let $u \in W^{1,p}(]0,1[)$

denote by $\tilde{u}(x) := \begin{cases} u(x) & 0 < x < 1 \\ 0 & x > 1 \end{cases}$

and $\widetilde{u'}(x) = \begin{cases} u'(x) & 0 < x < 1 \\ 0 & x > 1 \end{cases}$

consider $v(x) = \theta(x) \cdot \tilde{u}(x)$

Then $v \in W^{1,p}(]0, +\infty[)$

and $v' = \theta' \cdot \tilde{u} + \theta \cdot \widetilde{u'}$.

proof (lemma)

$\theta \tilde{u} \in L^p((0, +\infty))$, $\quad \theta' \tilde{u} + \theta \cdot \tilde{u} \in L^p((0, +\infty))$

it remains to prove that $\displaystyle\int_0^{+\infty} (\theta \tilde{u}) \cdot \psi' = - \int_0^{+\infty} (\theta' \tilde{u} + \theta \widetilde{u'}) \psi$

$\forall \psi \in \mathcal{C}_0^\infty(0, +\infty)$

let $\psi \in \mathcal{C}_0^\infty((0, +\infty))$

consider $\theta \cdot \psi$ and $\theta' \psi$ ($\theta\psi'$) this are $\mathcal{C}_0^\infty(0, +\infty)$ but also $\mathcal{C}_0^\infty((0,1))$

so that

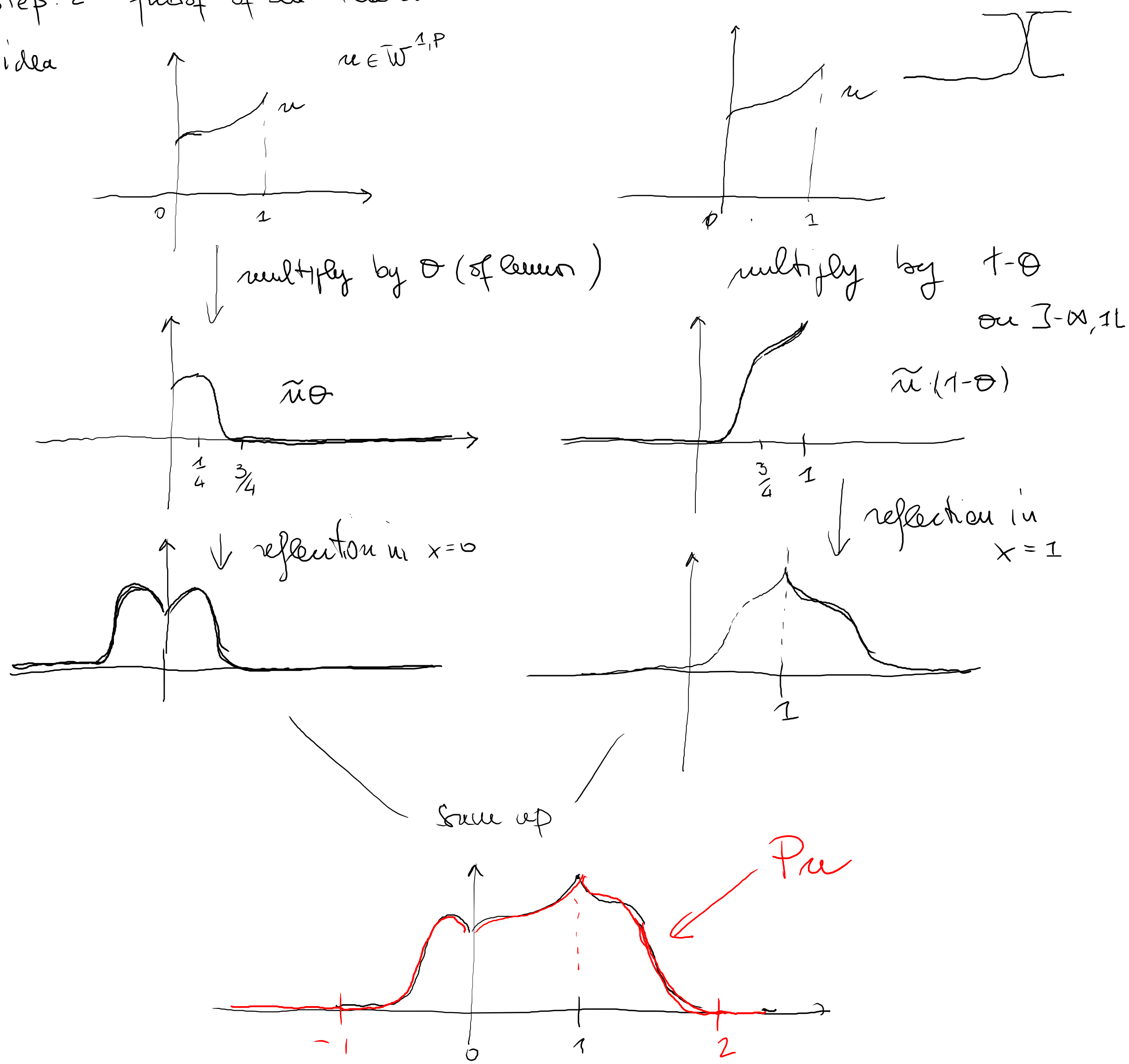
$$\int_0^{+\infty} (\theta \tilde{u}) \psi' = \int_0^{+\infty} \tilde{u}\, (\theta \psi') = \int_0^1 u\, (\theta \psi') = \int_0^1 u\, ((\theta\psi)' - \theta'\psi) \overset{u \in W^{1,p}(0,1)}{=} - \int_0^1 u'(\theta\psi) - \int_0^1 u\, \theta' \psi$$

$$= - \int_0^{+\infty} (\widetilde{u'} \theta + \tilde{u} \theta') \psi$$

QED

Step. 2 proof of the theorem

idea



obviously $\quad Pu|_{[0,1]} = u$

exercise : check that $\exists\, C_0, C_1$ which depend only on $|I|$

Approximation results.

Lemma let $f \in L^1(\mathbb{R})$ and let $u \in W^{1,p}(\mathbb{R})$

then $f * u \in W^{1,p}(\mathbb{R})$ and $(f*u)' = f * u'$

proof. Consider first $f \in L^1(\mathbb{R})$, with compact support

I prove that $f*u'$ is the weak derivative of $(f*u)$

i.e.

$$\int_{\mathbb{R}} (f*u)\varphi' = -\int (f*u')\cdot\varphi \qquad \forall \varphi \in C_0^\infty(\mathbb{R})$$

in fact

$$\int_{\mathbb{R}} (f*u)(x)\varphi'(x)\,dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-y)\,u(y)\,dy\right)\varphi'(x)\,dx$$

exchange the order of integration ?
(look at the supports $\Rightarrow$
$(x,y) \longmapsto f(x-y)u(y)\varphi'(x)$
is in $L^1(\mathbb{R}^2)$)

$$= \int_{\mathbb{R}} \Big(\underbrace{\int f(x-y)\varphi'(x)\,dx}_{\check{f}*\varphi'(y) = (\check{f}*\varphi)'(y)}\Big)\, u(y)\,dy$$

$$= \int (\check{f}*\varphi)'(y)\,u(y)\,dy = -\int (\check{f}*\varphi)(y)\,u'(y)\,dy$$

exchange the variables

$$= -\int (u' * f)\,\varphi\,dx$$

QED

Theorem. Let $u \in W^{1,p}(I)$ with $p \in [1, +\infty[$.

then $\exists (u_n)_n$ in $C_0^\infty(\mathbb{R})$ s.t.

$$u_n|_I \longrightarrow u \quad \text{in } W^{1,p}(I)$$

(Corollary if $I = \mathbb{R}$ then $C_0^\infty(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$ for $p < +\infty$.)

proof. consider $u \in W^{1,p}(I)$

use the extension thm. $\Rightarrow Pu \in W^{1,p}(\mathbb{R})$

If $(u_n)_n$ is in $C_0^\infty(\mathbb{R})$ and $u_n \longrightarrow Pu$ in $W^{1,p}(\mathbb{R})$

then $u_n|_I \longrightarrow u$ in $W^{1,p}(I)$

It will be sufficient to prove that given $u \in W^{1,p}(\mathbb{R})$ with $p < +\infty$

$\exists (u_n)_n$ in $C_0^\infty(\mathbb{R})$ s.t. $u_n \longrightarrow u$ in $W^{1,p}(\mathbb{R})$,

Consider $\chi(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| \ge 2 \end{cases}$ $\qquad \chi_n(x) = \chi\left(\frac{x}{n}\right)$

consider $(\rho_n)_n$ a family of mollifier ($\rho_n(x) = \frac{1}{n}\rho\left(\frac{x}{n}\right)$ etc.)

I set $\quad u_n(x) = \chi_n(x)\big((\rho_n * u)(x)\big)$

I have to estimate $\|u_n - u\|_{L^p(\mathbb{R})}$ and $\|u_n' - u'\|_{L^p(\mathbb{R})}$

$$\|u_n - u\|_{L^p} = \|\chi_n(\rho_n * u) - u\|_{L^p}$$

$$\le \|\chi_n(\rho_n*u) - \chi_n u\|_{L^p(\mathbb{R})} + \|\chi_n u - u\|_{L^p(\mathbb{R})} \; ?$$

$\|\chi_n u - u\|_{L^p(\mathbb{R})} \xrightarrow{n} 0$ (dominated convergence theorem)

$p \neq +\infty$

$$\|\chi_n(x)(\rho_n * u - u)\|_{L^p}$$

$$\le \underbrace{\|\chi_n\|_{L^\infty}}_{\le 1} \underbrace{\|\rho_n * u - u\|_{L^p}}_{\xrightarrow{n} 0}$$

$\Rightarrow \|u_n - u\|_{L^p} \xrightarrow{n} 0$

$$\|u_n' - u'\|_{L^p} = \left\|\frac{1}{n}\chi'\left(\frac{x}{n}\right)\cdot(\rho_n * u) + \chi_n(\rho_n * u') - u'\right\|_{L^p}$$

$$\le \frac{1}{n}\underbrace{\left\|\chi'\left(\frac{x}{n}\right)\right\|_{L^\infty}}_{\le C}\underbrace{\|\rho_n * u\|_{L^p}}_{\|\rho_n\|_{L^1}\|u\|_{L^p}} + \underbrace{\|\chi_n(\rho_n * u') - \chi_n u'\|_{L^p} + \|\chi_n u' - u'\|_{L^p}}_{\text{as before}}$$

$\frac{1}{n} \longrightarrow 0$

QED

Sobolev embeddings ($d = 1$)

Th. Let I interval in $\mathbb{R}$, let $p \in [1, +\infty]$, let $u \in W^{1,p}(I)$

(Sobolev) then $u \in L^\infty(I)$ and $\exists C$ (depending only on $|I|$)

s.t.
$$\|u\|_{L^\infty(I)} \le C \|u\|_{W^{1,p}(I)}$$

proof. we start with $I = \mathbb{R}$

if $W^{1,p}(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$ and $\|u\|_{L^\infty} \le C \|u\|_{W^{1,p}}$

then

$$u \in W^{1,p}(I) \Rightarrow \overset{\text{extension}}{\overset{\downarrow}{Pu}} \in W^{1,p}(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$$

so then $u = Pu|_I \in L^\infty(I)$

and
$$\|u\|_{L^\infty(I)} \le \|Pu\|_{L^\infty(\mathbb{R})} \le \|Pu\|_{W^{1,p}(\mathbb{R})} \le C \|u\|_{W^{1,p}(I)}$$

extension thm.