May 17th — Sobolev embeddings

Th let I be an open interval in $\mathbb{R}$, $p \in [1, +\infty]$, $u \in W^{1,p}(I)$.
Then $u \in L^\infty(I)$ and $\exists\, C > 0$ (depending only on $|I|$)
s.t. $\|u\|_{L^\infty(I)} \le C \|u\|_{W^{1,p}(I)}$.

(i.e. $W^{1,p}(I) \hookrightarrow L^\infty(I)$ continuous emb.)

proof.

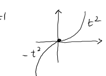
step 1) Let $I = \mathbb{R}$

if $p = +\infty$ nothing to prove

if $1 \le p < +\infty$ consider $G : \mathbb{R} \to \mathbb{R}$, $G(t) = t|t|^{p-1}$

G is a $\mathcal{C}^1(\mathbb{R})$ and $G'(t) = p|t|^{p-1}$

(e.g. $t \mapsto t|t|$



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consider $u \in \mathcal{C}_0^\infty(\mathbb{R})$, define $\Psi(t) = G(u(t))$

we have $\Psi \in \mathcal{C}_0^1(\mathbb{R})$, $\Psi'(t) = G'(u(t)) \cdot u'(t) = p\,|u(t)|^{p-1} u'(t)$

$\left(\Psi(t) = u(t)\,|u(t)|^{p-1}\right)$

$$\|u\|_{L^\infty}^p = \sup_{t\in\mathbb{R}} |\Psi(t)| = \sup_{t\in\mathbb{R}} |u(t)|^p = \|u\|_{L^\infty}^p$$

$$= \sup_{t\in\mathbb{R}} \left(\left| \int_{-\infty}^{t} \Psi'(s)\,ds \right| \right) \le \int_{-\infty}^{+\infty} |\Psi'(s)|\,ds$$

$$\le \int_{-\infty}^{+\infty} p \underbrace{|u(s)|^{p-1}}_{\in L^{p'}(\mathbb{R})} \cdot \underbrace{|u'(s)|}_{\in L^p(\mathbb{R})} ds$$

with $\frac{1}{p} + \frac{1}{p'} = 1$

$p' = \frac{p}{p-1}$

Hölder

$$\le p \left(\int_{\mathbb{R}} \left(|u(s)|^{\cancel{p-1}}\right)^{\cancel{\frac{p}{p-1}}} ds \right)^{\frac{p-1}{p}} \cdot \left(\int_{-\infty}^{+\infty} |u'(s)|^p \right)^{\frac{1}{p}}$$

$$\|u\|_{L^\infty}^p \le p \, \|u\|_{L^p(\mathbb{R})}^{p-1} \cdot \|u'\|_{L^p} \qquad \text{for all } u \in \mathcal{C}_0^\infty(\mathbb{R})$$

$$\|u\|_{L^\infty} \le p^{\frac{1}{p}} \|u\|_{L^p(\mathbb{R})}^{1-\frac{1}{p}} \|u'\|_{L^p(\mathbb{R})}^{\frac{1}{p}}$$

I use Young's inequality ($\forall a, b > 0$ $\forall \alpha, \beta \ge 0$, $\alpha + \beta = 1$,
$a^\alpha b^\beta \le \alpha a + \beta b \le a + b$

$$(*) \quad \|u\|_{L^\infty} \le p^{\frac{1}{p}} \left(\|u\|_{L^p} + \|u'\|_{L^p} \right) = C_p \cdot \|u\|_{W^{1,p}(\mathbb{R})}$$

suppose now $u \in W^{1,p}(\mathbb{R})$

$\exists\, (u_n)_n$ in $\mathcal{C}_0^\infty(\mathbb{R})$ s.t. $u_n \to u$ in $W^{1,p}(\mathbb{R})$

in particular
$u_n \to u$ in L^p
$u_n' \to u'$ in L^p
$u_n \to u$ a.e. (eventually taking a subsequence.)

now $(u_n)_n$ is a Cauchy seq. in $W^{1,p}(\mathbb{R})$

and since $\|u_n - u_m\|_{L^\infty} \le C \|u_n - u_m\|_{W^{1,p}}$

$(u_n)_n$ is Cauchy in L^∞

so $u_n \to v$ in L^∞, with $v \in L^\infty$ but $u_n \to u$ a.e.

$\Rightarrow u = v \in L^\infty$.

we can pass to the limit in

$$\|u_n\|_{L^\infty} \le C \|u_n\|_{W^{1,p}} \longrightarrow \|u\|_{L^\infty} \le C \|u\|_{W^{1,p}}.$$

second step. let $u \in W^{1,p}(I)$ with $I \subseteq \mathbb{R}$

take $P : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$ extension operator

$$\|u\|_{L^\infty(I)} = \|Pu_{|I}\|_{L^\infty} \le \|Pu\|_{L^\infty(\mathbb{R})} \le C_0 \|Pu\|_{W^{1,p}(\mathbb{R})} \le C_0 C_1(|I|, p) \|u\|_{W^{1,p}(I)}$$

sobolev constant in $\mathbb{R}$

extension constant

QED

Th. (Rellich)

Let I be an open interval in $\mathbb{R}$, with $|I| < +\infty$.

i) if $1 < p \leq +\infty$ then $W^{1,p}(I) \hookrightarrow \mathscr{C}(\bar{I})$

with compact immersion

(bounded sets in $W^{1,p}(I)$ sets are relatively compact in $\mathscr{C}(\bar{I})$ with sup-norm)

ii) if $p = 1$, then $W^{1,1}(I) \hookrightarrow L^q(\bar{I})$ for all $q < +\infty$

with compact immersion

proof. i) Ascoli - Arzela

I take $B \subseteq W^{1,p}(I)$, B bounded ($\exists C > 0$ $\forall u \in B$, $\|u\|_{W^{1,p}(I)} \leq C$)

I have to prove that B is a ~~set~~ a bounded set in $\mathscr{C}(\bar{I})$ and also "equicontinuous")

$u \in \mathscr{C}(\bar{I})$ from the cont. representative

boundedness for free $\|u\|_{L^\infty} \leq C_0 \underbrace{\|u\|_{W^{1,p}}}_{\leq C}$

for the equicontinuity

$$|u(x) - u(y)| \leq \left| \int_x^y 1 \cdot |u'(t)| \, dt \right| \underset{\text{Hölder}}{\leq} \left(\int_x^y 1^{p'} \, dt \right)^{\frac{1}{p'}} \left(\int_x^y |u'(t)|^p \right)^{\frac{1}{p}}$$

$\frac{1}{p'} = 1 - \frac{1}{p} =$

$$\leq |x-y|^{1-\frac{1}{p}} \cdot \underbrace{\|u\|_{W^{1,p}(I)}}_{\leq C}$$

$$\leq C |x-y|^{1-\frac{1}{p}}$$

the same for all $u \in B$

ii) let B bounded in $W^{1,1}(I)$

fix $q \in [1, +\infty[$, we have to prove that B is rel. compact in L^q

(Riesz - Fréchet - Kolmogoroff)

a) $\forall \varepsilon > 0$, $\forall \omega \subset\subset I$, $\exists \delta > 0$ (with $\delta < \operatorname{dist}(\partial I, \omega)$)

s.t. $\forall |h| < \delta$

$$\|\tau_h u - u\|_{L^q(\omega)} \leq \varepsilon \qquad \forall u \in B$$

b) $\forall \varepsilon > 0$, $\exists \omega \subset\subset I$ s.t

$$\|u\|_{L^q(I \setminus \omega)} \leq \varepsilon \qquad \forall u \in B$$

Let's prove a) I take $u \in B$ (B bounded in $W^{1,1}(I)$)

I remember the remark after the th. of characterization of $W^{1,p}(I)$ (for $p > 1$)

$\rightarrow$ $u \in W^{1,1}(I) \Rightarrow \forall \omega \subset\subset I$, $\forall |h| \leq \operatorname{dist}(\partial I, \omega)$

$$\|\tau_h u - u\|_{L^1(\omega)} \leq C |h|$$

consequently

$$\boxed{\int_\omega |u(x-h) - u(x)| \, dx \leq C |h|} \quad *$$

so

$$\int_\omega |u(x-h) - u(x)|^q \, dx = \int_\omega |u(x-h) - u(x)| \underbrace{|u(x-h) - u(x)|^{q-1}}_{\leq (2\|u\|_{L^\infty})} dx$$

I can do this since $u \in L^\infty$ by Sobolev

$$\leq \underbrace{\int_\omega |u(x-h) - u(x)| \, dx}_{* \leq Ch} \cdot \left(2 \|u\|_{L^\infty}\right)^{q-1}$$

$$\left(\int_\omega |u(x-h) - u(x)|^q \, dx \right)^{\frac{1}{q}} \leq \left(C h \left(2 \|u\|_{L^\infty}\right)^{q-1} \right)^{\frac{1}{q}}$$

$$\leq \tilde{C} |h|^{\frac{1}{q}}$$

this is fixed in B

It is sufficient to take $\tilde{C} |h|^{\frac{1}{q}} < \varepsilon$

so that $\boxed{|h| \leq \underbrace{\left(\frac{\varepsilon}{\tilde{C}} \right)^q}_{= \delta}}$.

point b) $\forall \varepsilon > 0 \quad \exists\, \omega \subset\subset I$ s.t.

$$\forall u \in B \qquad \|u\|_{L^q(I \setminus \omega)} \leq \varepsilon$$

$$\|u\|^q_{L^q(I\setminus\omega)} = \int_{I\setminus\omega} |u(x)|^q \, dx$$

$\underline{\underline{u \in L^\infty}}$ thanks to Sobolev

$$\leq |I\setminus\omega| \cdot \|u\|^q_{L^\infty}$$

$\underline{\underline{\text{a priori bounded}}}$

$$\|u\|_{L^q(I\setminus\omega)} \leq |I\setminus\omega|^{\frac{1}{q}} \|u\|_{L^\infty(I)} \leq \varepsilon$$

It is easy to find $\omega \subset\subset I$ s.t.

$|I\setminus\omega|^{\frac{1}{q}}$ small

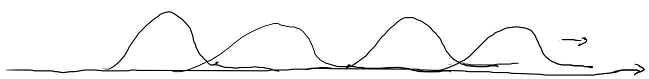
(thanks to the fact that $|I| < +\infty$)

QED

Remark a) $|I| < +\infty$ this is not avoidable in Rellich

$u \in \mathcal{C}_0^\infty(\mathbb{R})$

$u_n(x) = u(x-n)$ translation



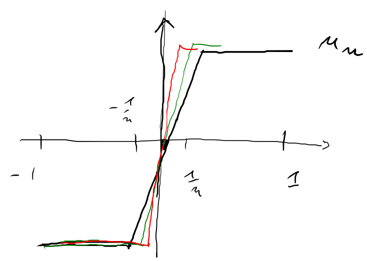
$(u_n)_n$ is bounded in $W^{1,p}(\mathbb{R})$

$(u_n)_n$ has not converging subsequence in $\mathcal{C}(\mathbb{R})$

b) in $W^{1,1}(I)$ there is not compact immersion in $\mathcal{C}(I)$

Ex. $I =]-1, 1[$

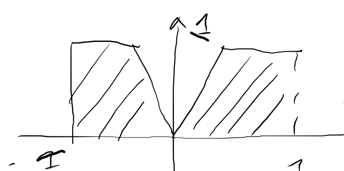
$$u_n(x) = \begin{cases} -1 & -1 \leq x \leq -\frac{1}{n} \\ nx & -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 1 & \frac{1}{n} \leq x \leq 2 \end{cases}$$



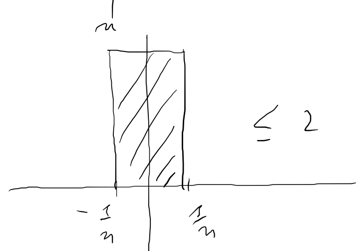
$$\|u_n\|_{W^{1,1}} \leq 4$$

$$\| \; = \; \|u_n\|_{L^1(-1,1)} + \|u_n'\|_{L^1(-1,1)}$$

$$u_n'(x) = \begin{cases} 0 & x < -\frac{1}{n} \\ n & -\frac{1}{n} < x < \frac{1}{n} \\ 0 & x > \frac{1}{n} \end{cases}$$



≤ 2



≤ 2

Try to prove that $(u_n)_n$ does not have a subsequence which converges in L^∞ norm to a continuous function

Corollaries to Sobolev Th.

1) Suppose $u \in W^{1,p}(\mathbb{R})$ with $p < +\infty$

then $\lim_{t \to +\infty} u(t) = \lim_{t \to -\infty} u(t) = 0$

proof. Fix $\varepsilon > 0$.

take $u \in W^{1,p}(\mathbb{R})$, $p < +\infty$.

we know that $\mathcal{D}(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R})$

(remember $p < +\infty$)

$\exists (u_n)_n$ in $\mathcal{D}(\mathbb{R})$

$u_n \longrightarrow u$ in $W^{1,p}(\mathbb{R})$

From Sobolev $\|u_n - u\|_{L^\infty} \leq C \|u_n - u\|_{W^{1,p}}$

so $u_n \longrightarrow u$ in L^∞ (i.e. uniformly)

so $\exists \bar{n}$ s.t $\|u_{\bar{n}} - u\|_{L^\infty} < \varepsilon$

$u_{\bar{n}} \in \mathcal{C}_0^\infty(\mathbb{R})$ $\exists R > 0$ s.t $|x| > R$

$u_{\bar{n}}(x) = 0$

$\forall \varepsilon, \exists R$ s.t

conclusion $|u(x)| < \varepsilon$ if $|x| > R$

$\Rightarrow$ limit ...

2) $\forall u, v \in W^{1,p}(I)$ $\quad (p \in [1,+\infty])$

then $u \cdot v \in W^{1,p}(I)$ and $(uv)' = u'v + u \cdot v'$

(i.e. $W^{1,p}(I)$ is an algebra)

<u>sketch of the proof</u>.

$$u, v \in W^{1,p}(I) \overset{\text{Sobolev}}{\Longrightarrow} u, v \in L^\infty(I)$$

so that $\underset{L^p}{u} \cdot \underset{L^\infty}{v} \in L^p(I)$

and $\underset{L^p}{u'}\underset{L^\infty}{v} + \underset{L^\infty}{u}\underset{L^p}{v'} \in L^p(I)$

it remains to prove that $(uv)' = (u'v + uv')$

i.e. $\forall \varphi \in C_0^\infty(I)$

$$\int_I (uv)\varphi' = -\int_I (u'v + uv')\varphi$$

suppose first $1 \leq p < +\infty$.

$\exists (u_n)_n, (v_n)_n$ in $C_0^\infty(\mathbb{R})$ s.t.

$u_n|_I \longrightarrow u$ in $W^{1,p}(I)$ $\quad$ this sequence is bounded in $W^{1,p}(I)$ $\Downarrow$ bounded in L^∞

$v_n|_I \longrightarrow v$ in $W^{1,p}(I)$

$$\|u_n|_I v_n|_I - uv\|_{L^p(I)}$$

$$\leq \|u_n|_I v_n|_I - u_n|_I \cdot v\|_{L^p(I)} + \|u_n|_I v - uv\|_{L^p(I)}$$

$$\leq \underbrace{\|u_n|_I\|_{L^\infty}}_{\leq C} \underbrace{\|v_n|_I - v\|_{L^p}}_{\to 0} + \underbrace{\|u_n - u\|_{L^p}}_{\to 0} \underbrace{\|v\|_{L^\infty}}_{\leq C}$$

so $u_n|_I v_n|_I \longrightarrow uv$ in L^p

$$\|u_n'|_I v_n|_I - u'v\|_{L^p} \longrightarrow 0$$

$$\leq \|u_n'|_I v_n|_I - u' v_n|_I\|_{L^p} + \|u' v_n|_I - u'v\|_{L^p}$$

$$\leq \underbrace{\|v_n|_I\|_{L^\infty}}_{\text{bounded}} \underbrace{\|u_n'|_I - u'\|_{L^p}}_{\to 0} + \|u'\|_{L^p} \underbrace{\|v_{n_I} - v\|_{L^\infty}}_{\to \text{ from Sobolev}}$$

so $u_n'v_n + u_n v_n' \longrightarrow u'v + uv'$ in L^p

if $p = +\infty$ we obtain a <u>similar identity</u> (trick)

3) let $u \in W^{1,p}(I)$

let $G : \mathbb{R} \to \mathbb{R}$ be a $C^1(\mathbb{R})$ function

with $G(0) = 0$

then $v(t) = G(u(t))$ is in $W^{1,p}(I)$

and $v'(t) = G'(u(t)) \cdot u'(t)$

<u>proof</u>. similar to the previous one.

<u>example</u> $G(u(t)) \in L^p(I)$

since $|G(u(t))| \leq C\, |u(t)|$

$\uparrow$

since $u(t) \in L^\infty$

$$G(s) = G(0) + s\, G'(\theta)$$

$$0 < \theta < s$$

so that $|G(s)| \leq |s| \underbrace{\max_{|\theta| \leq 1} |G'|}_{C}$

• the space $W^{m,p}(I)$

let $m \in \mathbb{N}$, $m \geq 2$.

def. $u \in W^{m,p}(I)$ if $u \in W^{m-1,p}(I)$ and $u' \in W^{m-1,p}(I)$

so $W^{2,p}(I) = \{ u \in W^{1,p}(I) : u' \in W^{1,p}(I) \}$

at the end

$$W^{m,p}(I) = \{ u \in L^p(I) : \underbrace{u', u'', u^{(3)}, \dots, u^{(m)}}_{\text{in the sense of distribution}} \in L^p(I) \}$$

$$\|u\|_{W^{m,p}} = \left(\|u\|_{L^p}^p + \|u'\|_{L^p}^p + \dots + \|u^{(m)}\|_{L^p}^p \right)^{\frac{1}{p}}$$

$W^{m,2} = H^m =$

$$H^m(\mathbb{R}) = \{ u \in L^2(\mathbb{R}) : \hat{u}(\xi) \cdot (1 + |\xi|^2)^{m/2} \in L^2 \}$$

The space $W_0^{1,p}(I)$

def. $|I| < +\infty$, $p \in [1, +\infty[$.

$W_0^{1,p}(I)$ = closure of $C_0^\infty(I)$ in $W^{1,p}(I)$.