Thm: $X \underset{\text{closed}}{\subset} \mathbb{P}^n$ projective variety, k algebraically closed

Then X is a complete variety.

Pf: 1) it is enough to prove that $\underbrace{p_2 : \mathbb{P}^n \times \mathbb{A}^m \longrightarrow \mathbb{A}^m}$ is closed.

Ass. p_2 is closed, $Z \subset X \times \mathbb{A}^m$ closed $\Rightarrow Z$ is closed in $\mathbb{P}^n \times \mathbb{A}^m$ $\Rightarrow$ $p_2(Z)$ is closed in $\mathbb{A}^m$.

So also $p_2\big|_{X \times \mathbb{A}^m}$ is closed.

Y any quasi proj. var., $Z \subseteq X \times Y \longrightarrow Y$

Y has an open covering with aff. varieties $Y = \bigcup U_i$

$X \times Y = \bigcup (X \times U_i)$ open covering of $X \times Y$

Z is closed in $X \times Y \Leftrightarrow Z \cap (X \cap U_i)$ is closed in $X \cap U_i$ $\forall i$

$p_2(Z)$ is closed in $Y \Leftrightarrow Z \cap U_i$ is closed in U_i

we can replace Y with U_i i.e. we can ass. that Y is affine and closed in $\mathbb{A}^m$, up to isom. Then we can replace Y with $\mathbb{A}^m$

Z closed in $\mathbb{P}^m \times Y \underset{\text{closed}}{\subset} \mathbb{P}^m \times \mathbb{A}^m$

$p_2 : \mathbb{P}^m \times \mathbb{A}^m \longrightarrow \mathbb{A}^m$ $\quad$ $p_2(Z)$ is closed in $\mathbb{A}^m$, $p_2(Z) \subseteq Y$

$\Rightarrow$ closed in Y.

2) $p_2 : \mathbb{P}^n \times \mathbb{A}^m \longrightarrow \mathbb{A}^m$ $\quad Z \subseteq \mathbb{P}^n \times \mathbb{A}^m$

We want to express that Z is closed in $\mathbb{P}^n \times \mathbb{A}^m$ using coordinates $[x_0, \dots, x_n]$ on $\mathbb{P}^n$, $(y_1, \dots, y_m)$ on $\mathbb{A}^m$

$\mathbb{A}^m = U_0 \subseteq \mathbb{P}^m$

$\sigma : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \Sigma \subseteq \mathbb{P}^N$

$[x_0 \dots x_n][y_0 \dots y_m] \longrightarrow [w_{ij} = x_i y_j]$

Z is closed : $Z = \sigma(\mathbb{P}^n \times \mathbb{A}^m) \cap V$ ← V closed in $\mathbb{P}^N$

$V = V_p(F_h(w_{ij}) \mid h = 1, \dots, r)$

$V \cap \sigma(\mathbb{P}^n \times \mathbb{P}^m) = V \cap \Sigma = \{[w_{ij}] \mid w_{ij} = x_i y_j, \; F_h(x_i y_j) = 0$

$F_h(x_i y_j)_{ij \dots} \in K[x_0 \dots x_n, y_0, \dots y_m]$ homogeneous in $x_0 \dots x_n$ and in $y_0 \dots y_m$ of the same degree

$V \cap \Sigma$: is the set of point in $\mathbb{P}^n \times \mathbb{P}^m$ satisfying a system of equations homog. of same degree in $x_0 \dots, y_0 \dots$.

If we have pol. $F(x_0 \dots x_n y_0 \dots y_m)$ homog. of deg d in $x_0 \dots$ and $y_0 \dots$ $\Longrightarrow$ $F' \in K[w_{ij}]$: when a product $x_i y_j$ appears we replace it with w_{ij}.

$x_1 y_2 x_3 y_4$ $\begin{cases} w_{12} w_{34} \\ w_{14} w_{32} \end{cases}$

The ambiguity in the choice disappears if we take $V_p(F') \cap \Sigma$

Closed subvarieties of $\Sigma \longleftrightarrow$ set of zeros of bihomogeneous polyn. in $x_0 \dots x_n, y_0 \dots y_m$ of same degree

$F(x_0 \dots, y_0 \dots)$ homog of deg d in $x_0 \dots x_n$
" d' $y_0 \dots y_m$

$d > d'$ $\quad F(\underbrace{x_0 \dots x_n}_{d}, \underbrace{y_0 \dots y_m}_{d'}) = 0$:

$$\begin{cases} y_0 F(x_0 \dots x_n \dots \dots) = 0 \\ y_1 F(\quad) = 0 \\ \vdots \\ y_m F(\quad) = 0 \end{cases}$$

the solutions with $[y_0 \dots y_m] \neq (0 \dots 0)$ are the same

Repeating from F, get an equivalent system

$\{ y_0^{i_0} \dots y_m^{i_m} F = 0 \qquad i_0 + \dots + i_m = d - d'$

Any system of equat. of polyn. homog. in $x_0 \dots x_n$, and $y_0 \dots y_m$ separately defines a closed subvariety in $\Sigma \longleftrightarrow \mathbb{P}^n \times \mathbb{P}^m$

$\mathbb{P}^n \times \mathbb{A}^m = \mathbb{P}^n \times U_0 \subset \mathbb{P}^n \times \mathbb{P}^m$

$[y_0 \dots y_m] \longrightarrow [1, y_1 \dots y_m]$

$F(x_0 \dots x_n, y_0 \dots y_m) \longrightarrow F(x_0 \dots x_n; 1, y_1 \dots y_m)$

homog. in $x_0 \dots x_n$

$Z \subseteq \mathbb{P}^n \times \mathbb{A}^m$: set of zeros of

$F_1(x_0 \dots x_n, y_1 \dots y_m)$, $\deg_x F_1 = d_1$. homog. in $x_0 \dots x_n$

$p_2(Z) \subseteq \mathbb{A}^m$: claim this is closed in $\mathbb{A}^m$

3) $p_2(Z) = \{(\bar y_1, \dots, \bar y_m) \in A^m \mid \exists P \in \mathbb{P}^n \text{ s.t. } \forall i \; F_i(P, \bar y_1 \dots \bar y_m) = 0\} =$

$= \{(\bar y_1 \dots \bar y_m) \in A^m \mid \langle F_i(x_0 \dots x_n, \bar y_1 \dots \bar y_m) \mid i = 1 \dots t \rangle$ has at least one zero in $\mathbb{P}^n \}$

If $I \subset K[x_0 \dots x_n]$ is homog. $V_P(I) = \emptyset \iff$ $\exists\, d \geq 1$ s.t. $I \supseteq K[x_0 \dots x_n]_d$

$$p_2(Z) = \{\bar y \in A^m \mid \forall d \geq 1 \; \langle F_i(x_0 \dots x_n, \bar y_1 \dots \bar y_m) \mid i \dots \rangle \underset{K[x_0 \dots x_n]_d}{\cancel{\supseteq}} \}$$

$$= \bigcap_{d \geq 1} \underbrace{\{\bar y \in A^m \mid \langle F_i(x_0 \dots x_n, \bar y_1 \dots) \rangle \not\supseteq K[x_0 \dots x_n]_d \}}_{T_d}$$

To prove the thm, it is enough to prove that T_d is closed in A^m $\forall d \geq 1$

$T_d = \{ \bar{y}(\bar{y}_1 \dots \bar{y}_m) \in A^m \mid \langle F_i(x_0 \dots x_n, \bar{y}_1 \dots \bar{y}_m) \rangle \not\supseteq K[x_0 \dots x_n]_d \}$

$\{M_\alpha\}_{\alpha = 1 \dots \binom{n+d}{d}}$ monomials of deg d : basis for K - vector space

d_i : what means that $\langle F_i(x_0 \dots \quad), i = 1 \dots t \rangle$?
$$\cup\!\!\mid$$
$$K[x_0 \dots x_n]_d$$

$\iff$ every M_α is a combination of the pol. $F_i(\dots)$.
if this happens the coeff. are homog. of deg $d - d_i$.

$\forall i$ $\{N_i^\beta\}$ monomials of deg $d - d_i$

$\iff$ M_α is a linear combination of

$N_i^\beta F_i(x_0 \dots x_n, \bar{y}_1 \dots \bar{y}_m)$, $i = 1 \dots t$

$\forall \alpha$

$\{N_i^\beta F_i\}$ generate the vector space $K[x_0 \dots x_n]_d$

Matrix M of the coordinates of the pol.

$N_i^\beta F_i$ w.r. to the basis $\{M_\alpha\}$.

its rank is maximal $\binom{d+n}{n}$ $\iff$ $\langle \dots \rangle \supseteq K[x_0 \dots x_n]_d$

$T_d = \{ \bar{y} \mid$ the matrix has rank $< \binom{n+d}{d} \}$

in the set of zeros of minors of order

$\binom{n+d}{d}$ of matrix : closed

Cor. $f : X \longrightarrow \mathbb{P}^n$, regular, X proj. variety

X complete $\Longrightarrow$ $f(X)$ is closed in $\mathbb{P}^n$: it is a proj. variety

$f : X \longrightarrow Y \subseteq \mathbb{P}^n$: $f(X)$ is a proj. variety

Cor. Y q. proj. var. $Y \simeq X$ proj. var. $\Longrightarrow$

Y is projective.