THE OSCILLATION AMPLITUDE $a_0 \equiv \frac{v_0}{c} = \frac{p_0}{m_e c} = \frac{eE_0}{m_e \omega c}$

IN MANY ARTICLES a_0 IS REFERRED AS "QUIVER" VELOCITY OR MOMENTUM. THE LASER INTENSITY, WAVELENGTH λ_L, E_0 AND ω ARE RELATED THROUGH

$$I_L = \frac{1}{2}\varepsilon_0 c E_0^2 , \quad \lambda_L = \frac{2\pi c}{\omega} \Rightarrow$$

$$a_0 \simeq 0.85 \left(I_{18} \lambda_\mu^2 \right)^{1/2} \text{ WHERE } I_{18} = \frac{I_L}{10^{18} (W/cm^{-2})}$$

$\lambda_\mu = \frac{\lambda_L}{\mu m}$. FROM THIS EXPRESSION IT CAN BE SEEN THAT WE WILL HAVE RELATIVISTIC v_e, OR $a_0 \sim 1$ FOR INTENSITIES $I_L \geq 10^{18} W/cm^2$ AT $\lambda_L \simeq 1 \mu m$.

• WAVE PROPAGATION IN PLASMA

THERE ARE SEVERAL POSSIBLE WAYS IN WHICH A PLASMA CAN SUPPORT WAVES, DEPENDING ON THE LOCAL CONDITIONS, THE PRESENCE OF EXTERNAL ELECTRIC OR MAGNETIC FIELDS ETC. TO DERIVE AND ANALYSE WAVE PHENOMENA THERE ARE SEVERAL THEORETICAL APPROACH:

1. FIRST PRINCPLE N-BODY MOLECULAR DYNAMICS
2. VLASOV-BOLTZMANN EQUATION (PHASE-SPACE METHODS)
3. TWO FLUID EQUATIONS
4. MAGNETOHYDRODYNAMICS (SINGLE MAGNETIZED FLUID).

THE FIRST IS RATHER COSTLY AND LIMITED TO MUCH SMALLER REGIONS OF PLASMA THAN USUALLY NEEDED TO DESCRIBE THE COMMON TYPES OF WAVE. INDEED, THE NUMBER OF PARTICLES NEEDED FOR MODELLING THE PLASMA IN A TOKAMAK IS $\sim 10^{21}$ AND FOR A LASER PLASMA $\sim 10^{20}$. IN FACT MANY PLASMA PHENOMENA CAN BE ANALYSED ASSUMING THAT EACH CHARGED PARTICLE COMPONENT OF DENSITY n_s AND VELOCITY $\bar{u}_s$ BEHAVES IN A FLUID-LIKE MANNER INTERACTING WITH OTHER SPECIES VIA THE $\bar{E}$ AND $\bar{B}$ FIELDS. THIS IS THE IDEA BEHIND APPROACH ③. THE RIGOROUS WAY TO DERIVE THE GOVERNING EQS. IN THIS APPROXIMATION IS VIA KINETIC THEORY, STARTING FROM METHOD ②, WHICH IS BEYOND THE SCOPE OF THESE LECTURES. FINALLY, SLOW WAVE PHENOMENA ON MORE MACROSCOPIC, ION-TIME SCALE CAN BE HANDLED WITH APPROACH ④.

FOR THE PRESENT PURPOSES WE START FROM THE TWO-FLUIDS EQUATIONS FOR A PLASMA WITH FINITE TEMPERATURE ($T_e > 0$) THAT IS ASSUMED TO BE COLLISIONLESS ($\nu_{ie} \to 0$) AND NON-RELATIVISTIC ($v \ll c$). THE EQUATIONS GOVERNING THE PLASMA DYNAMICS UNDER THESE CONDITIONS ARE

① $\partial_t n_s + \bar{\nabla} \cdot (n_s \bar{u}_s) = 0$ ③ $\frac{d}{dt}\left(P_s n_s^{-\gamma_s}\right) = 0$

② $n_s m_s \frac{d u_s}{dt} = n_s q_s \left(\bar{E} + \bar{u}_s \times \bar{B}\right) - \bar{\nabla} P_s$

WHERE "S" REFERS TO THE SPECIES "S"

P_s IS THE THERMAL PRESSURE (THIS IS A THERMODYNAMICS TERM, KNOWN ALSO AS THERMODYNAMIC PRESSURE COEFFICIENT, IS A MEASURE OF THE RELATIVE PRESSURE CHANGE OF A FLUID OR A SOLID AS A RESPONSE TO THE TEMPERATURE CHANGE AT CONSTANT VOLUME. IN GENERAL PRESSURE P CAN BE WRITTEN AS THE SUM $P_{TOT}(V,T) = P_{REF}(V,T) + \Delta P_{THER}(V,T)$ WHERE P_{REF} IS THE PRESSURE REQUIRED TO COMPRES THE MATTER FROM $V_0 \to V$ @ $T_0 = COST.$)

γ_s IS THE SPECIFIC HEAT RATIO [OR $(2+N)/N$ WITH N THE NUMBER OF DEGREES OF FREEDOM.

LET'S TRY NOW TO INTERPRET THE PLASMA DYNAMICS EQS. EQ. (1) HAS THE FORMAL STRUCTURE OF A CONTINITY EQ. AND IT TELLS US THAT (IN THE ABSENCE OF IONIZATION OR RECOMBINATION) THE NUMBER OF PARTICLES OF EACH SPECIES IS CONSERVED. WE CAN NOTE THAT UNDER THE ABOVE CONDITIONS $\rho_s = q_s n_s$ AND $\bar{J}_s = q_s n_s \bar{u}_s$ SO THAT MULTIPLING BY q_s

$$\left[\partial_t n_s + \bar{\nabla}\cdot(n_s \bar{u}_s)\right] q_s = 0 \Rightarrow \partial_t \rho_s + \bar{\nabla}\cdot\bar{J}_s = 0$$

THAT ~~EXPRESS~~ THE CHARGE CONSERVATION.

EQ. 2 GOVERNS THE MOTION OF A FLUID ELEMENT SPECIES IN THE PRESENCE OF ELECTRIC AND MAGNETIC FIELDS $\bar{E}$ AND $\bar{B}$. IT IS CLEARLY DERIVED FROM THE LORENTZ FORCE IF WRITTEN AS

$$m_s \frac{d\bar{u}_s}{dt} = q_s\left(\bar{E} + \bar{u}_s \times \bar{B}\right) - \frac{1}{n_s}\bar{\nabla} P_s$$

WHERE THE TERM $\frac{1}{n_s}\bar{\nabla}P_s$ REPPRESENTS THE FORCE ON A PARTICLE "S" DUE TO THE CHANGE OF THE THERMAL PRESSURE. IN THE ABSENCE OF THE FIELDS AND ASSUMING STRICT QUASI-NEUTRALITY ($n_e = Zn_i = n$; $\bar{u}_e = \bar{u}_i = \bar{u}$) WE RECOVER THE MORE FAMILIAR $\partial_t \bar{u} + (\bar{u}\cdot\bar{\nabla})\bar{u} = \frac{1}{\rho}\bar{\nabla}P$. THESE SEMPLIFIED ① AND ② EQUATIONS

$$\begin{cases} \partial_t \rho + \bar{\nabla}\cdot(\rho\bar{u}) = 0 \\ \partial_t \rho + (\bar{u}\cdot\bar{\nabla})\bar{u} = \frac{1}{\rho}\bar{\nabla}P \end{cases}$$

ARE KNOWN AS NAVIER-STOKES EQS. THE N-S EQS ARE A SET OF PARTIAL DIFFERENTIAL EQS. WHICH DESCRIBE THE MOTION OF VISCOUS FLUIDS. THESE EQ. MATHEMATICALLY EXPRESS THE MASS AND MOMENTUM CONSERVATION FOR NEWTONIAN FLUIDS. (FOR AN INFORMATIVE OVERVIEW ABOUT THE N-S EQS I RECCOMAND https://en.wikipedia.org/wiki/Navier-stokes_equations).

BY CONTRST IN THE PLASMA ACCELERATOR CONTEXT WE USUALLY DEAL WITH TIME SCALES OVER WHICH THE IONS CAN BE ASSUMED TO BE MOTIONLESS, i.e. $\bar{u}_i = 0$ AND ALSO UNMAGNETIZED PLASMA SO THAT THE MOMENTUM EQUATION READS

$$n_e m_e \frac{d\bar{u}_e}{dt} = -en_e\bar{E} - \bar{\nabla}P_e$$

($\bar{E}$ CAN INCLUDE BOTH EXTERNAL AND INTERNAL FIELDS COMPONENTS).

LONGITUDINAL (LANGMUIR) WAVES.

A CHARACTERISTIC PROPERTY OF PLASMA IS THEIR ABILITY TO TRANSFER MOMENTUM AND ENERGY VIA COLLECTIVE MOTION. ONE OF THE MOST IMPORTANT EXAMPLE OF THIS OSCILLATION OF ELECTRONS VERSUS A STATIONARY ION BACKGROUD (LATTICE IN A SOLID) ARE THE LANGMUIR WAVES. RETURNING TO THE TWO-MODEL FLUID EQS 1-3 CAN BE SEMPLIFIED. SETTING $\bar{u}_i = 0$ AND RESTRICTING THE ELECTRONS MOTION TO 1D → X, I.E. TAKING $\partial_y = \partial_z = 0$; WE OBTAIN

$$\partial_t n_e + \partial_x (n_e u_e) = 0 \quad (4)$$

$$n_e(\partial_t u_e + u_e \partial_x u_e) = -\frac{e}{m} n_e E - \frac{1}{m}\partial_x P_e \quad (5)$$

$$\frac{d}{dt}\left(\frac{P_e}{n_e^{\gamma_e}}\right) = 0 \quad (6)$$

THE SYSTEM 4-6 HAS 3 EQS AND 4 UNKNOWN TERMS TO CLOSE IT WE NEED AN EXPRESSION FOR THE $\bar{E}$-FIELD, SINCE $\bar{B} = 0$. IT CAN BE FOUND FROM THE POISSON EQUATION (GAUSS LAW) WITH $Zn_i = n_0$

$$\partial_x E = \frac{e}{\varepsilon_0}(n_0 - n_e) \quad (7)$$

THE SET OF EQS 4-7 IS NON-LINEAR (A PART FROM A FEW SPECIAL CASES) AND IT CANNOT BE SOLVED EXACTLY, A PRACTICAL TECHNIQUE IS TO LINEARISE THE EQS 4-7 ASSUMING THAT THE PERTURBED AMPLITUDES ARE SMALL COMPARED TO EQUILIBRIUM VALUES ⇒

8 $\begin{cases} n_e = n_0 + n_1 , \\ u_e = u_1 \\ P_e = P_0 + P_1 ; \\ E = E_1 \end{cases}$ WHERE $n_1 \ll n_0$ AND $P_1 \ll P_0$

BY SUBSTITUTING THESE EXPRESSION IN 4–7 AND NEGLECTING ALL PRODUCTS OF PERTURBATION SUCH AS $n_1 \partial_t u_1$ AND $u_1 \partial_x u_1$ WE OBTAIN A SET OF LINEAR EQS FOR THE PERTURBED QUANTITIES

(9) $\partial_t n_1 + n_0 \partial_t u_1 = 0$

(10) $n_0 \partial_t u_1 = -\frac{e}{m} n_0 E_1 - \frac{1}{m} \partial_x P_1$

(11) $\partial_x E_1 = -\frac{e}{\varepsilon_0} n_1$

(12) $P_1 = 3 K_B T_e n_1$

THE EXPRESSION FOR P_1 RESULTS FROM THE SPECIFIC HEAT RATIO $\gamma_e = 3$ AND FROM ASSUMING ISOTHERMAL BACKGROUD ELECTRONS, $P_0 = K_B T n_0$ (IDEAL GAS) (REF. W. KRUER, THE PHYSICS OF PLASMA LASER INTERACTIONS (ADDISON-WESLEY, BOSTON 1988), WE CAN NOW ELIMINATE E_1, P_1 AND u_1 FROM 9–12 TO GET

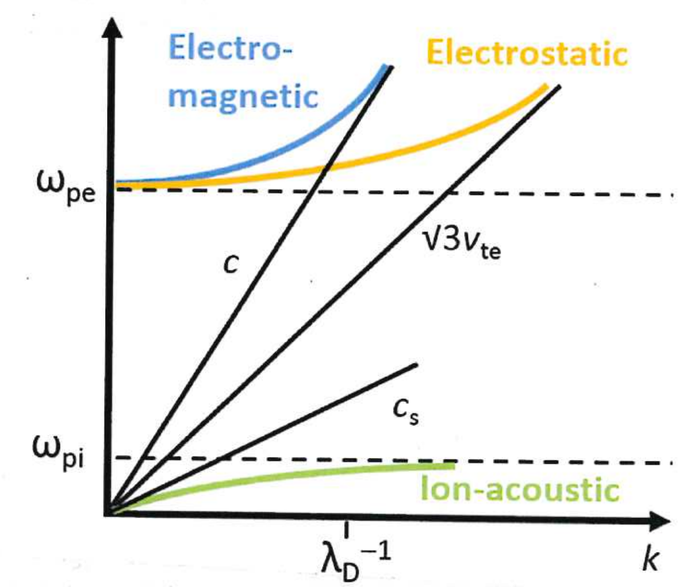
(13) $\left(\partial_t^2 - 3 V_{Te}^2 \partial_x^2 + \omega_p^2\right) n_1 = 0$

WHERE $V_{Te}^2 = K_B T_e / m_e$ AND $\omega_p = \left(\frac{e^2 n_e}{\varepsilon_0 m_e}\right)^{1/2}$

WE CAN EASELY RECOGNIZED IN (13) AN INHOMOGENEOUS WAVE EQUATION IN 1D WHERE THE $\square^2$ IS $\partial_t^2 - \partial_x^2$ WITH A KNOWN TERM GIVEN BY ω_p. AN EASY WAY TO SOLVE (13) IS TO ADOPT A GENERIC PLANE WAVE SOLUTIONS SUCH AS $A = A_0 \exp\{i(kx - \omega t)\}$ TO OBTAIN THE CORRESPONDENT ALGEBRIC EQ, WHICH IS KNOWN AS THE BOHM-GROSS DISPERSION RELATION

$$\omega^2 = \omega_p^2 + 3k^2 V_{Te}^2 \quad (14)$$

(A MUCH MORE EXTENSIVE ANALITICAL FORMULATIO CAN BE FOUND IN MOODLE AED_LECTURE#5_APPE 1 PLASMA), A SCHEMATIC PICTURE OF THESE DISPERSION RELATIONS IS GIVEN IN THE FOLLOWING FIGURE



THIS FIGURE GIVES AN OVERVIEW OF WHICH PROPAGATION MODES ARE PERMITTED FOR LOW AND HIGH-WAVELENGTH LIMITS.

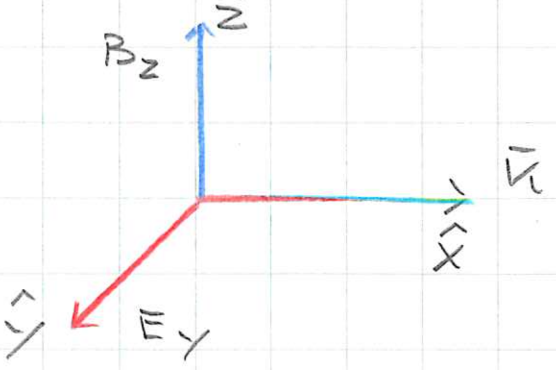
TRANVERSE WAVES TO DESCRIBE TRANSVERSE E.M. WAVES WE NEED TWO ADDITIONAL MAXWELL EQS: FARADAY AND AMPÈRE-MAXWELL LAWS THAT WILL BE INTRODUCED LATER. FOR THE TIME BEING LET'S START BY MAKING USE, AS WE DID IN THE PREVIOUS ANALYSIS, OF SMALL DISPLACEMENTS RESPECT TO THE EQUILIBRIUM POSITION. THEREFORE, WE LINEARISE AND APPLY THE HARMONIC APPROXIMATIONS $(\partial_t \to i\omega ; \partial_x \to -ik)$ TO GET

$$\bar{\nabla} \times \bar{E}_1 = -i\omega \bar{B}_1 \qquad (15)$$

$$\bar{\nabla} \times \bar{B}_1 = \mu_0 \bar{J}_1 + i\, \varepsilon_0 \mu_0 \omega \bar{E}_1 \qquad (16)$$

WHERE THE TRANSVERSE CURRENT DENSITY IS GIVEN BY $\bar{J}_1 = -n_0 e \bar{u}_1$

THIS TIME WE LOOK FOR PURE E.M. PLANE-WAVE SOLUTIONS WITH $\bar{E}_1 \perp \bar{k}$ $\bar{B}_1 \perp \bar{k}$ AND $\bar{E}_1 \perp \bar{B}_1$



WE ALSO ASSUME THAT THE GROUP AND PHASE VELOCITIES ARE LARGE ENOUGH, $V_p, V_g \gg V_{Te}$ SO THAT WE HAVE A COLD PLASMA WITH $P_e = n_0 K_B T_e \simeq 0$

THE LINEARIZED ELECTRON FLUID VELOCITY IS DERIVED FROM $m_e \frac{d\bar{u}_1}{dt} = -e\bar{E}_1$ WITH $\frac{d}{dt} \to i\omega$

(17) $-e\bar{E}_1 = m_e i\omega \bar{u}_1 \Rightarrow \bar{u}_1 = -\frac{e}{i\omega m_e}\bar{E}_1$ AND

(18) $\bar{J}_1 = -n_0 e u_1 = \frac{n_0 e^2}{i\omega m_e}\bar{E}_1 = \sigma \bar{E}_1$ WHERE σ_1 IS

THE AC ELECTRICAL CONDUCTIVITY, BY ANALOGY WITH DIELECTRIC MEDIA WHERE $\bar{\nabla}\times\bar{B}_1 = \mu_0 \partial_t D_1$ AND $\bar{D}_1 = \varepsilon_0 \varepsilon \bar{E}_1$ WITH $\varepsilon = 1 + \frac{\sigma}{i\omega\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}$ (19)

IT FOLLOWS THAT $\eta \equiv \sqrt{\varepsilon}$ (η IS HERE THE REFRACTIVE INDEX, USUALLY IDENTIFIED WITH n, IN ORDER TO AVOID CONFUSION WITH THE PARTICLES DENSITIES n_i, n_e, n_0 ... USED HERE)

$$\eta \equiv \sqrt{\varepsilon} = \frac{ck}{\omega} = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} \quad (20)$$

WITH $\omega^2 = \omega_p^2 + c^2k^2$. (21)

THIS EQUATION CAN ALSO BE FOUND BY ELIMINATIN $\bar{J}_1$ AND $\bar{E}_1$ FROM (15) AND (18). FROM THE DISPERSION RELATION (21) (AS SHOWN IN THE DISPERSION RELATION FIGURE ON PAG. 232) SEVERAL IMPORTANT FEATURES OF EM WAVE PROPAGATION IN PLASMAS CAN BE DEDUCED.

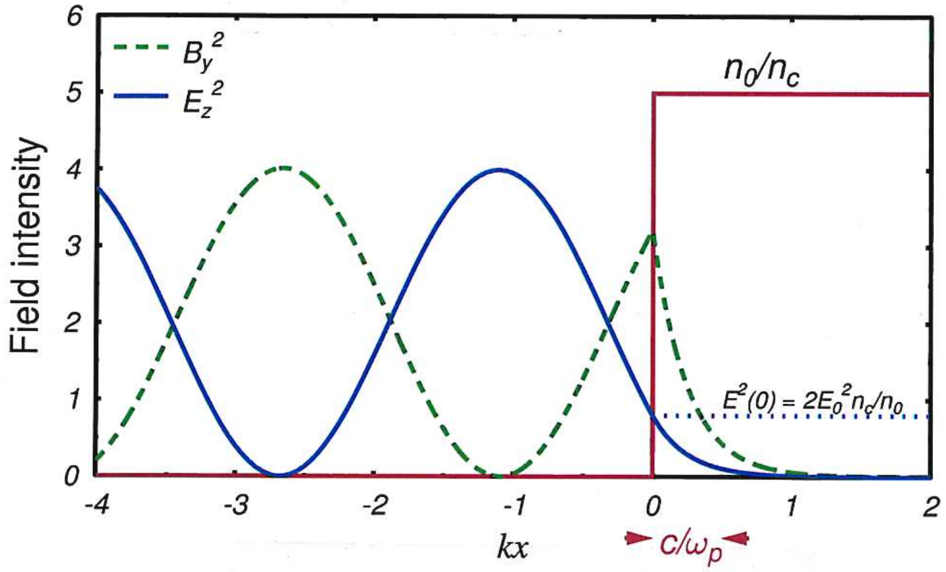
- FOR UNDER-DENSE PLASMA ($n_e \ll n_c$ - CRITICAL DENSITY)

PHASE VELOCITY $V_P = \frac{\omega}{k} \simeq c\left(1 + \frac{\omega_p^2}{2\omega^2}\right) > c$ (21)

GROUP VELOCITY $V_G = \frac{\partial\omega}{\partial k} \simeq c\left(1 - \frac{\omega_p^2}{2\omega^2}\right) < c$ (22)

IN THE OPPOSITE CASE OF A SUPERDENSE PLASMA

WHERE $n_e > n_c$ THE REFRECTIVE INDEX η BECOMES IMMAGINARY AND THE WAVE CANNOT LONGER PROPAGATE, BECOMING EVANESCENT INSTEAD, WITH A DECAY LENGTH DETERMINED BY THE COLLISIONLESS SKIN-DEPTH c/ω_p



- NON-LINEAR PROPAGATION

SO FAR WE HAVE CONSIDERED PURELY LONGITUDINAL OR TRANSVERSE WAVES DERIVED BY LINEARAZING THE WAVE EQS, THIS ENSURES THAT ANY NON-LINEARITIES OR COUPLING BETWEEN THESE TWO MODES IS EXCLUDED. ALTHOUGH THIS IS A REASONABLE APPROXIMATION FOR LOW AMPLITUDE WAVES IT IS INADEQUATE FOR DESCRIBING STRONGLY DRIVEN WAVES IN RELATIVISTIC REGIME.

THE STARTING POINT OF MOST ANALYSIS OF NON-LINEAR WAVE PROPAGATION PHENOMENA IS THE LORENTZ EQUATION OF MOTION FOR THE ELECTRON IN A COLD

$(T_e = 0)$ UNMAGNETIZED PLASMA ALONG WITH THE MAXWELL EQS. WE MAKE TWO FURTHER ASSUMPTION

i) THAT THE IONS ARE INITIALLY SINGLE CHARGED $(Z = 1)$ AND ARE TREATED AS AN IMMOBILE HOMOGENEOUS BACKGROUND $(V_i = 0)$ WITH $n_0 = Z n_i$

ii) THAT THE THERMAL MOTION CAN BE NEGLECTED SINCE THE TEMPERATURE REMAINS LOW COMPARED TO THE TYPICAL OSCILLATION ENERGY $(V_{osc} \gg V_{Te})$.

IT IS IMPORTANT HERE TO POINT OUT THAT NON-LINEAR OPTICAL EFFECTS ARE RELEVANT TO THE INTERACTION BETWEEN A PLASMA AND A STRONG LASER FIELD. THE STARTING EQS ARE THAN THE FOLLOWING

$$\partial_t \bar{p} + (\bar{V} \cdot \bar{\nabla}) \bar{p} = -e(\bar{E} + \bar{V} \times \bar{B}) \quad (23)$$

$$\bar{\nabla} \cdot \bar{E} = \frac{e}{\varepsilon_0}(n_0 - n_e) \quad (24)$$

$$\bar{\nabla} \times \bar{E} = -\partial_t \bar{B} \quad (25)$$

$$c^2 \bar{\nabla} \times \bar{B} = -\frac{e}{\varepsilon_0} n_e \bar{V} + \partial_t \bar{E} \quad (26)$$

$$\bar{\nabla} \cdot B = 0 \quad (27)$$

; WITH $\bar{p} = \gamma m_e \bar{V}$ AND $\gamma = (1 + p^2/m_e^2 c^2)^{-1/2}$

WE ASSUME A PLANE E.M. WAVE AS FOR THE FIG. ON PAGE 233 WITH THE TRANSVERSAL FIELDS ARE $\bar{E}_T = (0, E_y, 0)$ AND $\bar{B}_T = (0, 0, B_z)$. FROM EQ. 23 THE TRANSVERSE ELECTRON MOMENTUM

$$p_y = e A_y$$

AND $E_y = \partial_t A_y$, THIS RELATION EXPRESS THE CONSERVATION OF THE CANONICAL MOMENTUM SINCE $E_y = \partial_t \frac{P_y}{e} \Rightarrow eE_y = \partial_t P_y \Rightarrow F_y = \partial_t P_y$

BY SUBSTITUTING $\bar{E} = -\bar{\nabla}\phi - \partial_t \bar{A}$ AND $\bar{B} = \bar{\nabla}\times\bar{A}$ INTO (26) (THE AMPÈRE-MAXWELL EQ) YIELDS

$$c^2 \bar{\nabla}\times(\bar{\nabla}\times\bar{A}) + \partial_t^2 \bar{A} = \bar{J}/\varepsilon_0 - \nabla\partial_t \phi$$

WHERE $\bar{J} = -en_e\bar{v}$. NOW USING THE HELMHOLTZ THEOREM WE SPLIT THE CURRENT DENSITY VECTOR INTO ROTATIONAL (SOLENOIDAL) AND IRROTATIONAL (LONGITUDINAL) COMPONENTS

$$\bar{J} = \bar{J}_\perp + \bar{J}_\parallel = \underbrace{\bar{\nabla}\times\Pi}_{ROT} + \underbrace{\nabla\Psi}_{IRROT}$$

(SEE JACKSON'S BOOK AND AED LECTURE#5_APPEN_2.) WHERE $\bar{J}_\parallel = \frac{\Delta}{c^2}\bar{\nabla}\partial_t \bar{\phi}$ AND

$\bar{J}_\perp = \bar{J} - \bar{J}_\parallel$

IN THE COULOMB GAUGE $\bar{\nabla}\cdot\bar{A} = 0$ AND $v_y = eA_y/\gamma$

$$\Rightarrow \partial_t^2 A_y - c^2\nabla^2 A_y = \mu_0 \bar{J} = -\frac{e^2 n_e}{\varepsilon_0 m_e \gamma} A_y$$

THE NON-LINEAR TERM ON THE RIGHT-HAND SIDE CONTAINS TWO IMPORTANT BITS OF PHYSICS $n_e = n_0 + \delta n$, WHICH COUPLES THE E.M. WAVES TO THE PLASMA WAVES, AND $\gamma = \left(1 + P^2/m_e^2c^2\right)^{-1/2}$

WHICH INTRODUCES RELATIVISTIC EFFECTS THROUGH THE INCREASE ELECTRON INERTIA.

TAKING THE LONGITUDINAL COMPONENT OF THE MOMENTUM (23) GIVES

$$\frac{d}{dt}\left(\gamma m_e v_x\right) = -e E_x - \frac{e^2}{2 m_e \gamma} \partial_x^2 A_y^2$$

WE CAN ELIMINATE V_x USING THE X COMPONENT OF THE AMPÈRE-MAXWELL LAW ⇒ (26)

$$0 = -\frac{e}{\varepsilon_0} n_e v_x + \partial_t E_x$$

THESE EQS CAN BE SOLVED NUMERICALLY. TO OUR PURPOSES WE CAN SEPLIFY THE PHYSICS BY LINEARIZING THE PLASMA FLUID QUANTIES.

LET $n_e \simeq n_0 + n_1 + \cdots$; $V_x \simeq V_1 + V_2 + \ldots$

AND NEGLET PRODUCTS OF PERTURBATION SUCH AS $n_1 V_1$. THIS TO