The Non-Homogeneous Wave Equation

The wave equation, with sources, has the general form

$$\nabla^2\Psi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi(\mathbf{r},t) = F(\mathbf{r},t) \qquad \text{A}$$

Solutions to the homogeneous wave equation,

$$\nabla^2\Psi_0(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi_0(\mathbf{r},t) = 0$$

have the following solution:

$$\Psi_0(\mathbf{r},t) = h(t)\Psi_0(\mathbf{r})$$

Separating the variables and letting the separation constant be $(ik)^2$ where k is real :

$$\frac{\nabla^2\Psi_0(\mathbf{r})}{\Psi_0(\mathbf{r})} = \frac{1}{c^2h(t)}\frac{\partial^2}{\partial t^2}h(t) = (ik)^2$$

and

$$\nabla^2\Psi_0(\mathbf{r}) + k^2\Psi_0(\mathbf{r}) = 0$$

$$\frac{\partial^2}{\partial t^2}h(t) + k^2c^2h(t) = 0$$

Where we define:

$$\mathbf{k}\cdot\mathbf{k} = k_x^2 + k_x^2 + k_2^2$$

$$k^2c^2 = \omega^2$$

note that there are three separation constants and **k** is a vector.

$$\nabla^2\Psi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi(\mathbf{r},t) = 0$$

$$\left[-k^2 + \frac{\omega^2}{c^2}\right]\Psi(\mathbf{r},t) = 0$$

A solution to the homogeneous wave equation can be written as follows, where one sums over all values of the separation constant, **k**:

$$\Psi_0(\mathbf{r},t) = \sum_{\mathbf{k}} a(\mathbf{k}) \exp(i[\mathbf{k}\cdot\mathbf{r} - \omega t])$$

where for each **k**,

$$k^2c^2 = \omega^2$$

Note that $\Psi_0(\mathbf{r}) = C\exp(i[\mathbf{k}\cdot\mathbf{r}])$ is the solution to the Helmholtz equation (where k^2 is specified) in Cartesian coordinates In the present case, k is an (arbitrary) separation constant and must be summed over. If one assumes the general case with continuous values of the separation constant, **k** and the solution is normalized with $.\frac{1}{(2\pi)^4}$ we have the general solution:

$$\Psi_0(\mathbf{r},t) = \frac{1}{(2\pi)^4}\iiiint 2\pi\delta(\omega - |kc|)\psi(\mathbf{k},\omega)\exp(i[\mathbf{k}\cdot\mathbf{r} - \omega t])\,d^3k d\omega \qquad \text{B1}$$

$$= \frac{1}{(2\pi)^3}\iiiint \psi(\mathbf{k},\omega)\exp(i[\mathbf{k}\cdot\mathbf{r} - |kc|t])\,d^3k$$

This form for the solution is the Fourier expansion of the space-time solution, $\Psi_0(\mathbf{r},t)$.

For the **non-homogeneous differential equation** $k^2c^2 = \omega^2$ is not required and one must make a four-dimensional Fourier expansion:

$$\Psi_0(\mathbf{r},t) = \frac{1}{(2\pi)^4}\iiiint \psi(\mathbf{k},\omega)\exp(i[\mathbf{k}\cdot\mathbf{r} - \omega t])\,d^3k d\omega \qquad \text{B2}$$

. Similarly, one can expand the (non-homogeneous) source term as follows:

$$F(\mathbf{r},t) = \frac{1}{(2\pi)^4}\iiiint f(\mathbf{k},\omega)\exp(i[\mathbf{k}\cdot\mathbf{r} - \omega t])\,d^3k d\omega$$

where we use the relationship

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuv}dv = \delta(u)$$

$$\frac{c}{(2\pi)^4}\iiiint \exp\left(i\left[\mathbf{k}\cdot\mathbf{r} - \frac{\omega}{c}ct\right]\right) d^3k d(\omega/c) = c\delta(\mathbf{r})\delta(ct)$$

Note the product of two Dirac delta functions, and that the first delta function, $\delta(\mathbf{r})$ is three dimensional and

$$\delta(ct) = \frac{1}{|c|}\delta(t)$$

The Fourier transform of our non-homogeneous wave Eq. (49) converts it into an algebraic equation.

$$[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]\frac{1}{(2\pi)^4}\iiiint \psi(\mathbf{k},\omega)\exp(i[\mathbf{k}\cdot\mathbf{r}-\omega t])\,d^3k d\omega$$
$$= \frac{1}{(2\pi)^4}\iiiint f(\mathbf{k},\omega)\exp(i[\mathbf{k}\cdot\mathbf{r}-\omega t])\,d^3k d\omega$$

or

$$\iiiint [\psi(\mathbf{k},\omega)(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}) - f(\mathbf{k},\omega)]\exp(i[\mathbf{k}\cdot\mathbf{r}-\omega t])\,d^3k d\omega = 0$$
$$\iiiint [\psi(\mathbf{k},\omega)(-\mathbf{k}\cdot\mathbf{k} + \frac{\omega^2}{c^2}) - f(\mathbf{k},\omega)]\exp(i[\mathbf{k}\cdot\mathbf{r}-\omega t])\,d^3k d\omega = 0.$$

Since each $\exp(i[\mathbf{k}\cdot\mathbf{r}-\omega t])$ is linearly independent, the coefficients must all be zero. Hence the solution for the expansion coefficients, $\psi(\mathbf{k},\omega)$ can be done algebraically:

$$\psi(\mathbf{k},\omega) = \frac{f(\mathbf{k},\omega)}{-k^2 + (\omega/c)^2}. \qquad \text{C}$$

The problem is now reduced to taking the inverse transform. There is at least one difficulty. The integrand will have singularities at $\omega = \pm c|\mathbf{k}|$. To understand the source of this difficulty and to determine how it is handled we consider first the Green's function for the wave equation. That is, the case where $F(\mathbf{r},t) = \delta^{(3)}(\mathbf{r}-\mathbf{r}')\delta(t-t')$.

The Green's Function for the Non-Homogeneous Wave Equation

The Green's function is a function of two space-time points, $(\mathbf{r},t)$ and $(\mathbf{r}',t')$ so we write it $G(\mathbf{r},\mathbf{r}',t,t')$:

$$\nabla^2 G(\mathbf{r},\mathbf{r}',t,t') - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}G(\mathbf{r},\mathbf{r}',t,t') = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \qquad \text{D}$$

It is easy to see that in the above differential equation (by shifting the origin of the coordinate system to $\mathbf{r}'$ and the time by t') one could change to the following variables without altering the equation

$$\mathbf{r}'' = \mathbf{r} - \mathbf{r}'$$
$$t'' = t - t'$$

$$\nabla''^2 G(\mathbf{r}'',0,t'',0) - \frac{1}{c^2}\frac{\partial^2}{\partial t''^2} G(\mathbf{r}'',0,t''0) = \delta(\mathbf{r}'')\,\delta(t'').$$

and we have shown that

$$G(\mathbf{r},\mathbf{r}',t,t') = G(\mathbf{r}-\mathbf{r}',t-t')$$

Using the Fourier expansion for the Green's function,

$$G(\mathbf{r},\mathbf{r}',t,t') = \frac{1}{(2\pi)^4}\iiiint g(\mathbf{k},\omega,\mathbf{r}',t')\exp(i[\mathbf{k}\cdot\mathbf{r}) - \omega t)])\,d^3k d\omega. \qquad \text{E}$$

For the solution to the Green's function, the source term can be written

$$F(\mathbf{r},t) = \delta^{(3)}(\mathbf{r}-\mathbf{r}')\,\delta(t-t') = \frac{1}{[2\pi]^4}\iiiint \exp\Big(i\big[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') - \omega(t-t')\big]\Big)\,d^3k d\omega$$

One can find the $g(\mathbf{k},\omega,\mathbf{r}',t')$ (the Fourier transform of $G(\mathbf{r},\mathbf{r}',t,t')$) using the same method as above, where in the integrand we factor out the exponential in $\mathbf{r}$ and t.

$$\iiiint [g(\mathbf{k},\omega,\mathbf{r}',t')(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}) - \exp(-i[\mathbf{k}\cdot\mathbf{r}' - \omega t'])]\exp(i[\mathbf{k}\cdot\mathbf{r} - \omega t])\,d^3k d\omega = 0$$

$$\iiiint [g(\mathbf{k},\omega,\mathbf{r}',t')(-\mathbf{k}\cdot\mathbf{k} + \frac{\omega^2}{c^2}) - \exp(-i[\mathbf{k}\cdot\mathbf{r}' - \omega t'])]\exp(i[\mathbf{k}\cdot\mathbf{r} - \omega t])\,d^3k d\omega = 0$$

$$g(\mathbf{k},\omega,\mathbf{r}',t') = \frac{\exp(-i[\mathbf{k}\cdot\mathbf{r}' - \omega t'])}{-k^2 + (\omega/c)^2}.$$

In terms of space-time coordinates, then the Green's function for the wave equation is given explicitly in terms of $\mathbf{r}-\mathbf{r}'$ and $t-t'$.

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{1}{[2\pi]^4}\iiiint_{-\infty}^{\infty} \frac{\exp(i[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') - \omega(t-t')])}{-k^2 + (\omega/c)^2}\,d\omega d^3k \qquad \text{F}$$

To find $G(\mathbf{r}-\mathbf{r}',t-t')$ all one needs to do is carry out the integrations. First we shall do the first integration over ω using a contour. integration and making use of the singularities at $\omega = \pm ck$ (or $k = \pm\omega/c$). How these singularities are handled depends on the boundary conditions in time, $t-t'$, imposed on the system. They give rise to causality conditions. The basic approach involves the *Cauchy's Residue Theorem.*

More about the Cauchy's Residue Theorem

Assume an analytic function $F(z)$, $z = x + iy$, has an mth order pole at $z = z_0$. A function, $F(z)$, has an m^{th} order pole at $z = z_0$ if in the neighborhood of z_0 it has an expansion

$$F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n-m}$$

Then if $F(z)$ is integrated counter-clockwise around a contour enclosing z_0 we obtain the *residue of* $F(z)$ at $z = z_0$,

$$\oint_{\text{counter clockwise}} F(z)dz = 2\pi i \, res(z_0)$$

In the case of an mth order pole

$$res(z_0) = \frac{1}{(m-1)!}\left[\frac{d^{m-1}\{(z-z_0)^m F(z)\}}{dz^{m-1}}\right]_{z=z_0}$$

Note that the residue is always the coefficient, $a_{m-1}(z_0)$, of the simple pole term in the expansion of $F(z)$

$$F(z) = \frac{a_o(z_0)}{(z-z_0)^m} + \frac{a_1(z_0)}{(z-z_0)^{m-1}} + \ldots \frac{res(z_0)}{(z-z_0)^1} + \frac{a_m(z_0)}{(z-z_0)^0} + \ldots\ldots$$

No other term contributes to the contour integral.

The only term which contributes to the integral is the simple pole term!

The ω integral's integrand has two first order poles but it is an integral along the real ω axis.

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{-1}{(2\pi)^4}\iiint \frac{\exp(i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}'))}{2k} d^3k \cdot$$
$$\int_{-\infty}^{+\infty} c\left[\frac{\exp(-i\omega(t-t'))}{\omega+ck} - \frac{\exp((-i\omega(t-t'))}{\omega-ck}\right]d\omega$$

In order to apply Cauchy's theorem we must have a closed path, so we write G as follows,

$$G = \iiint f(\mathbf{k})\, d^3k\left[\oint_{closed} \frac{\exp(-i\omega(t-t'))}{\omega^2-(ck)^2}d\omega - \int_{semi-circle}\frac{\exp(-i\omega(t-t'))}{\omega^2-(ck)^2}d\omega\right]$$

We will try to close the path with a semi-circle, with infinite radius, in either the upper half complex ω plane or the lower complex ω half plane. The denominator will cause the integrand to vanish, on either semi-circle, as $|\omega|^{-2}$ for large $|\omega|$ if the numerator is well behaved. It suffices to check the numerator on the imaginary ω axis, i.e., for $\omega = i\,\mathrm{Im}\,\omega$. The value is determined by the exponential

$$\exp[-i\omega(t-t')] = \exp[\mathrm{Im}\,\omega(t-t')]$$

This diverges for $|\mathrm{Im}\,\omega| \to \infty$ if $\mathrm{Im}\,\omega(t-t') > 0$ and vanishes exponentially with $\mathrm{Im}\,\omega$ for $\mathrm{Im}\,\omega(t-t') < 0$.

For $t < t'$ we will close the path in the upper half complex ω plane where $\mathrm{Im}\,\omega > 0$ while for $t > t'$ the path will be closed in the lower half complex ω plane where .$\mathrm{Im}\,\omega < 0$

We now know how to close the path of integration but because the poles lie on the real axis the integral is not well-defined. That is, it is an improper integral. The ambiguity provides the freedom to impose temporal boundary conditions on the Green's function. One approach distorts the path so that it avoids the poles. It that case there are four possible paths. The pertinent paths for our problems are shown in Figure 1.

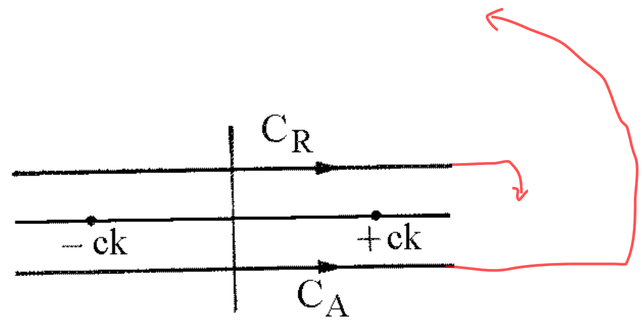

Figure 1.

Figure 1. shows the Location and direction of the contours C_R and C_A relative to the real ω axis (center line) Contour C_R is completed below the real axis and contour C_A is completed above the real axis so that in each case the poles at $\pm ck$ are inside the closed contour. It suffices to first consider the ω or angular frequency part of the integrand.

$$I(C,k) = \oint_C \left[\frac{\exp(-i\omega(t-t'))}{\omega + ck} - \frac{\exp((-i\omega(t-t'))}{\omega - ck} \right] d\omega$$

Note path is not counterclockwise, so we must multiply the residue by -1.

where C is the path of integration. If the path C_R in Figure 1 is used we obtain

$$\begin{aligned} I(C_R,k) &= -2\pi i\Theta(t-t')[res(-ck) - res(+ck)] \\ &= -2\pi i\Theta(t-t')[\exp(-i(-ck)(t-t')) - \exp((-ick(t-t'))] \\ &= 4\pi\Theta(t-t')\sin(ck(t-t')). \end{aligned}$$

The Green's function obtained with this path choice will be labelled G_R.

$$G_R(\mathbf{r}-\mathbf{r}', t-t') = \Theta(t-t')\frac{-4\pi c}{(2\pi)^4}\iiint \frac{\exp(i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}'))}{2k}\sin(ck(t-t'))d^3k$$

Now we do the k integration in spherical $\mathbf{k}$ space, letting $(\mathbf{r}-\mathbf{r}')$ be along the $\hat{\mathbf{z}}$ direction:

$$
\begin{aligned}
G_R &= \Theta(t-t')\frac{-c}{(2\pi)^3}\iiint \frac{\exp(ik|\mathbf{r}-\mathbf{r}'|\cos\theta_k)}{k}\sin(ck(t-t'))k^2dkd\phi_kd(-\cos\theta_k)\\
&= \Theta(t-t')\frac{-c2\pi}{(2\pi)^3}\int\int_{-1}^{1}\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|u)}{k}\sin(ck(t-t'))k^2dkdu\\
&= \Theta(t-t')\frac{-c}{(2\pi)^2}\int_0^{\infty}\frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)-\exp(-ik|\mathbf{r}-\mathbf{r}'|)}{ik^2|\mathbf{r}-\mathbf{r}'|}\sin(ck(t-t'))k^2dk\\
&= \Theta(t-t')\frac{-c}{i(2\pi)^2|\mathbf{r}-\mathbf{r}'|}\int_{-\infty}^{\infty}\exp(ik|\mathbf{r}-\mathbf{r}'|)\sin(ck(t-t'))dk\\
&= \Theta(t-t')\frac{c}{2(2\pi)^2|\mathbf{r}-\mathbf{r}'|}\int_{-\infty}^{\infty}\exp(ik|\mathbf{r}-\mathbf{r}'|)[\exp(ick(t-t'))\ -\exp(-ick(t-t'))]dk\\
&= \Theta(t-t')\frac{c}{2(2\pi)^2|\mathbf{r}-\mathbf{r}'|}\int_{-\infty}^{\infty}[\exp(ik[|\mathbf{r}-\mathbf{r}'|+c(t-t')])-\exp(ik[|\mathbf{r}-\mathbf{r}'|-c(t-t')])]dk
\end{aligned}
$$

Retarded Green's function:

Finally, we have :

$$
\begin{aligned}
G_R(\mathbf{r}-\mathbf{r}',t-t') &= \frac{c}{(4\pi)|\mathbf{r}-\mathbf{r}'|}\Theta(t-t')[\delta(|\mathbf{r}-\mathbf{r}'|+c(t-t'))\ -\delta(|\mathbf{r}-\mathbf{r}'|-c(t-t'))] \qquad \text{GR}\\
&= \frac{-c}{(4\pi)|\mathbf{r}-\mathbf{r}'|}\Theta(t-t')\delta(|\mathbf{r}-\mathbf{r}'|-c(t-t'))
\end{aligned}
$$

In the last step the additional term involving $\delta(|\mathbf{r}-\mathbf{r}'|+c(t-t'))$ was dropped since the argument of this delta function could not vanish for $t > t'$. The Green's function G_R is called the retarded Green's function. It gives a disturbance (created at time t') propagating outward from a source point $\mathbf{r}'$. The condition $(t-t') > 0$ is often interpreted as a causality condition. That is, the disturbance is detected at a time, t, which must be larger than the creation time, t', at the source. The Green's function is also called a propagator as it ”propagates” the disturbance from point $(\mathbf{r}', t')$ to the point $(\mathbf{r}, t)$ within the integral expression for the solution to the wave equation

Advanced Green's Function:

The Green's function obtained using the path C_A is similar, but corresponds to $t < t'$. Note that one must repeat all the steps leading to Eq. (67b), but use the contour C_A, rather than C_R.

$$\begin{aligned} I(C_A,k) &= 2\pi i\Theta(t'-t)[res(-ck)-res(+ck)] \\ &= 2\pi i\Theta(t'-t)[\exp(-i(-ck)(t-t'))-\exp((-ick(t-t'))] \\ &= -4\pi\Theta(t'-t')\sin(ck(t'-t)). \end{aligned}$$

Finally, we have :

$$\begin{aligned} G_A(\mathbf{r}-\mathbf{r}',t-t') &= \frac{-c}{(4\pi)|\mathbf{r}-\mathbf{r}'|}\Theta(t'-t)[\delta(|\mathbf{r}-\mathbf{r}'|+c(t-t'))\ -\delta(|\mathbf{r}-\mathbf{r}'|-c(t-t'))] \qquad \text{GA} \\ &= \frac{-c}{(4\pi)|\mathbf{r}-\mathbf{r}'|}\Theta(t'-t)[\delta(|\mathbf{r}-\mathbf{r}'|-c(t'-t))\ -\delta(|\mathbf{r}-\mathbf{r}'|+c(t'-t))] \\ &= \frac{-c}{(4\pi)|\mathbf{r}-\mathbf{r}'|}\Theta(t'-t)\delta(|\mathbf{r}-\mathbf{r}'|-c(t'-t)) \end{aligned}$$

note sign

This is the advanced Green's function and corresponds to the time reversed retarded Green's function. The interpretation of this Green's function is that the disturbance is 'converging on its source'

Green's Function Solution to Wave Equation

Given

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Psi(\mathbf{r},t) = F(\mathbf{r},t) \qquad \text{A1}$$

$$\mathcal{L}\Psi(\mathbf{r},t) = F(\mathbf{r},t)$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]G(\mathbf{r},\mathbf{r}';t,t') = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t') \qquad \text{A2}$$

$$\mathcal{L}G(\mathbf{r},\mathbf{r}';t,t') = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t')$$

Use the wave equations to evaluate this.

One can write:

$$\iiiint \Psi(\mathbf{r}',t')\left[\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right]G(\mathbf{r},\mathbf{r}';t,t')d^3x'd(ct') \qquad \text{A3}$$

$$-\iiiint G(\mathbf{r},\mathbf{r}';t,t')\left[\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right]\Psi(\mathbf{r}',t')d^3x'd(ct')$$

$$= \iiiint \Psi(\mathbf{r}',t')\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')d^3x'd(ct') - \iiiint G(\mathbf{r},\mathbf{r}';t,t')F(\mathbf{r}',t')d^3x'd(ct')$$

$$= \Psi(\mathbf{r},t) - \iiiint G(\mathbf{r},\mathbf{r}';t,t')F(\mathbf{r}',t')d^3x'd(ct') \qquad \text{A4}$$

This expressions in A3 can also be written in terms of "surface" integrals as follows:

$$\iiiint [\Psi(\mathbf{r}',t')\nabla'^2 G(\mathbf{r},\mathbf{r}';t,t') - G(\mathbf{r},\mathbf{r}';t,t')\nabla'^2\Psi(\mathbf{r}',t')]d^3x'd(ct') \qquad \text{A5a}$$

$$= \int d(ct') \iiint \nabla' \cdot [\Psi(\mathbf{r}',t')\nabla G(\mathbf{r},\mathbf{r}';t,t') - G(\mathbf{r},\mathbf{r}';t,t')\nabla\Psi(\mathbf{r}',t')]d^3x'$$

$$= \int d(ct') \iint_{r' \to \infty} [\Psi(\mathbf{r}',t')\nabla G(\mathbf{r},\mathbf{r}';t,t') - G(\mathbf{r},\mathbf{r}';t,t')\nabla\Psi(\mathbf{r}',t')] \cdot \mathbf{n}_s dS'$$

If we want the solution for all space, and both $\Psi(\mathbf{r}',t')$ and $G(\mathbf{r},\mathbf{r}';t,t') -> 0$ at $r' -> \infty$ at least like $\frac{1}{r'}$ then the surface integrals vanish.

$$\iiiint \Psi(\mathbf{r}',t')\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right]G(\mathbf{r},\mathbf{r}';t,t')d^3x'd(ct') \qquad \text{A5b}$$
$$-\iiiint G(\mathbf{r},\mathbf{r}';t,t')\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right]\Psi(\mathbf{r}',t')d^3x'd(ct')$$
$$= -\iiint\left[\int \frac{\partial}{\partial ct'}[\Psi(\mathbf{r}',t')\frac{\partial}{\partial ct'}G(\mathbf{r},\mathbf{r}';t,t') - G(\mathbf{r},\mathbf{r}';t,t')\frac{\partial}{\partial ct'}\Psi(\mathbf{r}',t')]d(ct')\right]d^3x'$$
$$= -\iiint\left[[\Psi(\mathbf{r}',t')\frac{\partial}{\partial ct'}G(\mathbf{r},\mathbf{r}';t,t') - G(\mathbf{r},\mathbf{r}';t,t')\frac{\partial}{\partial ct'}\Psi(\mathbf{r}',t')]\right]\Bigg|_{t'\to-\infty}^{t'\to\infty} d^3x'$$

a) Causality (via the Retarded Green's function) provides a means of evaluating the $t' \to \infty$ limit:

$$G(\mathbf{r},\mathbf{r}';t,t') = 0 \text{ for } t < t' \qquad \text{(violates Causality as } t'\to\infty)$$
$$\frac{\partial}{\partial ct'}G(\mathbf{r},\mathbf{r}';t,t') = 0 \text{ for } t < t'$$

b) Intial conditions provide a means of evaluating the $t' \to -\infty$ (one usually assumes the source is turned on at, say, $t' = 0$::

$$\Psi(\mathbf{r}',t') = 0 \text{ at } t' = -\infty \qquad \text{(no signal before source is turned on)}$$
$$\frac{\partial}{\partial ct'}\Psi(\mathbf{r}',t') = 0 \text{ at } t' = -\infty$$

With these boundary conditions both the spatial and time "surface" terms vanish and we

have:

$$\Psi(\mathbf{r},t) - \iiiint G(\mathbf{r},\mathbf{r}';t,t')F(\mathbf{r}',t')d^3x'd(ct') = 0$$

and

$$\boxed{\Psi(\mathbf{r},t) = \iiiint G(\mathbf{r},\mathbf{r}';t,t')F(\mathbf{r}',t')d^3x'd(ct')} \qquad \text{A6}$$

Note that if a homogeneous background wave (not due to the source) exists, one could superimpose it onto our solution without any loss of generality. Usually this is not considered part of the problem.

Using the Green's Function to find the Solution to the Wave Equation:

Example 1:

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Psi(\mathbf{r},t) = -\frac{q}{\epsilon_0}e^{i\omega t}[\delta(\mathbf{r} - a\hat{\mathbf{z}}) - \delta(\mathbf{r} + a\hat{\mathbf{z}})]$$

$$\Psi(\mathbf{r},t)_{r\to\infty} \to \frac{1}{r};\ \frac{\partial}{\partial r}\Psi(\mathbf{r},t)_{r\to\infty} \to \frac{1}{r^2}$$

The Green's function solution is given by

$$\begin{aligned}\Psi(\mathbf{r},t) &= \iiiint G_R(\mathbf{r}-\mathbf{r}',t-t')F(\mathbf{r}',t')d^3r'dt' \qquad \text{A7}\\ &= \iiiint \Theta(t-t')\frac{-c}{4\pi|\mathbf{r}-\mathbf{r}'|}\delta(|\mathbf{r}-\mathbf{r}'|-c(t-t'))F(\mathbf{r}',t')d^3r'dt'\\ &= \iiiint \Theta(t-t')\frac{-c}{4\pi|\mathbf{r}-\mathbf{r}'|}\frac{1}{c}\delta\left(t'-[t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}]\right)F(\mathbf{r}',t')d^3r'dt'\\ &= \iiint F\left(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}d^3r'\end{aligned}$$

$$\Psi(\mathbf{r},t) = \iiint -\frac{q}{4\pi\epsilon_0}e^{i\omega(t'-\frac{|\mathbf{r}-\mathbf{r}'|}{c})}[\delta(\mathbf{r}'-a\hat{\mathbf{z}}) - \delta(\mathbf{r}'+a\hat{\mathbf{z}})]\frac{-1}{|\mathbf{r}-\mathbf{r}'|}d^3r'$$

$$= \frac{q}{4\pi\epsilon_0}\left[\exp(i\omega(t-\frac{|\mathbf{r}-\mathbf{a}\hat{\mathbf{z}}|}{c})\frac{1}{|\mathbf{r}-\mathbf{a}\hat{\mathbf{z}}|} - \exp(i\omega(t-\frac{|\mathbf{r}+\mathbf{a}\hat{\mathbf{z}}|}{c})\frac{1}{|\mathbf{r}+\mathbf{a}\hat{\mathbf{z}}|}\right]$$

Normally the Green's function solution would have "surface integral terms" evaluated at $r' = \infty$ and at $t' = \pm\infty$. They would be of the form:

$$\int_{-\infty}^{\infty} d(ct')\iint_{r'\to\infty}[\Psi\nabla' G - G\nabla'\Psi]\cdot d\mathbf{S}' + \iiint dV'[\Psi\frac{\partial}{\partial ct'}G - G\frac{\partial}{\partial ct'}\Psi]_{t'\to-\infty}.$$

The surface integrals do not contribute since both the solution and the Green's function vanish as $r' \to \infty$ .The time derivatives at $t' = \infty$. do not appear because of the $\Theta(t-t')$ in the Green's function and at $t' = -\infty$ one assumes that there is no time derivative of the source signal in the Green's function, and no time derivative of the "wave" function, Ψ. The $t' = \infty$ boundary condition has been taken care of by our choice of contour. In most applications of the Green's function the disturbance is assumed to take place near $t' = 0$ and to "turn off" at $t' << \infty$ In this case one can see that the boundary conditions in time are automatically taken care of.

Maxwell's Equations

Generally one wants to find $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, but in practice it is easier to find $\mathbf{A}(\mathbf{r},t)$ and $\phi(\mathbf{r},t)$.first and then determine $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ from the following:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

Since the above defines **A** uniquely, one has to supply another condition on **A**. There are two commonly used choices or gauges:

Lorentz Gauge

$$\nabla \cdot \mathbf{A}(\mathbf{r},t) + \frac{1}{c}\frac{\partial}{\partial t}\phi(\mathbf{r},t) = 0 \qquad \text{A8}$$

$$\nabla^2\phi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi(\mathbf{r},t) = -\rho(\mathbf{r},t)/\epsilon_0 \quad \text{S.I. units}$$

$$\nabla^2\mathbf{A}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}(\mathbf{r},t) = -\mu_0\mathbf{J}(\mathbf{r},t) \quad \text{S.I. units}$$

Coulomb/Radiation Gauge

$$\nabla \cdot \mathbf{A}(\mathbf{r},t) = 0 \qquad \text{A9}$$

$$\nabla^2\phi(\mathbf{r},t) = -\rho(\mathbf{r},t)/\epsilon_0$$

$$\nabla^2\mathbf{A}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}(\mathbf{r},t) = -\mu_0\mathbf{J}_t(\mathbf{r},t)$$

$$\mathbf{J} = \mathbf{J}_\ell + \mathbf{J}_t$$

$$\nabla \cdot \mathbf{J}_t = 0; \quad \nabla \times . \mathbf{J}_\ell = 0$$

In the Lorentz gauge:

$$\begin{aligned}
\mathbf{A}(\mathbf{r},t) &= -\iiiint G_R(\mathbf{r}-\mathbf{r}',t-t')\mu_0\mathbf{J}(\mathbf{r}',t')'d^3r'dt' \qquad \text{A10}\\
&= \mu_0 \iiiint \Theta(t-t')\frac{c}{4\pi|\mathbf{r}-\mathbf{r}'|}\delta(|\mathbf{r}-\mathbf{r}'|-c(t-t'))\mathbf{J}(\mathbf{r}',t')d^3r'dt'\\
&= \mu_0 \iiiint \Theta(t-t')\frac{-c}{4\pi|\mathbf{r}-\mathbf{r}'|}\frac{1}{c}\delta\left(t'-[t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}]\right)\mathbf{J}(\mathbf{r}',t')d^3r'dt'\\
&= \mu_0 \iiint \mathbf{J}\left(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}d^3r'
\end{aligned}$$

Example: A single frequency current density source

$$\mathbf{J}_t(\mathbf{r},t) = \mathbf{J}(\mathbf{r})\,\mathbf{exp}(-i\omega_0 t)$$

$$\begin{aligned}
\mathbf{A}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\iiint \mathbf{J}\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3r' \\
&= \frac{\mu_0}{4\pi}\iiint \mathbf{J}(\mathbf{r}')\,\mathbf{exp}\left[-i\omega_0(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})\right]\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3r' \\
&= \frac{\mu_0}{4\pi}e^{-i\omega_0 t}\iiint \mathbf{J}(\mathbf{r}')\,\mathbf{exp}\left[\frac{i\omega_0|\mathbf{r}-\mathbf{r}'|}{c})\right]\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3r' \\
&= \frac{\mu_0}{4\pi}e^{-i\omega_0 t}\iiint \mathbf{J}(\mathbf{r}')\frac{\mathbf{exp}[ik_0|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|}d^3r'; \quad k_0 = \frac{\omega_0}{c}
\end{aligned}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}e^{-i\omega_0 t}\iiint \mathbf{J}(\mathbf{r}')\frac{\mathbf{exp}[ik_0|\mathbf{r}-\mathbf{r}'|]}{|\mathbf{r}-\mathbf{r}'|}d^3r'; \quad k_0 = \frac{\omega_0}{c} \tag{A11}$$

Using the expansion for the Helmholtz Equation Green's function we have

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} 4\pi ik j_\ell(kr_<)h_\ell(kr_>)Y_\ell^m(\theta',\phi')^* Y_\ell^m(\theta,\phi)$$

and finally we have the multipole expansion of the vector potential, **A**:

$$\mathbf{A}(\mathbf{r},t) = ik\mu_0 e^{-i\omega_0 t}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\iiint \mathbf{J}(\mathbf{r}')j_\ell(kr_<)h_\ell(kr_>)Y_\ell^m(\theta',\phi')^* Y_\ell^m(\theta,\phi)d^3r' \tag{A12}$$

Since the source current is localized near $r' = 0$, the $j_\ell(kr')$ can be used in the integral and the solution for $r > r'$ is given by:

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}_0(\mathbf{r})e^{-i\omega_0 t}$$

$$\mathbf{A}_0(\mathbf{r}) = ik\mu_0\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}[j_\ell(kr) + in_\ell(kr)]Y_{\ell m}(\theta,\phi)\iiint \mathbf{J}_0(\mathbf{r}')j_\ell(kr')Y_{\ell m}(\theta',\phi')^* r'^2 d\Omega' dr'$$

where $k = k_0 = \frac{\omega_0}{c}$. and $\mathbf{J}(\mathbf{r}') = \mathbf{J}_0(\mathbf{r}')$

Approximating $j_\ell(kr')$ with the form near $r' = 0$:

$$j_\ell(kr) \approx \frac{1}{(2\ell+1)!!}(kr)^{\ell}$$

$$\mathbf{A}_0(\mathbf{r}) \approx k\mu_0 \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}[ij_\ell(kr) - n_\ell(kr)]Y_{\ell m}(\theta,\phi)\iiint \mathbf{J}_0(\mathbf{r}')k^{\ell}\frac{1}{(2\ell+1)!!}r'^{\,\ell+2}Y_{\ell m}(\theta',\phi')^{*}d\Omega' dr'.$$

$$kr = \frac{\omega_0 r}{c} = \frac{r}{c/2\pi f} = \frac{r}{2\pi\lambda} \approx \frac{r}{\lambda}$$

In the 'near field' region, $kr \ll 1 \Rightarrow r \ll \lambda$, and $\mathbf{A}_0(\mathbf{r})$ is approximated by letting:

$$[ij_\ell(kr) - n_\ell(kr)] \approx -n_\ell(kr) = \left[-\frac{(2\ell-1)!!}{kr^{\ell+1}}\right]$$

Note: r' (source) is close to 0 and less than r and $r \ll \lambda$

Note extent of source limits integration over r'.

$$\mathbf{A}_0(\mathbf{r}) \approx k\mu_0 \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[\frac{(2\ell-1)!!}{(kr)^{\ell+1}}\right]Y_{\ell m}(\theta,\phi)\iiint \mathbf{J}_0(\mathbf{r}')k^{\ell}\frac{1}{(2\ell+1)!!}r'^{\,\ell+2}Y_{\ell m}(\theta',\phi')^{*}d\Omega' dr' \qquad \text{A13}$$

$$= k\mu_0 \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{r^{\ell+1}}Y_{\ell m}(\theta,\phi)\iiint \mathbf{J}_0(\mathbf{r}')\frac{1}{(2\ell+1)}r'^{\,\ell+2}Y_{\ell m}(\theta',\phi')^{*}d\Omega' dr'.$$

$r > r'$

In the 'far field' region, $kr \gg 1$ and $r \gg \lambda$, the radiation form for $\mathbf{A}_0(\mathbf{r})$ is approximated using

$$[ij_\ell(kr) - n_\ell(kr)] \approx \frac{\exp\left[i\left(kr - \frac{\ell\pi}{2}\right)\right]}{kr}$$

Note: r' (source) is close to 0 and less than r and $r \gg \lambda$

$r > r'$

$$\mathbf{A}_0(\mathbf{r}) \approx k\mu_0 \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{k^{\ell}}{(2\ell+1)!!}\frac{\exp\left[i\left(kr - \frac{\ell\pi}{2}\right)\right]}{kr}Y_{\ell m}(\theta,\phi)\iiint \mathbf{J}_0(\mathbf{r}')r'^{\,\ell+2}Y_{\ell m}(\theta',\phi')^{*}d\Omega' dr'. \qquad \text{A1}$$

This can be evaluated term by term. When $kd \ll 1$ the series is generally dominated by the lowest non-zero term. It is also a useful expression if $\mathbf{J}_0(\mathbf{r})$ is described by a superposition of one or two spherical harmonics.

More general time dependence

In this section we consider the potentials generated by a moving charge located at $\mathbf{R}(t)$. The charge and current densities are

$$\rho(\mathbf{r},t) = q\delta(\mathbf{r}-\mathbf{R}(t))$$
$$\mathbf{J}(\mathbf{r},t) = q\delta(\mathbf{r}-\mathbf{R}(t))\left[\frac{d}{dt}\mathbf{R}(t)\right]$$

These can be used to obtain the potentials, at $\mathbf{r}$ and time t (Gaussian units) from:

$$\nabla\cdot\mathbf{A}(\mathbf{r},t) + \frac{1}{c}\frac{\partial}{\partial t}\phi(\mathbf{r},t) = 0 \qquad \text{A8b}$$
$$\nabla^2\phi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi(\mathbf{r},t) = -4\pi\rho(\mathbf{r},t)/\epsilon_0$$
$$\nabla^2\mathbf{A}(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A}(\mathbf{r},t) = -\frac{4\pi}{c}\mathbf{J}(\mathbf{r},t)$$

:

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{c}\int\iiint\Theta(t-t')\frac{c}{|\mathbf{r}-\mathbf{r}'|}\delta(|\mathbf{r}-\mathbf{r}'|-c(t-t'))\frac{d}{dt'}\mathbf{R}(t')\delta(\mathbf{r}'-\mathbf{R}(t'))d^3r'dt'$$
$$\phi(\mathbf{r},t) = q\iiint\int\Theta(t-t')\frac{c}{|\mathbf{r}-\mathbf{r}'|}\delta(|\mathbf{r}-\mathbf{r}'|-c(t-t'))\delta(\mathbf{r}'-\mathbf{R}(t'))d^3r'dt'.$$

The $\mathbf{r}'$ integrations can be done first, using $\delta(\mathbf{r}'-\mathbf{R}(t'))$:

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{c}\int\frac{c}{|\mathbf{r}-\mathbf{R}(t')|}\frac{d}{dt'}\mathbf{R}(t')\delta\Big(\left|\mathbf{r}-\mathbf{R}(t')'\right|-c(t-t')\Big)dt';$$
$$\phi(\mathbf{r},t) = q\int\frac{c}{|\mathbf{r}-\mathbf{R}(t')|}\delta(|\mathbf{r}-\mathbf{R}(t')|-c(t-t'))dt'$$

Finally, the t' integration for $\phi(\mathbf{r},t)$ can be done with the $\delta(|\mathbf{r}-\mathbf{R}(t')|-c(t-t'))$:

$$\phi(\mathbf{r},t) = q\int\frac{c}{|\mathbf{r}-\mathbf{R}(t')|}\frac{\delta(t'-t_o)}{|\frac{d}{dt'}[\mathbf{r}-\mathbf{R}(t')-c(t-t')]|_{t'=t_o}}dt'$$
$$= q\frac{c}{|\mathbf{r}-\mathbf{R}(t_o)|}\frac{1}{|\frac{d}{dt'}[|\mathbf{r}-\mathbf{R}(t')|+c]|_{t'=t_o}},$$

where t_o represents the values of $t' < t$ for which

$$|\mathbf{r}-\mathbf{R}(t')|-c(t-t') = 0$$
$$(\mathbf{r}-\mathbf{R}(t'))\cdot(\mathbf{r}-\mathbf{R}(t')) = c^2(t-t')^2.$$

Similarly

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{c}\frac{c}{|\mathbf{r}-\mathbf{R}(t_o)|}\frac{\frac{d}{dt'}\mathbf{R}(t')|_{t'=t_o}}{|\frac{d}{dt'}[\left|\mathbf{r}-\mathbf{R}(t')'\right|+c]|_{t'=t_o}}$$

As a special case, if $\mathbf{R}(t) = \mathbf{v}_o t$ one obtains t_o as follows:

$$(\mathbf{r}-\mathbf{v_o}t')\cdot(\mathbf{r}-\mathbf{v}_o t') = c^2(t-t')^2 \equiv W^2$$

$$(\mathbf{R}_o-\mathbf{b}c(t'-t))\ \cdot(\mathbf{R}_o-\mathbf{b}c(t'-t)') = c^2(t-t')^2 \equiv W\,; \qquad \mathbf{b} = \frac{\mathbf{v_o}}{c}$$

$$(\mathbf{R}_o-\mathbf{b}W)\cdot(\mathbf{R}_o-\mathbf{b}W) = W^2, \qquad \mathbf{R}_o = \mathbf{r}-\mathbf{v_o}t$$

with solution

$$W_\pm = \gamma^2\mathbf{b}\cdot\mathbf{R}_o[1\pm[1+\frac{R_o^2}{(\mathbf{b}\cdot\mathbf{R}_o)^2\gamma^2}]^{1/2} = ct-ct_o,$$

where

$$\gamma^2 = \frac{1}{1-b^2}.$$

Only one value of $W_\pm > 0$ satisfies causality. The solution for $\mathbf{R}(t') = \mathbf{v}_o t'$ is (after some algebra)

$$\phi(\mathbf{r},t) == \frac{q}{|\mathbf{r}-\mathbf{v_o}t|}\cdot\frac{1}{\sqrt{1-b^2\sin^2\theta}} \text{ where } \frac{\mathbf{b}\cdot\mathbf{R}_o}{bR_o} = \cos\theta;$$

$$\mathbf{A}(\mathbf{r},t) = \frac{q}{|\mathbf{r}-\mathbf{v_o}t|}\cdot\frac{\mathbf{v_o}}{\sqrt{1-b^2\sin^2\theta}}.$$

In order to calculate the fields in the general case for $\mathbf{R}(t')$ one needs a relationship

between the derivatives with respect to $\mathbf{r}$ and t and the derivative with respect to t'. We begin with the equality $c(t-t') = |\mathbf{r}-\mathbf{R}(t')|$.

$$\frac{d}{dt'}|\mathbf{r}-\mathbf{R}(t')| = -\frac{\mathbf{r}-\mathbf{R}(t')}{|\mathbf{r}-\mathbf{R}(t')|}\cdot\mathbf{u}(t'),$$

where $\mathbf{u}(t') = \frac{dR}{dt'}$.

Relativistic four vector notation: Covariant form for Maxwell's Equations

Under a Lorentz transformation, L^{μ}_{ν},

$$x'^{\mu} = L^{\mu}_{\nu}x^{\nu} \text{ with the following choice of metric tensor}$$
$$g_{\mu\nu} = [L^T L]_{\mu\nu} = -\delta_{\mu\nu} \text{ for } \mu,\nu = 1,2,3$$
$$= \delta_{0\nu} = \delta_{\mu 0} \text{ otherwise.}$$

$$x^{\mu} = (ct, \mathbf{r}) \qquad \mu = 0,1,2,3$$
$$x_{\mu} = (ct, -\mathbf{r})$$
$$\partial_{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right)$$
$$\partial^{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right)$$
$$\partial_{\mu}\partial^{\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

The vector potential is a four vector (transforms under Lorentz transformation as a four vector),

$$A^{\mu} = (\phi(\mathbf{r},t), \mathbf{A})$$
$$A_{\mu} \equiv (\phi(\mathbf{r},t), -\mathbf{A});$$
$$A'^{\mu} = A^{\mu} - \partial^{\mu}\Lambda$$

gauge transformation on A

$$= \left(\left(\phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial\Lambda}{\partial t}\right), \mathbf{A} + \boldsymbol{\nabla}\boldsymbol{\Lambda}\right)$$
$$= (\phi'(\mathbf{r},t), \mathbf{A}')$$
$$A'_{\mu} = A_{\mu} - \partial_{\mu}\Lambda$$
$$= \left(\left(\phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial\Lambda}{\partial t}\right), -\mathbf{A} - \boldsymbol{\nabla}\boldsymbol{\Lambda}\right)$$

Eq. 31 (the Lorentz gauge condition) becomes

$$\partial_{\mu}A'^{\mu} = \partial_{\mu}A^{\mu} - \partial_{\mu}\partial^{\mu}\Lambda \qquad \text{R5}$$
$$= \frac{1}{c}\frac{\partial}{\partial t}\phi(\mathbf{r},t) + \boldsymbol{\nabla}\cdot\mathbf{A} - \partial_{\mu}\partial^{\mu}\Lambda$$
$$= 0$$

Note that the general condition on Λ which ensures the Lorentz gauge is

$$\partial_{\mu}\partial^{\mu}\Lambda = -\nabla^2\Lambda(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Lambda(\mathbf{r},t) = \boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{r},t) + \frac{1}{c}\frac{\partial}{\partial t}\phi(\mathbf{r},t), = 0$$

Finally,

$$\partial_{\mu}\partial^{\mu}A^{\nu} = \frac{4\pi}{c}J^{\nu} \text{ where} \qquad \text{R6a}$$
$$J^{\nu} = (c\rho, \mathbf{J}) \qquad \text{R6b}$$

This box contains all of Maxwells equations (both wave equations with sources). Note four-vector notation for the current density.

Conservation of charge is given by

$$\partial_\mu J^\mu = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \qquad \text{R7}$$