WHICH INTRODUCES RELATIVISTIC EFFECTS THROUGH THE INCREASE ELECTRON INERTIA.
TAKING THE LONGITUDINAL COMPONENT OF THE MOMENTUM (23) GIVES

$$\frac{d}{dt}\left(\gamma m_e v_x\right) = -eE_x - \frac{e^2}{2m_e\gamma}\partial_x A_y^2$$

WE CAN ELIMINATE V_x USING THE X COMPONENT OF THE AMPÈRE-MAXWELL LAW ⇒ (26)

$$0 = -\frac{e}{\varepsilon_0} n_e v_x + \partial_t E_x$$

THESE EQS CAN BE SOLVED NUMERICALLY. TO OUR PURPOSES WE CAN SEPLIFY THE PHYSICS BY LINEARIZING THE PLASMA FLUID QUANTIES.
LET $n_e \simeq n_0 + n_1 + \cdots$; $v_x \simeq v_1 + v_2 + \ldots$
AND NEGLET PRODUCTS OF PERTURBATION SUCH AS $n_1 v_1$. THIS TO

- OBSERVATION SIMPLE WAVE REPRESENTATION

IN THE FOLLOWING WE RECALL SOME BESIC CONCEPTS USEFULL TO INTERPRET THE PHYSICS OF THE PLASMA WAVES. A GENERAL REPRESENTATION OF A MONOCHROMATIC, PLANE PROGRESSIVE WAVE IN $\mathbb{C}$ IS GIVEN BY

$$\tilde{W}(\bar{r},t) = \tilde{A}_W \exp\left[i(\bar{k}\cdot\bar{r} - \omega t)\right]$$

WHERE WE ASSUME ALL THE SIMBLE ARE KNOWN. THE COMPLEX AMPLITUDE $\tilde{A}_W = A_W \cdot \hat{\varepsilon}(t)$, WHERE $\hat{\varepsilon}$ IS THE UNITARY VECTOR GIVING THE DIRECTION THE AMPLITUDE A_W OSCILLATE.

A 1D $\Psi(x,t)$ WILL BE

$$\Psi(x,t) = \tilde{A}_{\Psi} \exp\left[i(kx - \omega t)\right]$$

WHERE X IS THE PROPAGATION DIRECTION.

THE CONSTANT PHASE IS MAINTAINED FOR A POINT ON THE WAVE WHEN

$$\frac{d}{dt}(kx - \omega t) = 0$$

$$\Rightarrow \frac{d(kx)}{dt} - \omega = 0 \Rightarrow \quad (27)$$

$$\frac{dx}{dt} = \frac{\omega}{k} = V_P \quad \text{PHASE VELOCITY} \quad (28)$$

- PHASE SPEED IS NOT THE RATE OF INFORMATION (ENERGY) TRANSFERRED.
- GROUP SPEED IS SIMILARLY DEFINED BUT FOR CONSTANT PHASE ON A MODULATED WAVE ENVELOPE.

$$\Psi \propto \exp\left[i(\Delta k x - \Delta\omega t)\right]$$

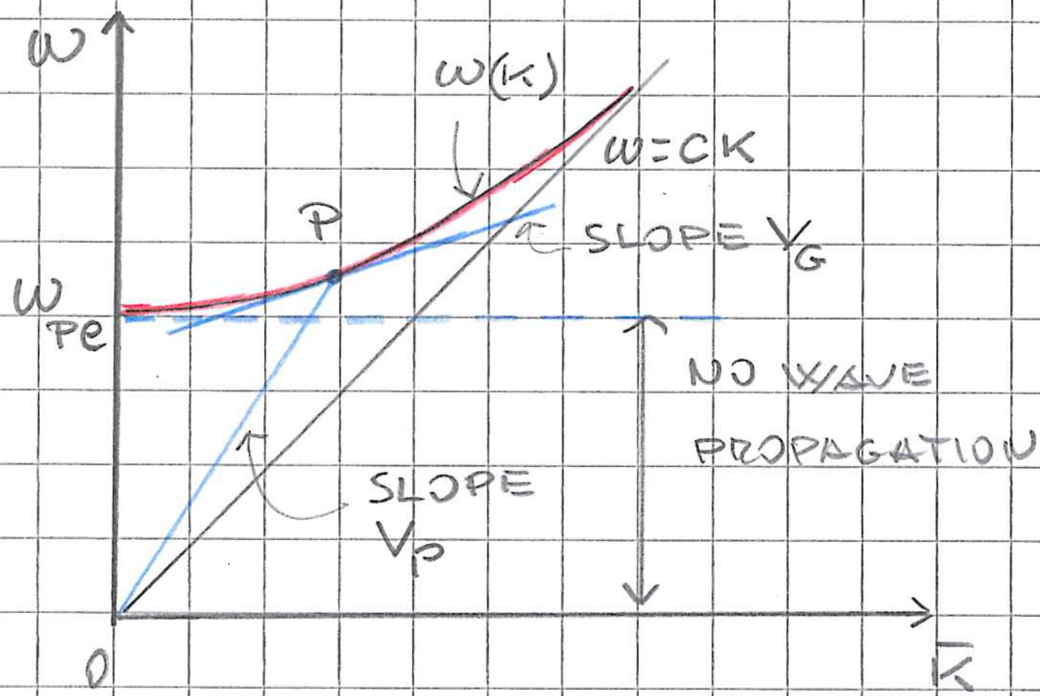
$$\frac{d}{dt}(\Delta k x - \Delta\omega t) = 0 \Rightarrow \frac{dx}{dt} = \frac{\Delta\omega}{\Delta k}$$

$$\lim_{\Delta\omega \to 0}\left(\frac{\Delta\omega}{\Delta k}\right) = \frac{d\omega}{dk} = V_G \quad (29)$$

THIS HELPS US TO UDERSTAND THE WAVE DISPERSION RELATION $\omega = \omega(k)$.

- $\bar{k}$ IS OFTEN COMPLEX BUT WAVES PROPAGATE ONLY FOR $\mathbb{R}(\bar{k}) > 0$
- DISPERSION RELATIONS INDICATES

CUTOFFS AND RESONANCES



- WHAT MAKES PLASMA WAVES DIFFERENT?

PLASMA PROPERTIES

- GAS LIKE → FLUID EQUATIONS
- CHARGES → EM EQUATIONS
- MAGNETIC FIELD (COMPLICATES EVERYTHING) → EM EQUATIONS

- FLUID EQUATIONS
  - MASS CONTINUITY
  - EQ. OF MOTION
  - ENERGY EQ.
  - IDEAL GAS LAW

- EM EQUATIONS
  - MAXWELL EQS + LORENTZ FORCE
  - INDUCTION EQ
  - OHM LAW

SINGLE-PARTICLE MOTION

MOTION IN A UNIFORM $\bar{B}$ FIELD

IN DENSE PLASMA COULOMB FORCES COUPLE PARTICLES, SO BULK MOTION OF PLASMA IS SIGNIFICANT.

IN RAREFIED PLASMAS CHARGE PARTICLES DO NOT INTERACT WITH ONE ANOTHER SIGNIFICANTLY $\Rightarrow$ THE MOTION OF EACH PARTICLE CAN BE TREATED INDEPENDENTLY.

IN GENERAL WE CAN START FROM THE LORENTZ FORCE

$$m\frac{d\bar{v}}{dt} = q\left(\bar{E} + \bar{v}\times\bar{B}\right)$$

WHERE FOR RELATIVISTIC MOTION WE ASSUME

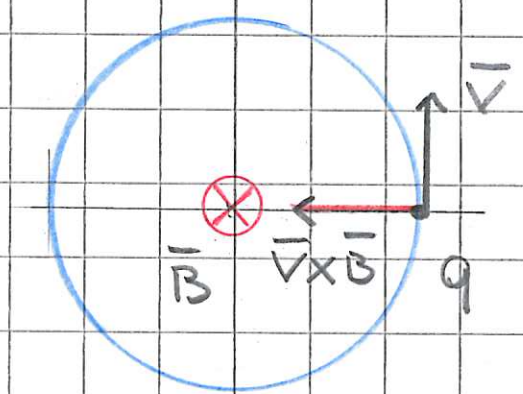
$$m = \frac{m_0}{\left[1-\left(v/c\right)^2\right]^{1/2}}$$

WITH m_0 THE REST MASS.

WE ALREADY KNOW THAT $\bar{B}$ DOES NOT MAKE ANY WORK $\Rightarrow$ FOR $\bar{E}=0$ $\frac{d}{dt}\left(\frac{mv^2}{2}\right)=0$ FOR

$$dW = q\left(\bar{v}\times\bar{B}\right)\cdot d\bar{s} = 0 \quad \left(\bar{v} \parallel d\bar{s}\right)$$

- OBSERVATION THIS IS ALSO TRUE FOR $\bar{B}$ NON UNIFORM.



$$\bar{V} = \bar{V}_{\parallel} + \bar{V}_{\perp} \Rightarrow \frac{dV_{\parallel}}{dt} + \frac{dV_{\perp}}{dt} = \frac{q}{m}\left(\bar{V}_{\perp} \times \bar{B}\right) \quad (30)$$

FOR $\frac{d\bar{V}_{\parallel}}{dt} = 0 \Rightarrow \bar{V}_{\parallel} = \text{COST}$

- UNIFORM $\bar{B}$ FIELDS: CYCLOTRON FREQUENCY.

$$\bar{B} \equiv (0, 0, B_z) \equiv B\hat{z} \Rightarrow$$

$$(31) \quad \begin{cases} m\frac{dV_x}{dt} = qBV_y \\ m\frac{dV_y}{dt} = -qBV_x \\ m\frac{dV_z}{dt} = 0 \end{cases}$$

DERIVING

$$\frac{d^2V_x}{dt^2} = \frac{qB}{m}\frac{dV_y}{dt} = \frac{qB}{m}\left(-\frac{qB}{m}\right)V_x$$

$$\frac{d^2V_x}{dt^2} = -\left(\frac{qB}{m}\right)^2 V_x \quad \text{WITH } -\frac{qB}{m} = \omega_c \quad (32)$$

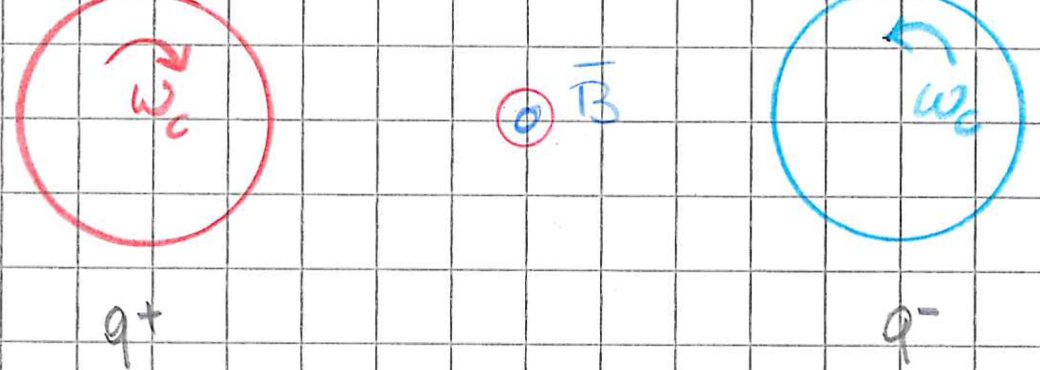
$$(33) \quad \begin{cases} \frac{d^2V_x}{dt^2} + \omega_c^2 V_x = 0 \\ \frac{d^2V_y}{dt^2} + \omega_c^2 V_y = 0 \end{cases}$$

THE SAME FOR

WHERE ω_c = GYROFREQ. OR CYCLOTRON FREQUENCY.

- ω_c IS RELATED TO THE FIELD STRENGTH AND TO THE PARTICLE MASS. IT DOES NOT DEPEND ON THE KINETIC ENERGY.

- FOR q^- AND q^+ ω_c HAS OPPOSITE DIRECTIONS



- PLASMA CAN HAVE SEVERAL CYCLOTRON FREQUENCIES
- THE $\bar{V} \times \bar{B}$ FORCE IS CENTRIPETAL

$$-\frac{mv_\perp^2}{r} = q\bar{V}\times\bar{B} = qV_\perp B \Rightarrow r_L = \frac{mV_\perp}{|q|B} = \frac{V_\perp}{\omega_c} \qquad (34)$$

IS KNOWN AS LARMOR RADIUS (OR GYRO RADIUS) $\Rightarrow$ PARTICLES WITH HIGHER $|\bar{V}|$ FORM CIRCLES OF LARGER r_L.

- UNIFORM FIELDS: HELICAL MOTION

SOLUTIONS TO EQS 3.4 AND 3.5 ARE HARMONIC

$$(35) \quad \begin{cases} V_x = V_\perp \exp i\omega_c t = \dot{x} \\ V_y = \dfrac{m}{qB} \dot{v}_x = \pm \dfrac{1}{\omega_c} \dot{V}_x = \pm i V_\perp \exp i\omega_c t = \dot{y} \end{cases}$$

WHERE $V_\perp = \sqrt{V_x^2 + V_y^2}$ IS A CONSTANT SPEED IN A PLAN $\perp$ TO $\bar{B}$.

BY INTEGRATING WE HAVE

INTEGRATING (35) WE HAVE

$$\boxed{x - x_0 = -i\frac{V_\perp}{\omega_c} e^{i\omega_c t}; \quad y - y_0 = \pm\frac{V_\perp}{\omega_c} e^{i\omega_c t}} \qquad (36)$$

USING r_L (LARMOR RADIUS) AND TAKING THE REAL PART WE HAVE

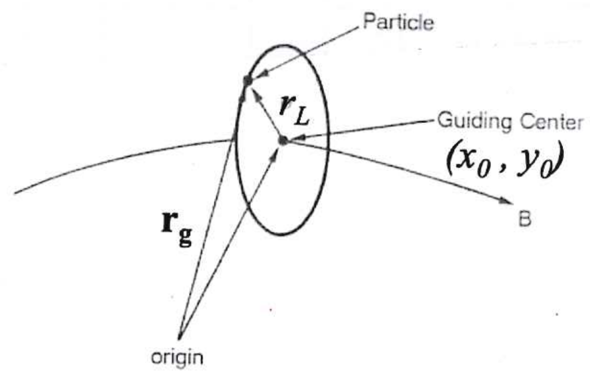
$$\begin{cases} x - x_0 = r_L \sin(\omega_c t) \\ y - y_0 = \pm r_L \cos(\omega_c t) \end{cases} \quad (37)$$

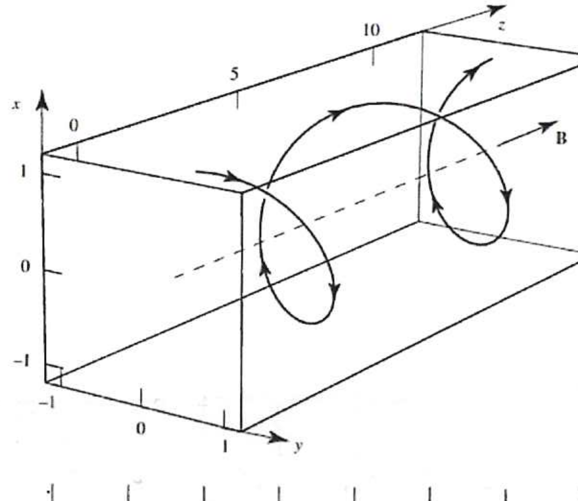
- THIS DESCRIBE A CIRCULAR ORBIT AROUND A GUIDING CENTER (x_0, y_0).

IN ADDITION TO THIS MOTION THERE IS A VELOCITY V_z NON AFFECTED BY $\bar{B}$ BUT PARALEL TO $\bar{B}$ COMBINED WITH (36) THIS GIVES RISE TO HELICAL MOTION ABOUT A GUIDING CENTER

$$\boxed{\underline{\bar{r}}_g = \hat{x} x_0 + \hat{y} y_0 + \hat{z}(z_0 + v_{\parallel} t)} \quad (38)$$

- OBSERVATION. GUIDING CENTER MOVES LINEARLY ALONG $\hat{z}$ WITH CONSTANT VELOCITY $(v_{\parallel})$.

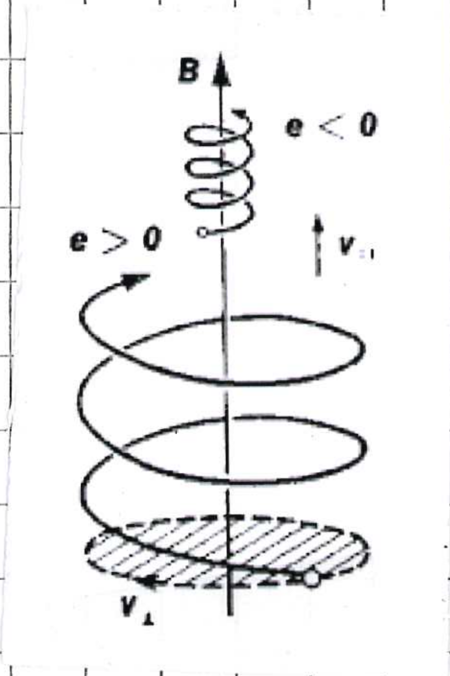




HELICAL MOTION: PARTICLE PITCH ANGLE

THE PITCH ANGLE IS THE ANGLE FORMED BY THE CHARGED PARTICLE VELOCITY VECTOR AND THE LOCAL MAGNETIC FIELD VECTOR, THIS IS A COMMON MEASUREMENT AND TOPIC WHEN STUDYING THE MAGNETOSPHERE, FOR EXAMPLE.
THE PITCH ANGLE OF HELIX IS DEFINED AS

$$\alpha = \tan^{-1}\left(\frac{V_\perp}{V_{\parallel}}\right)$$



- REVOLUTION OF IONS AND ELECTRONS IN LARMOR SPIRALS WEAKENS THE EXTERNAL MAGNETIC FIELD.
- THE RADIUS OF REVOLUTION OF IONS q^+ IS > THAN THAT OF AN e^- IF $m^+ > m_e^-$
- CHARGED PARTICLES IN MOTION ABOUT $\bar{B}$ HAS A MAGNETIC MOMENT $|\bar{\mu}| = IA = \frac{q}{\tilde{\tau}_c} \pi r_L^2$

$$= \frac{q\omega_c}{2\pi} \pi r_L^2 \rightarrow$$

$$\bar{\mu} = 1/2\, m V_\perp^2\, \bar{B}^{-1}$$

UNIFORM $\bar{E}$ AND $\bar{B}$ FIELD: $\bar{E}\times\bar{B}$ DRIFT

IF $\bar{E} \neq 0$ THE RESULTING TRAJECTORY IS THE SUM OF TWO MOTION; CIRCULAR LARMOR GYRATION PLUS DRIFT OF THE GUIDING CENTER.
LET'S CHOOSE $\bar{E}$ IN THE (x,z) PLANE $(E_y = 0)$
THE EQ OF MOTION IS

$m\frac{dv}{dt} = q(\bar{E} + \bar{v} \times \bar{B})$. THE V_z IS THEN

$$\frac{dV_z}{dt} = \frac{q}{m}E_z \Rightarrow V_z = \frac{qE_z}{m}t + V_{z0}$$

THIS IS A STRAIGHT ACCELERATION ALONG $\bar{B}$, THE TRANSVERSE COMPONENTS ARE

$$\frac{dV_x}{dt} = \frac{q}{m}E_x \pm \omega_c V_y$$

$$\frac{dV_y}{dt} = 0 \pm \omega_c V_x$$

TAKING THE TIME DERIVATIVE

$$\ddot{V}_x = -\omega_c^2 V_x \; ; \; \ddot{V}_y = \pm\omega_c\left(\frac{q}{m}E_x \pm \omega_c V_y\right)$$

$$= -\omega_c^2\left(\frac{E_x}{B} + V_y\right)$$