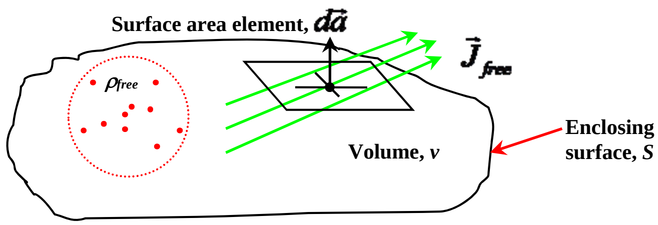
LECTURE NOTES 1

CONSERVATION LAWS

Conservation of energy E, linear momentum $\vec{p}$, angular momentum $\vec{L}$ and electric charge q are of fundamental importance in electrodynamics (*n.b.* this is *also* true for ***all*** fundamental forces of nature – the weak, strong, *EM* and gravitational force, both microscopically (locally), and hence macroscopically (globally - *i.e.* the entire universe)!

Electric Charge Conservation

Previously (*i.e.* last semester in Physics 435), we discussed electric charge conservation:



Electric current flowing ***outward*** from volume v through closed bounding surface S at time t:

$$I_{free}(t) = \oint_S \vec{J}_{free}(\vec{r},t)\cdot d\vec{a} \quad (Amperes)$$

Electric charge contained in volume v at time t:

$$Q_{free}(t) = \int_v \rho_{free}(\vec{r},t)\, d\tau \quad (Coulombs)$$

An ***outward*** flow of current through surface S corresponds to a ***decrease*** in charge in volume v:

$$I_{free}(t) = -\frac{dQ_{free}(t)}{dt} \quad (Amperes = Coulombs/sec) \qquad i.e.\ \frac{dQ_{free}(t)}{dt} < 0,\ I_{free}(t) = -\frac{dQ_{free}(t)}{dt} > 0$$

Global conservation of electric charge:

$$I_{free}(t) = \oint_S \vec{J}_{free}(\vec{r},t)\cdot d\vec{a} = -\frac{dQ_{free}(t)}{dt}$$

But:

$$\frac{dQ_{free}(t)}{dt} = \frac{d}{dt}\int_v \rho_{free}(\vec{r},t)\, d\tau = \int_v \frac{\partial \rho_{free}(\vec{r},t)}{\partial t}\, d\tau$$

Use the divergence theorem on the LHS of the global conservation of charge equation:

$$\int_v \vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t)\, d\tau = -\int_v \frac{\partial \rho_{free}(\vec{r},t)}{\partial t}\, d\tau$$

⇐ Integral form of the continuity equation.

This relation ***must*** hold for ***any*** arbitrary volume v associated with the enclosing surface S; hence the integrands in the above equation ***must*** be equal – we thus obtain the continuity equation (in differential form), which expresses local conservation of electric charge at $(\vec{r},t)$:

$$\vec{\nabla}\cdot\vec{J}_{free}(\vec{r},t) = -\frac{\partial \rho_{free}(\vec{r},t)}{\partial t}$$

⇐ Differential form of the continuity equation.

n.b. The continuity equation doesn't explain ***why*** electric charge is conserved
– it merely describes mathematically that electric charge ***is*** conserved!!

Poynting's Theorem and Poynting's Vector $\vec{S}(\vec{r},t)$

We know that the work required to assemble a ***static*** charge distribution is:

$$W_E(t) = \frac{\varepsilon_o}{2}\int_v E^2(\vec{r},t)\,d\tau = \frac{\varepsilon_o}{2}\int_v \left(\vec{E}(\vec{r},t)\bullet\vec{E}(\vec{r},t)\right)d\tau = \frac{1}{2}\int_v \left(\vec{D}(\vec{r},t)\bullet\vec{E}(\vec{r},t)\right)d\tau$$

SI units: *Joules*

***Linear* Dielectric Media**

Likewise, the work required to get electric currents flowing, *e.g.* against a back *EMF* is:

$$W_M(t) = \frac{1}{2\mu_o}\int_v B^2(\vec{r},t)\,d\tau = \frac{1}{2\mu_o}\int_v \left(\vec{B}(\vec{r},t)\bullet\vec{B}(\vec{r},t)\right)d\tau = \frac{1}{2}\int_v \left(\vec{H}(\vec{r},t)\bullet\vec{B}(\vec{r},t)\right)d\tau$$

SI units: *Joules*

***Linear* Magnetic Media**

Thus the <u>total</u> energy, U_{EM} stored in *EM* field(s) is (by energy conservation) = total work done:

$$U_{EM}(t) = W_{tot}(t) = W_{EM}(t) = W_E(t) + W_M(t) = \frac{1}{2}\int_v \left(\varepsilon_o E^2(\vec{r},t) + \frac{1}{\mu_o}B^2(\vec{r},t)\right)d\tau = \int_v u_{EM}(\vec{r},t)\,d\tau$$

SI units: *Joules*

$$U_{EM}(t) = \int_v u_{EM}(\vec{r},t)\,d\tau = \frac{1}{2}\int_v \left(\varepsilon_o E^2(\vec{r},t) + \frac{1}{\mu_o}B^2(\vec{r},t)\right)d\tau$$

SI units: *Joules*

where u_{EM} = total energy <u>density</u>: $u_{EM}(\vec{r},t) = \frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \frac{1}{\mu_o}B^2(\vec{r},t)\right)$ (SI units: $Joules/m^3$)

Suppose we have some charge density $\rho(\vec{r},t)$ and current density $\vec{J}(\vec{r},t)$ configuration(s) that at time t produce *EM* fields $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$. In the next instant dt, *i.e.* at time $t + dt$, the charge moves around. What is the amount of infinitesimal work dW done by *EM* forces acting on these charges / currents, in the time interval dt ?

The Lorentz Force Law is: $\vec{F}(\vec{r},t) = q\left(\vec{E}(\vec{r},t) + \vec{v}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)$

The infinitesimal amount of work dW done on an electric charge q moving an infinitesimal distance $d\vec{\ell} = \vec{v}dt$ in an infinitesimal time interval dt is:

$$dW = \vec{F}\bullet d\vec{\ell} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)\bullet d\vec{\ell} = q\vec{E}\bullet\vec{v}dt + \underbrace{q\big(\overbrace{\vec{v}\times\vec{B}}^{n.b.\perp\text{to }\vec{v}\,!!!}\big)\bullet\vec{v}dt}_{=0} = q\vec{E}\bullet\vec{v}dt$$

(*n.b.* magnetic forces do <u>no</u> work!!)

<u>But</u>: $q_{free}(\vec{r},t) = \rho_{free}(\vec{r},t)\,d\tau$ and: $\rho_{free}(\vec{r},t)\vec{v}(\vec{r},t) = \vec{J}_{free}(\vec{r},t)$

The (instantaneous) rate at which (total) work is done on all of the electric charges within the volume v is:

$$\frac{dW(t)}{dt} = \int_v \vec{F}(\vec{r},t)\cdot\left(d\vec{\ell}(\vec{r},t)/dt\right) = \int_v \vec{F}(\vec{r},t)\cdot\vec{v}(\vec{r},t) = \int_v q_{free}(\vec{r},t)\vec{E}(\vec{r},t)\cdot\vec{v}(\vec{r},t)$$

$$= \int_v \rho_{free}(\vec{r},t)\,d\tau\,\vec{E}(\vec{r},t)\cdot\vec{v}(\vec{r},t) \quad \text{using:} \quad q_{free}(\vec{r},t) = \rho_{free}(\vec{r},t)\,d\tau$$

$$= \int_v \left(\vec{E}(\vec{r},t)\cdot\rho_{free}(\vec{r},t)\vec{v}(\vec{r},t)\right)d\tau \quad \text{but:} \quad J_{free}(\vec{r},t) = \rho_{free}(\vec{r},t)\vec{v}(\vec{r},t)$$

$$\therefore \quad \boxed{\frac{dW(t)}{dt} = \int_v \left(\vec{E}(\vec{r},t)\cdot\vec{J}_{free}(\vec{r},t)\right)d\tau = P(t)} = \text{instantaneous power (SI units: } \textit{Watts}\text{)}$$

The quantity $\vec{E}(\vec{r},t)\cdot\vec{J}_{free}(\vec{r},t)$ is the (instantaneous) work done per unit time, per unit volume – *i.e*. the instantaneous *power* delivered *per unit volume* (*aka* the power *density*).

Thus:
$$\boxed{P(t) = \frac{dW(t)}{dt} = \int_v \left(\vec{E}(\vec{r},t)\cdot\vec{J}_{free}(\vec{r},t)\right)d\tau} \quad \left(\text{SI units: } \textit{Watts} = \frac{\textit{Joules}}{\textit{sec}}\right)$$

We can express the quantity $\left(\vec{E}\cdot\vec{J}_{free}\right)$ in terms of the *EM* fields (alone) using the Ampere-Maxwell law (in differential form) to eliminate $\vec{J}_{free}$.

Ampere's Law with Maxwell's Displacement Current correction term (in differential form):

$$\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t) = \mu_o\left\{\vec{J}_{free}(\vec{r},t) + \vec{J}_D(\vec{r},t)\right\} = \mu_o\vec{J}_{free}(\vec{r},t) + \mu_o\varepsilon_o\frac{\partial\vec{E}(\vec{r},t)}{\partial t}}$$

Thus:
$$\boxed{\vec{J}_{free}(\vec{r},t) = \frac{1}{\mu_o}\left(\vec{\nabla}\times\vec{B}(\vec{r},t)\right) - \varepsilon_o\frac{\partial\vec{E}(\vec{r},t)}{\partial t}}$$

Then:
$$\vec{E}(\vec{r},t)\cdot\vec{J}_{free}(\vec{r},t) = \vec{E}(\vec{r},t)\cdot\left\{\frac{1}{\mu_o}\left(\vec{\nabla}\times\vec{B}(\vec{r},t)\right) - \varepsilon_o\frac{\partial\vec{E}(\vec{r},t)}{\partial t}\right\}$$
$$= \frac{1}{\mu_o}\vec{E}(\vec{r},t)\cdot\left(\vec{\nabla}\times\vec{B}(\vec{r},t)\right) - \varepsilon_o\vec{E}(\vec{r},t)\cdot\frac{\partial\vec{E}(\vec{r},t)}{\partial t}$$

Now: $\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right) = \vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right) - \vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right)$ Griffiths Product Rule #6 (see inside front cover)

Thus: $\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right) = \vec{B}\cdot\left(\vec{\nabla}\times\vec{E}\right) - \vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right)$

But Faraday's Law (in differential form) is: $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t) = -\frac{\partial\vec{B}(\vec{r},t)}{\partial t}}$

$$\therefore \quad \vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right) = -\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right)$$

However: $\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\vec{B}\cdot\vec{B}\right) = \frac{1}{2}\frac{\partial}{\partial t}\left(B^2\right)$ and similarly: $\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\vec{E}\cdot\vec{E}\right) = \frac{1}{2}\frac{\partial}{\partial t}\left(E^2\right)$

Therefore:

$$\vec{E}(\vec{r},t)\cdot\vec{J}_{free}(\vec{r},t) = -\frac{1}{\mu_o}\left\{-\frac{1}{2}\frac{\partial}{\partial t}\left(B^2(\vec{r},t)\right) - \vec{\nabla}\cdot\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)\right\} - \varepsilon_o\left\{\frac{1}{2}\frac{\partial}{\partial t}\left(E^2(\vec{r},t)\right)\right\}$$

$$= -\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_o E^2(\vec{r},t) + \frac{1}{\mu_o}B^2(\vec{r},t)\right) - \frac{1}{\mu_o}\vec{\nabla}\cdot\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)$$

Then:

$$P(t) = \frac{dW(t)}{dt} = \int_v \left(\vec{E}(\vec{r},t)\cdot\vec{J}_{free}(\vec{r},t)\right)d\tau$$

$$= -\frac{1}{2}\frac{d}{dt}\int_v\left(\varepsilon_o E^2(\vec{r},t) + \frac{1}{\mu_o}B^2(\vec{r},t)\right)d\tau - \frac{1}{\mu_o}\underbrace{\int_v \vec{\nabla}\cdot\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)d\tau}$$

Apply the divergence theorem to this term, get:

Poynting's Theorem = "Work-Energy" Theorem of Electrodynamics:

$$\boxed{P(t) = \frac{dW(t)}{dt} = -\frac{d}{dt}\int_v\left\{\frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t) + \frac{1}{\mu_o}B^2(\vec{r},t)\right)\right\}d\tau - \frac{1}{\mu_o}\oint_S\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)\cdot d\vec{a}}$$

Physically, $\frac{1}{2}\int_v\left(\varepsilon_o E^2(\vec{r},t) + \frac{1}{\mu_o}B^2(\vec{r},t)\right)d\tau$ = instantaneous energy stored in the *EM* fields $\left(\vec{E}(\vec{r},t) \text{ and } \vec{B}(\vec{r},t)\right)$ within the volume v (SI units: Joules)

Physically, the term $-\frac{1}{\mu_o}\oint_S\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)\cdot d\vec{a}$ = instantaneous rate at which *EM* energy is carried / flows out of the volume v (carried microscopically by virtual (and/or real!) photons across the bounding/enclosing surface S by the *EM* fields $\vec{E}$ and $\vec{B}$ – *i.e.* this term represents/is the instantaneous *EM* power flowing across/through the bounding/enclosing surface S (SI units: *Watts* = *Joules/sec*).

Poynting's Theorem says that:

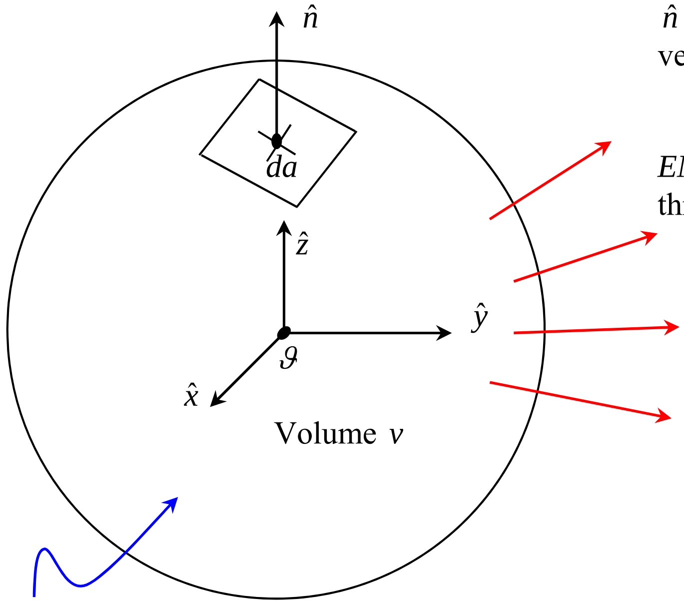
The instantaneous work done on the electric charges in the volume v by the *EM* force is equal to the decrease in the instantaneous energy stored in *EM* fields ($\vec{E}$ and $\vec{B}$), minus the energy that is instantaneously flowing out of/through the bounding surface S.

We define ***Poynting's vector***: $\boxed{\vec{S}(\vec{r},t) \equiv \frac{1}{\mu_o}\left(\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)\right)}$ = energy / unit time / unit area, transported by the *EM* fields ($\vec{E}$ and $\vec{B}$) across/through the bounding surface S

n.b. Poynting's vector $\vec{S}$ has SI units of *Watts/m²* – *i.e.* an energy flux *density*.

Thus, we see that: $$P(t)=\frac{dW(t)}{dt}=-\frac{dU_{EM}(t)}{dt}-\oint_S \vec{S}(\vec{r},t)\cdot d\vec{a}$$

where $\vec{S}(\vec{r},t)\cdot d\vec{a}$ = instantaneous power (energy per unit time) crossing/passing through an infinitesimal surface area element $d\vec{a}=\hat{n}da$, as shown in the figure below:



$\hat{n}$ = outward pointing unit normal vector (everywhere $\perp$ to surface S)

EM energy flowing out of volume v through enclosing surface S

Enclosing surface S

Poynting's vector: $\vec{S}(\vec{r},t)\equiv\frac{1}{\mu_o}\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t)$ = Energy *Flux* Density (SI units: *Watts*/m^2)

The work W done on the electrical charges contained within the volume v will increase their mechanical energy – kinetic and/or potential energy. Define the (instantaneous) mechanical energy *density* $u_{mech}(\vec{r},t)$ such that:

$$\frac{du_{mech}(\vec{r},t)}{dt}=\vec{E}(\vec{r},t)\cdot\vec{J}_{free}(\vec{r},t)$$ Hence: $$\frac{dU_{mech}}{dt}=\int_v\left(\vec{E}(\vec{r},t)\cdot\vec{J}_{free}(\vec{r},t)\right)d\tau$$

Then: $$P(t)=\frac{dW(t)}{dt}=\frac{dU_{mech}}{dt}=\frac{d}{dt}\int_v u_{mech}(\vec{r},t)\,d\tau=\int_v\left(\vec{E}(\vec{r},t)\cdot\vec{J}_{free}(\vec{r},t)\right)d\tau$$

However, the (instantaneous) *EM* field energy density is:

$$u_{EM}(\vec{r},t)=\frac{1}{2}\left(\varepsilon_o E^2(\vec{r},t)+\frac{1}{\mu_o}B^2(\vec{r},t)\right)$$ (*Joules*/m^3)

Then the (instantaneous) *EM* field energy contained within the volume v is:

$$U_{EM}(t)=\int_v u_{EM}(\vec{r},t)\,d\tau$$ (*Joules*)

Thus, we see that: $$\frac{d}{dt}\int_v \left(u_{mech}(\vec{r},t)+u_{EM}(\vec{r},t)\right)d\tau = -\oint_S \vec{S}(\vec{r},t)\cdot d\vec{a} = -\int_v \left(\vec{\nabla}\cdot\vec{S}(\vec{r},t)\right)d\tau$$ **Using the Divergence theorem**

The integrands of LHS *vs*. {far} RHS of the above equation ***must*** be equal for each/every space-time point $(\vec{r},t)$ within the source volume v associated with bounding surface *S*. Thus, we obtain:

The Differential Form of Poynting's Theorem: $$\frac{\partial}{\partial t}\left[u_{mech}(\vec{r},t)+u_{EM}(\vec{r},t)\right] = -\vec{\nabla}\cdot\vec{S}(\vec{r},t)$$

Poynting's theorem = **Energy Conservation** "book-keeping" equation, *c.f.* with the
Continuity equation = **Charge Conservation** "book-keeping" equation:

The Differential Form of the Continuity Equation: $$\frac{\partial}{\partial t}\rho(\vec{r},t) = -\vec{\nabla}\cdot\vec{J}(\vec{r},t)$$

Since $\frac{\partial u_{mech}(\vec{r},t)}{\partial t} = \vec{E}(\vec{r},t)\cdot\vec{J}_{free}(\vec{r},t)$, we can write the differential form of Poynting's theorem as:

$$\vec{E}(\vec{r},t)\cdot\vec{J}_{free}(\vec{r},t)+\frac{\partial u_{EM}(\vec{r},t)}{\partial t} = -\vec{\nabla}\cdot\vec{S}(\vec{r},t)$$

Or: $$\vec{E}(\vec{r},t)\cdot\vec{J}_{free}(\vec{r},t)+\frac{\partial u_{EM}(\vec{r},t)}{\partial t}+\vec{\nabla}\cdot\vec{S}(\vec{r},t) = 0$$

Poynting's Theorem / Poynting's vector $\vec{S}(\vec{r},t)$ represents the (instantaneous) flow of *EM* energy in exactly the same/analogous way that the free current density $\vec{J}_{free}(\vec{r},t)$ represents the (instantaneous) flow of electric charge.

In the presence of *linear* dielectric / *linear* magnetic media, if one is ONLY interested in FREE charges and FREE currents, then:

$$u_{EM}^{free}(\vec{r},t) = \frac{1}{2}\left(\vec{E}(\vec{r},t)\cdot\vec{D}(\vec{r},t)+\vec{B}(\vec{r},t)\cdot\vec{H}(\vec{r},t)\right) \qquad \vec{D}(\vec{r},t) = \varepsilon\vec{E}(\vec{r},t) \qquad \varepsilon = \varepsilon_o(1+\chi_e)$$

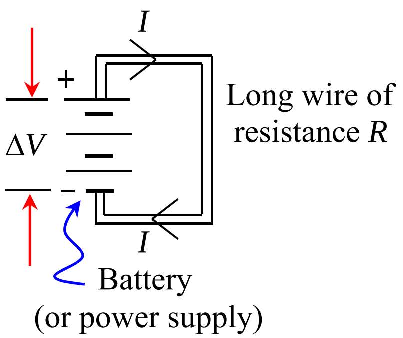
$$\vec{S}(\vec{r},t) = \tfrac{1}{\mu}\vec{E}(\vec{r},t)\times\vec{B}(\vec{r},t) = \vec{E}(\vec{r},t)\times\vec{H}(\vec{r},t) \qquad \vec{B}(\vec{r},t) = \mu\vec{H}(\vec{r},t) \qquad \mu = \mu_o(1+\chi_m)$$

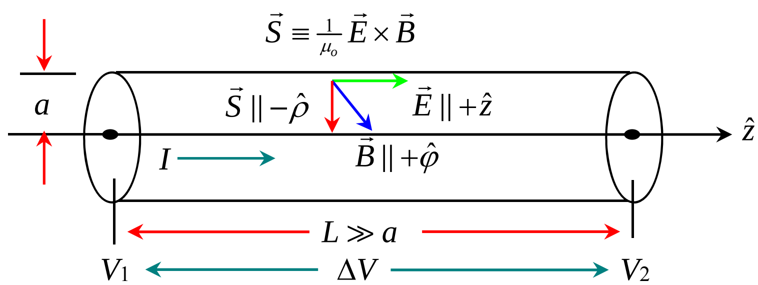
Griffiths Example 8.1:

Poynting's vector $\vec{S}$, power dissipation and Joule heating of a long, current-carrying wire.

When a steady, free electrical current I ($\neq$ function of time, t) flows down a long wire of length $L \gg a$ (a = radius of wire) and resistance $R \left(= L/\pi a^2 \sigma_C\right)$, the electrical energy is dissipated as heat (*i.e.* thermal energy) in the wire.

Electrical power dissipation: $\boxed{P = \Delta V \cdot I = I^2 R}$

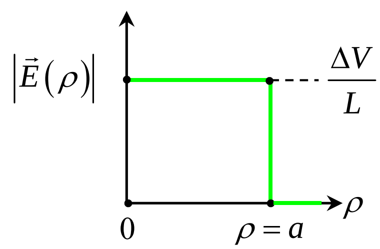




Free Current Density: $\boxed{\vec{J}_{free} = \sigma_C \vec{E} = \left(I/\pi a^2\right)\hat{z} \;\; \left(Amps/m^2\right)}$

Longitudinal Electric Field: $\boxed{\vec{E} = \dfrac{\vec{J}_{free}}{\sigma_C} = \dfrac{\Delta V}{L}\hat{z} \;\; \left(Volts/m\right)}$

Potential Difference: $\boxed{\Delta V = V_1 - V_2 \;\; (>0) \; (Volts)}$



n.b. The {steady} free current density $\vec{J}_{free} \left(= \sigma_C \vec{E} = I/\pi a^2\right)$ and the longitudinal electric field $\vec{E} = \left(\Delta V/L\right)\hat{z}$ are uniform across (and along) the long wire, everywhere within the volume of the wire $(\rho < a)$. $\Rightarrow$ Thus, this particular problem has no time-dependence…

From Ampere's Law: $\boxed{\vec{B}^{inside}\left(\rho < a\right) = \dfrac{\mu_o I \rho}{2\pi a^2}\hat{\varphi}}$ $\quad \rho = \sqrt{x^2 + y^2}$ in cylindrical coordinates

$\left\{\oint_C \vec{B}(\vec{r})\cdot d\vec{\ell} = \mu_o I_{encl}\right\}$ $\quad \boxed{\vec{B}^{outside}\left(\rho \geq a\right) = \dfrac{\mu_o I}{2\pi \rho}\hat{\varphi}}$ $\quad (Tesla)$

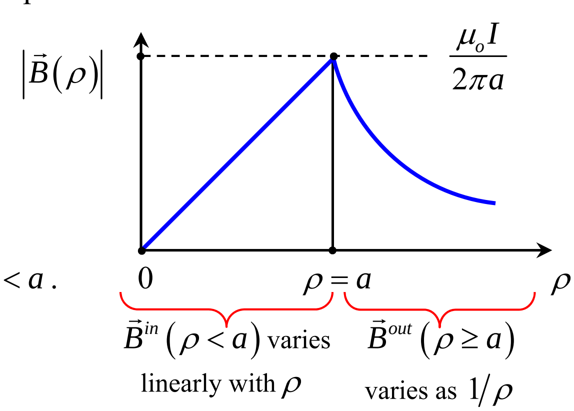
n.b. for simplicity's sake, we have approximated the finite length wire by an ∞-length wire. This will have unphysical, but understandable consequences later on….

Poynting's Vector: $\boxed{\vec{S}(\vec{r}) = \dfrac{1}{\mu_o}\vec{E}(\vec{r}) \times \vec{B}(\vec{r})}$

$$\boxed{\vec{S}^{inside}\left(\rho < a\right) = \frac{\Delta V \cdot I \rho}{2\pi a^2 L}\overbrace{\left(\hat{z}\times\hat{\varphi}\right)}^{=-\hat{\rho}} = \frac{\Delta V \cdot I \rho}{2\pi a^2 L}\left(-\hat{\rho}\right)}$$

Poynting's vector $\vec{S}$ oriented radially inward for $\rho < a$.

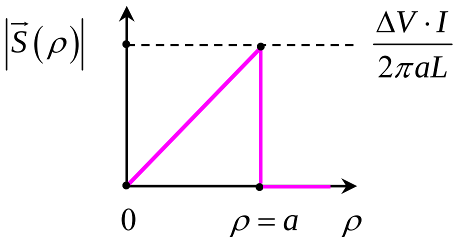
$\boxed{\vec{S}^{outside}\left(\rho > a\right) = 0}$ {because $\vec{E}\left(\rho > a\right) = 0$!!!}



Note the following result for Poynting's vector evaluated at the surface of the long wire, *i.e.* @ $\rho = a$:

$$\boxed{\vec{S}^{inside}(\rho=a) = \frac{\Delta V \cdot I}{2\pi a L}(-\hat{\rho})}$$ (SI units: *Watts/m*^{2})

Since $\vec{E}^{outside}(\rho \ge a) = 0$: $\boxed{\vec{S}^{outside}(\rho=a)=0}$ $\Rightarrow \exists$ a ***discontinuity*** in $\vec{S}$ at $\rho = a$!!!

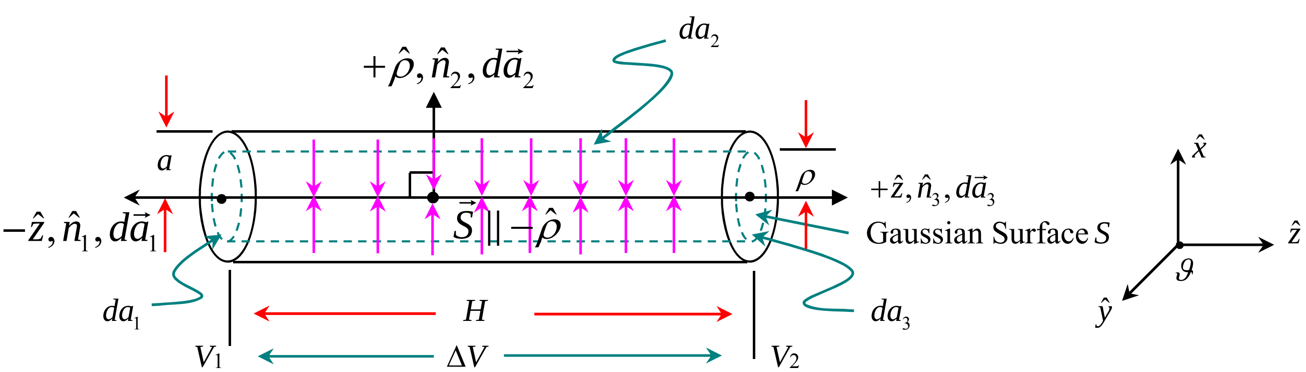


$$\boxed{\vec{S}_{\substack{long\\wire}}(\rho \le a) = \frac{\Delta V \cdot I\rho}{2\pi a^2 L}(-\hat{\rho}) = -\frac{\Delta V \cdot I\rho}{2\pi a^2 L}\hat{\rho}}$$

Now let us use the *integral* version of Poynting's theorem to determine the *EM* energy flowing through an imaginary Gaussian cylindrical surface S of radius $\rho < a$ and length $H \ll L$:

$$\boxed{\begin{aligned} P(t) &= \frac{dW(t)}{dt} = \frac{dU_{mech}}{dt} = \frac{d}{dt}\int_v u_{mech}(\vec{r},t)\,d\tau = \int_v \left(\vec{E}(\vec{r},t)\cdot\vec{J}_{free}(\vec{r},t)\right)d\tau \\ &= -\frac{dU_{EM}(t)}{dt} - \oint_S \vec{S}(\vec{r},t)\cdot d\vec{a} = -\frac{d}{dt}\int_v u_{EM}(\vec{r},t)\,d\tau - \int_v \left(\vec{\nabla}\cdot\vec{S}(\vec{r},t)\right)d\tau \end{aligned}}$$

Since this is a static/steady-state problem, we assume that $dU_{EM}(t)/dt = 0$.



Then for an imaginary Gaussian surface taken *inside* the long wire ($\rho < a$):

$$P_{wire} = -\oint_S \vec{S}_{wire}\cdot d\vec{a} = -\underbrace{\int_{\substack{LHS\\endcap}} \overset{=0}{\vec{S}\cdot d\vec{a}_1}}_{d\vec{a}_1 = da_1(-\hat{z})} - \underbrace{\int_{\substack{cyl\\surface}} \vec{S}\cdot d\vec{a}_2}_{d\vec{a}_2 = da_2\hat{\rho}} - \underbrace{\int_{\substack{RHS\\endcap}} \overset{=0}{\vec{S}\cdot d\vec{a}_3}}_{d\vec{a}_3 = da_3(+\hat{z})}$$

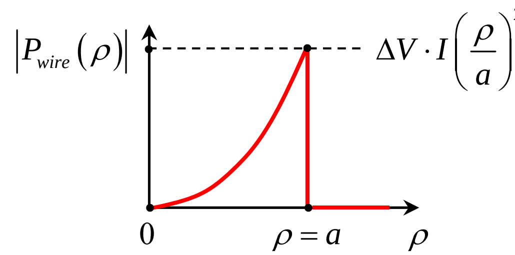
$\vec{S}(\parallel -\hat{\rho})$ is $\perp$ to $d\vec{a}_1(\parallel -\hat{z})$; $\vec{S}(\parallel -\hat{\rho})$ is anti-$\parallel$ to $d\vec{a}_2(\parallel +\hat{\rho})$; $\vec{S}(\parallel -\hat{\rho})$ is $\perp$ to $d\vec{a}_3(\parallel +\hat{z})$

<u>Only surviving term is:</u>

$$\boxed{P_{wire}(\rho) = -\int_{\substack{cyl\\surface}} \vec{S}(\rho)\cdot d\vec{a}_2 = -\int_{z=-H/2}^{z=+H/2}\int_{\varphi=0}^{\varphi=2\pi}\left(-\frac{\Delta V \cdot I\rho}{2\pi a^2 H}\hat{\rho}\right)\rho\, d\varphi\, dz\,\hat{\rho} = \left(\frac{\Delta V \cdot I}{2\pi a^2 H}\rho\right)(2\pi\rho H) = \Delta V \cdot I\left(\frac{\rho^2}{a^2}\right)}$$

Thus: $\boxed{P_{wire}(\rho) = \Delta V \cdot I\left(\frac{\rho}{a}\right)^2}$ $(Watts)$

And: $\boxed{P_{wire}(\rho = a) = \Delta V \cdot I}$ $(Watts)$



This *EM* energy is dissipated as heat (thermal energy) in the wire – also known as Joule heating of the wire. Since $|P_{wire}(\rho)| \propto \rho^2$, note also that the Joule heating of the wire occurs primarily at/on the outermost portions of the wire.

From Ohm's Law: $\boxed{\Delta V = I \cdot R_{wire}}$ where $\boxed{R_{wire} = \text{resistance of wire} = \rho_C^{wire} L / A_\perp^{wire} = L / \sigma_C^{wire} A_\perp^{wire}}$

Joule Heating of current-carrying wire	$P_{wire}(\rho) = -I^2 R_{wire}\left(\frac{\rho}{a}\right)^2$ $P_{wire}(\rho = a) = -I^2 R_{wire}$	**Power losses in wire show up / result in Joule heating of wire. Electrical energy is converted into heat (thermal) energy – At the microscopic level, this is due to *kinetic* energy losses associated with the ensemble of individual drift/conduction/free electron scatterings inside the wire!**

Again use the integral version of Poynting's theorem to determine the *EM* field energy flowing through an imaginary Gaussian cylindrical surface S of radius $\rho \geq a$ and length $H \ll L$.

We expect that we ***should*** get the same answer as that obtained above, for the $\rho < a$ Gaussian cylindrical surface. However, for $\rho \geq a$, $\vec{S}^{outside}(\rho > a) = 0$, because $\vec{E}^{outside}(\rho > a) = 0$!!!

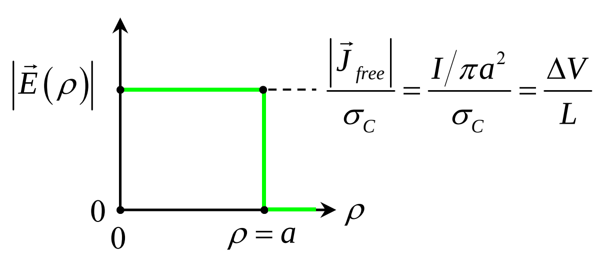
Thus, for a Gaussian cylindrical surface S taken with $\rho \geq a$ we obtain: $P_{wire} = -\oint_S \vec{S}_{wire} \cdot d\vec{a} = 0$!!!

What??? How can we get two ***different*** P_{wire} answers for $\rho < a$ *vs.* $\rho \geq a$??? This ***can't*** be!!!

$\Rightarrow$ We need to re-assess our assumptions here…

It turns out that we have neglected an important, and somewhat subtle point...

The longitudinal electric field $\vec{E} = (\Delta V / L)\hat{z}$ formally/mathematically has a ***discontinuity*** at $\rho = a$:

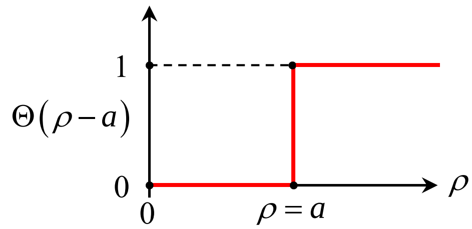


i.e. The tangential ($\hat{z}$) component of $\vec{E}$ is ***discontinuous*** at $\rho = a$.

Formally/mathematically, we need to write the longitudinal electric field for this situation as:

$$\boxed{\vec{E}(\rho) = \frac{\vec{J}_{free}}{\sigma_C}\left[1-\Theta(\rho-a)\right] = \frac{\left|\vec{J}_{free}\right|}{\sigma_C}\left[1-\Theta(\rho-a)\right]\hat{z}}$$

where the ***Heaviside step function*** is defined as: $\Theta(\rho-a) \equiv \begin{cases} 0 \ \text{ for } \ \rho < a \\ 1 \ \text{ for } \ \rho \geq a \end{cases}$ as shown below:



Furthermore, note that: $\Theta(x) = \int_{-\infty}^{x} \delta(t)\,dt$ and that: $\frac{d}{dx}\Theta(x) = \delta(x)$,

where $\delta(x)$ is the Dirac delta function.

Now, in the process of ***deriving*** Poynting's theorem (above), we used Griffith's Product Rule # 6 to obtain $\vec{E}\bullet\left(\vec{\nabla}\times\vec{B}\right) = \vec{B}\bullet\left(\vec{\nabla}\times\vec{E}\right) - \vec{\nabla}\bullet\left(\vec{E}\times\vec{B}\right)$, and then used Faraday's law (in differential form) $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$ and then used $\vec{B}\bullet\frac{\partial\vec{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\vec{B}\bullet\vec{B}\right) = \frac{1}{2}\frac{\partial}{\partial t}\left(B^2\right)$ and $\vec{E}\bullet\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\vec{E}\bullet\vec{E}\right) = \frac{1}{2}\frac{\partial}{\partial t}\left(E^2\right)$ with $u_{EM} = \tfrac{1}{2}\left(\varepsilon_o E^2 + \tfrac{1}{\mu_o}B^2\right)$ to finally obtain:

$$\begin{aligned} P(t) &= \frac{dW(t)}{dt} = \frac{dU_{mech}}{dt} = \frac{d}{dt}\int_v u_{mech}\,d\tau = \int_v \vec{E}\bullet\vec{J}_{free}\,d\tau \\ &= -\frac{dU_{EM}(t)}{dt} - \oint_S \vec{S}\bullet d\vec{a} = -\frac{d}{dt}\int_v u_{EM}\,d\tau - \int_v \vec{\nabla}\bullet\vec{S}(\vec{r},t)\,d\tau \end{aligned}$$

So here, in this specific problem, what is $\vec{\nabla}\times\vec{E}$???

In cylindrical coordinates, the only non-vanishing term is:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial\rho}E_z\hat{\varphi} = \frac{\partial}{\partial\rho}\left\{-\frac{\left|\vec{J}_{free}\right|}{\sigma_C}\left[1-\Theta(\rho-a)\right]\right\}\hat{\varphi} = +\frac{\left|\vec{J}_{free}\right|}{\sigma_C}\frac{\partial\Theta(\rho-a)}{\partial\rho}\hat{\varphi} = \frac{\left|\vec{J}_{free}\right|}{\sigma_C}\delta(\rho-a)\hat{\varphi} = -\frac{\partial\vec{B}}{\partial t}$$

In other words: $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = \begin{cases} 0 & \text{for } \ \rho < a \\ \infty\cdot\left\{\frac{\left|\vec{J}_{free}\right|}{\sigma_C}\right\}\hat{\varphi} & \text{for } \ \rho = a \\ 0 & \text{for } \ \rho > a \end{cases}$

Thus, {only} for $\rho > a$ integration volumes, we {very definitely} need to {explicitly} include the δ-function such that its contribution to the integral at $\rho = a$ is properly taken into account!

$$
\begin{aligned}
P(t) = \frac{dW(t)}{dt} &= -\frac{d}{dt}\int_v u_{EM}\,d\tau - \oint_S \vec{S}\bullet d\vec{a} \\
&= -\frac{d}{dt}\int_v \tfrac{1}{2}\left(\varepsilon_o E^2 + \tfrac{1}{\mu_o}B^2\right)d\tau - \oint_S \vec{S}\bullet d\vec{a} \\
&= -\tfrac{1}{2}\varepsilon_o\int_v \frac{d}{dt}E^2\,d\tau - \tfrac{1}{2\mu_o}\int_v \frac{d}{dt}B^2\,d\tau - \oint_S \vec{S}\bullet d\vec{a} \\
&= -\varepsilon_o\int_v \vec{E}\bullet\frac{d\vec{E}}{dt}\,d\tau - \tfrac{1}{\mu_o}\int_v \vec{B}\bullet\frac{d\vec{B}}{dt}\,d\tau - \oint_S \vec{S}\bullet d\vec{a} \\
&= -\varepsilon_o\int_v \vec{E}\bullet\frac{d\vec{E}}{dt}\,d\tau + \tfrac{1}{\mu_o}\int_v \vec{B}\bullet\vec{\nabla}\times\vec{E}\,d\tau - \oint_S \vec{S}\bullet d\vec{a} \\
&= -\varepsilon_o\int_v \vec{E}\bullet\frac{d\vec{E}}{dt}\,d\tau + \frac{\left|\vec{J}_{free}\right|}{\mu_o\sigma_C}\int_v \vec{B}\bullet\delta(\rho-a)\hat{\varphi}\,d\tau - \oint_S \vec{S}\bullet d\vec{a}
\end{aligned}
$$

For this specific problem: $d\vec{E}/dt = 0$ and for $\rho > a$, $\vec{S}(\rho > a) = \tfrac{1}{\mu_o}\underbrace{\vec{E}(\rho > a)}_{=0}\times\vec{B}(\rho > a) = 0$.

Thus for $\rho > a$:

$$P(t) = \frac{\left|\vec{J}_{free}\right|}{\mu_o\sigma_C}\int_v \vec{B}\bullet\delta(\rho-a)\hat{\varphi}\,d\tau = 2\pi a L\frac{\left|\vec{J}_{free}\right|}{\mu_o\sigma_C}\left|\vec{B}(\rho=a)\right| = \cancel{2\pi a}L\frac{\left|\vec{J}_{free}\right|}{\cancel{\mu_o}\sigma_C}\frac{\cancel{\mu_o}I}{\cancel{2\pi a}} = \frac{\left|\vec{J}_{free}\right|}{\sigma_C}I\cdot L$$

But: $\vec{E} = \dfrac{\vec{J}_{free}}{\sigma_C} = \dfrac{\Delta V}{L}\hat{z}$, and thus, finally we obtain, for $\rho > a$: $P(t) = \dfrac{\Delta V}{\cancel{L}}I\cdot\cancel{L} = \Delta V\cdot I$,

which agrees precisely with that obtained earlier for $\rho < a$: $P(t) = \Delta V\cdot I$!!!

For an *E&M* problem that nominally has a ***steady-state*** current *I* present, it is indeed curious that $\vec{\nabla}\times\vec{E} = \dfrac{\left|\vec{J}_{free}\right|}{\sigma_C}\delta(\rho-a)\hat{\varphi} = -\dfrac{\partial\vec{B}}{\partial t}$ is non-zero, and in fact singular {at $\rho = a$}! The singularity is a consequence of the discontinuity in $\vec{E}$ on the $\rho = a$ surface of the long, current-carrying wire.

The relativistic nature of the 4-dimensional space-time world that we live in is ***encrypted*** into Faraday's law; here is one example where we come face-to-face with it!

Let's pursue the physics of this problem a bit further – and calculate the magnetic vector potential $\vec{A}(\vec{r})$ inside $(\rho < a)$ and outside $(\rho > a)$ the long wire…

In general, we know/anticipate that {here}: $\vec{A}(\vec{r}) \parallel \vec{J}(\vec{r}) \parallel +\hat{z}$ since: $\vec{A}(\vec{r}) = \dfrac{\mu_o}{4\pi}\int_{v'} \dfrac{\vec{J}(\vec{r}')}{\mathrm{r}} d\tau'$ where $\mathrm{r} = |\vec{\mathrm{r}}| \equiv |\vec{r} - \vec{r}'|$.

We don't need to carry out the above integral to obtain $\vec{A}(\vec{r})$ – a simpler method is to use $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$ in cylindrical coordinates. Since $\vec{A}(\vec{r}) = A_z(\vec{r})\hat{z}$ (only, here), the only non-zero contribution to this curl is: $\vec{B}(\vec{r}) = -\dfrac{\partial A_z(\vec{r})}{\partial \rho}\hat{\varphi}$.

For $\rho < a$: $\vec{B}(\rho < a) = \dfrac{\mu_o I \rho}{2\pi a^2}\hat{\varphi} = \dfrac{1}{2}\mu_o J \rho \hat{\varphi} = -\dfrac{\partial A_z(\rho < a)}{\partial \rho}\hat{\varphi} \Rightarrow \dfrac{\partial \vec{A}(\rho < a)}{\partial \rho} = -\dfrac{1}{2}\mu_o J \rho \hat{z}$

For $\rho \geq a$: $\vec{B}(\rho \geq a) = \dfrac{\mu_o I}{2\pi \rho}\hat{\varphi} = \dfrac{1}{2}\mu_o J a^2\left(\dfrac{1}{\rho}\right)\hat{\varphi} = -\dfrac{\partial A_z(\rho \geq a)}{\partial \rho}\hat{\varphi} \Rightarrow \dfrac{\partial \vec{A}(\rho \geq a)}{\partial \rho} = -\dfrac{1}{2}\mu_o J a^2\left(\dfrac{1}{\rho}\right)\hat{z}$

Using $\rho = a$ as our reference point for carrying out the integration {and noting that as in the case for the scalar potential $V(\vec{r})$, we similarly have the freedom to *e.g.* add ***<u>any</u>*** constant vector to $\vec{A}(\vec{r})$}:

$$\vec{A}(\rho < a) = -\frac{1}{2}\mu_o J \int \rho d\rho\, \hat{z} = -\frac{1}{2}\mu_o J \frac{1}{2}\left(\rho^2 - c_1^2\right)\hat{z} = -\frac{1}{4}\mu_o J\left(\rho^2 - c_1^2\right)\hat{z}$$

$$\vec{A}(\rho \geq a) = -\frac{1}{2}\mu_o J a^2 \int \left(\frac{1}{\rho}\right) d\rho\, \hat{z} = -\frac{1}{2}\mu_o J a^2 \ln(\rho/c_2)\hat{z}$$

where c_1 and c_2 are constants of the integration(s).

Physically, we demand that $\vec{A}(\rho)$ be continuous at $\rho = a$, thus we ***<u>must</u>*** have:

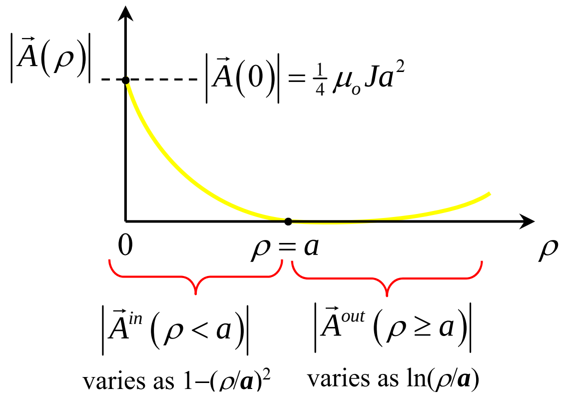
$$\vec{A}(\rho = a) = -\frac{1}{4}\mu_o J\left(a^2 - c_1^2\right)\hat{z} = -\frac{1}{2}\mu_o J a^2 \ln(a/c_2)\hat{z}$$

Obviously, the ***<u>only</u>*** way that this relation can be satisfied is if $c_1 = c_2 = \pm a$, because then $\vec{A}(\rho = a) = 0$ {*n.b.* $\ln(1) = \ln e^0 = 0$ }.

<u>Additionally</u>, we demand that $\vec{A}(\vec{r}) \parallel \vec{J}(\vec{r}) \parallel +\hat{z}$, hence ***<u>the</u>*** physically acceptable solution is $c_1 = c_2 = -a$, and thus the solutions for the magnetic vector potential $\vec{A}(\vec{r})$ for this problem are:

$$\vec{A}(\rho < a) = -\frac{1}{4}\mu_o J\left(\rho^2 - a^2\right)\hat{z} = +\frac{1}{4}\mu_o J\left(a^2 - \rho^2\right)\hat{z}$$

$$\vec{A}(\rho \geq a) = -\frac{1}{2}\mu_o J a^2 \ln(\rho/-a)\hat{z} = +\frac{1}{2}\mu_o J a^2 \ln(\rho/a)\hat{z}$$



Note that: $\vec{A}(\rho \geq a) = \dfrac{1}{2}\mu_o J \ln(\rho/a)\hat{z}$ has a {logarithmic} divergence as $\rho \to \infty$, whereas:

$$\vec{B}(\rho \to \infty) = \nabla \times \vec{A}(\rho \to \infty) = \frac{1}{2}\mu_o J a^2 \left(\frac{1}{\rho}\right)\hat{\varphi} \to 0$$

This is merely a consequence associated with the {calculationally-simplifying} choice that we made at the beginning of this problem, that of an *infinitely* long wire – which is ***unphysical***. It takes *infinite EM* energy to power an *infinitely* long wire… For a *finite* length wire carrying a steady current *I*, the magnetic vector potential is mathematically well-behaved {but has a correspondingly more complicated mathematical expression}.

It is easy to show that both of the solutions for the magnetic vector potential $\vec{A}(\rho \lesseqgtr a)$ satisfy the Coulomb gauge condition: $\vec{\nabla} \bullet \vec{A}(\vec{r}) = 0$, by noting that since $\vec{A}(\rho \lesseqgtr a) = A_z(\rho \lesseqgtr a)\hat{z}$ are functions ***only*** of ρ, then in cylindrical coordinates: $\vec{\nabla} \bullet \vec{A}(\rho \lesseqgtr a) = \partial A_z(\rho \lesseqgtr a)/\partial z = 0$.

Let us now investigate the ramifications of the non-zero curl result associated with Faraday's law at $\rho = a$ for the $\vec{A}$-field at that radial location:

$$\boxed{\vec{\nabla} \times \vec{E} = \frac{\left|\vec{J}_{free}\right|}{\sigma_C}\delta(\rho - a)\hat{\varphi} = -\frac{\partial \vec{B}}{\partial t}}$$

Since $\vec{B} = \vec{\nabla} \times \vec{A} = -\dfrac{\partial A_z}{\partial \rho}\hat{\varphi}$ {here, in this problem}, then:

$$\frac{\partial \vec{B}}{\partial t} = \frac{\partial\left(\vec{\nabla} \times \vec{A}\right)}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{\partial A_z}{\partial \rho}\right)\hat{\varphi} = -\frac{\left|\vec{J}_{free}\right|}{\sigma_C}\delta(\rho - a)\hat{\varphi} \quad \text{or:} \quad \frac{\partial}{\partial t}\left(\frac{\partial A_z}{\partial \rho}\right) = \frac{\left|\vec{J}_{free}\right|}{\sigma_C}\delta(\rho - a)$$

Then: $\dfrac{\partial A_z}{\partial t} = \dfrac{\left|\vec{J}_{free}\right|}{\sigma_C}\underbrace{\int \delta(\rho - a)\,d\rho}_{\equiv \Theta(\rho - a)} = \dfrac{\left|\vec{J}_{free}\right|}{\sigma_C}\Theta(\rho - a)$ or: $\boxed{\dfrac{\partial \vec{A}}{\partial t} = \dfrac{\left|\vec{J}_{free}\right|}{\sigma_C}\Theta(\rho - a)\hat{z}}$.

Now, recall that the {correct!} electric field for this problem is:

$$\boxed{\vec{E}(\rho)=\frac{\left|\vec{J}_{free}\right|}{\sigma_C}\left[1-\Theta(\rho-a)\right]\hat{z}}$$

However, in general, the electric field is defined in terms of the scalar and vector potentials as:

$$\boxed{\vec{E}(\vec{r},t)=-\vec{\nabla}V(\vec{r},t)-\frac{\partial\vec{A}(\vec{r},t)}{\partial t}}$$

Since {here, in this problem}: $\boxed{\frac{\partial\vec{A}}{\partial t}=\frac{\left|\vec{J}_{free}\right|}{\sigma_C}\Theta(\rho-a)\hat{z}}$, we see that: $\boxed{-\vec{\nabla}V=\frac{\left|\vec{J}_{free}\right|}{\sigma_C}\hat{z}}$

and hence {in cylindrical coordinates} that: $\boxed{V(z)=-\frac{\left|\vec{J}_{free}\right|}{\sigma_C}z}$, then:

$$\boxed{-\vec{\nabla}V=+\frac{\partial}{\partial z}\left(\frac{\left|\vec{J}_{free}\right|}{\sigma_C}z\right)\hat{z}=\frac{\left|\vec{J}_{free}\right|}{\sigma_C}\frac{\partial}{\partial z}(z)\hat{z}=\frac{\left|\vec{J}_{free}\right|}{\sigma_C}\hat{z}}.$$

Note that the {static} scalar field $\boxed{V(z)=-\frac{\left|\vec{J}_{free}\right|}{\sigma_C}z}$ pervades ***all*** space, as does $\vec{A}(\rho \lessgtr a)\parallel +\hat{z}$.

Explicitly, due to the behavior of the Heaviside step function $\Theta(\rho-a)$ we see that the electric field contribution $\boxed{\frac{\partial\vec{A}}{\partial t}=\frac{\left|\vec{J}_{free}\right|}{\sigma_C}\Theta(\rho-a)\hat{z}}$ is: $\boxed{\frac{\partial\vec{A}}{\partial t}=\begin{cases}0 & \text{for } \rho<a\\ \frac{\left|\vec{J}_{free}\right|}{\sigma_C}\hat{z} & \text{for } \rho\geq a\end{cases}}$.

Explicitly writing out the electric field in this manner, we see that:

$$\boxed{\vec{E}(\rho \lessgtr a)=-\vec{\nabla}V(\rho \lessgtr a)-\frac{\partial\vec{A}(\rho \lessgtr a)}{\partial t}=\begin{cases}\frac{\left|\vec{J}_{free}\right|}{\sigma_C}\hat{z}+0=\frac{\left|\vec{J}_{free}\right|}{\sigma_C}\hat{z} & \text{for } \rho<a\\ \frac{\left|\vec{J}_{free}\right|}{\sigma_C}\hat{z}-\frac{\left|\vec{J}_{free}\right|}{\sigma_C}\hat{z}=0 & \text{for } \rho\geq a\end{cases}}$$

Thus, for $\rho\geq a$ we see that the $-\partial\vec{A}(\rho\geq a)/\partial t$ contribution to the $\vec{E}$-field outside the wire {which arises from the non-zero $\vec{\nabla}\times\vec{E}$ of Faraday's law at $\rho=a$ } ***exactly*** cancels the $-\vec{\nabla}V(\rho\geq a)$ contribution to the $\vec{E}$-field outside the wire, ***everywhere*** in space outside the wire, despite the fact that $\vec{A}(\rho\geq a)$ varies logarithmically outside the wire!!!!

The long, current-carrying wire can thus also be equivalently viewed as an ***electric flux tube***:

$$\Phi_E = \int_S \vec{E} \bullet d\vec{a} = \left(\left| \vec{J}_{free} \right| / \sigma_C \right) \int_S \left[1 - \Theta(\rho - a) \right] \hat{z} \bullet d\vec{a} = I/\sigma_C$$

The electric field $\vec{E}$ is ***confined*** within the tube (= the long, current carrying wire) by the $-\partial \vec{A}(\rho \geq a)/\partial t$ contribution arising from the Faraday's law effect on the $\rho = a$ boundary of the flux tube, due to the {matter geometry-induced} discontinuity in the electric field at $\rho = a$!

The $\vec{\nabla} \times \vec{E} = \left(\left| \vec{J}_{free} \right| / \sigma_C \right) \delta(\rho - a) \hat{\varphi} = -\partial \vec{B} / \partial t$ effect at $\rho = a$ also predicts a ***non-zero*** "induced" *EMF* in a loop/coil of wire: $\varepsilon = -\partial \Phi_m / \partial t$. The magnetic flux through a loop of wire is: $\Phi_m = \oint_C \vec{A} \bullet d\ell = \int_S \vec{B} \bullet d\vec{a} \simeq B \cdot A_\perp^{loop}$ where $A_\perp^{loop}$ is the cross-sectional area of a loop of wire {whose plane is perpendicular to the magnetic field at that point}. Note further that the width, *w* of the coil only needs to be large enough for the coil to accept the $\partial \vec{B} / \partial t$ contribution from the δ-function at $\rho = a$. Then, here in ***this*** problem, since the magnetic field at the surface of the wire is oriented in the $\hat{\varphi}$-direction, and:

$$\frac{\partial \vec{B}}{\partial t} = -\frac{\left| \vec{J}_{free} \right|}{\sigma_C} \delta(\rho - a) \hat{\varphi}$$

, then we see that:

$$\varepsilon = -\frac{\partial \Phi_m}{\partial t} = -\frac{\partial \vec{B} \cdot A_\perp^{loop}}{\partial t} = \frac{\left| \vec{J}_{free} \right| \cdot A_\perp^{loop}}{\sigma_C} \delta(\rho - a)$$

For a ***real*** wire, *e.g.* made of copper, how large will this *EMF* be – is it something *e.g.* that we could actually measure/observe in the laboratory with garden-variety/every-day lab equipment???

A number 8 *AWG* (*A*merican *W*ire *G*auge) copper wire has a diameter *D* = 0.1285" = 0.00162 *m* (~ 1/8" = 0.125") and can easily carry *I* = 10 Amps of current through it.

The current density in an 8 AWG copper wire carrying a steady current of *I* = 10 Amps is:

$$J_{8AWG} = \frac{I}{\pi a^2} = \frac{4 \cdot I}{\pi D^2} = \frac{4 \cdot 10}{\pi (0.001632)^2} \simeq 4.8 \times 10^6 \ \left(Amps/m^2 \right)$$

The electrical conductivity of {pure} copper is: $\sigma_C^{Cu} = 5.96 \times 10^7 \ \left(Siemens/m \right)$.

If our "long" 1/8" diameter copper wire is $L = 1\,m$ long, and if we can *e.g.* make a loop of ultra-fine gauge wire that penetrates the surface of the wire and runs parallel to the surface, then if we approximate the radial delta function $\delta(\rho - a)$ at $\rho = a$ as ~ a narrow Gaussian of width $w \sim 10\ \text{Å} = 1\,nm = 10^{-9}\,m$ (*i.e.* ~ the order of the inter-atomic distance/spacing of atoms in the copper lattice {$3.61\ \text{Å}$}), noting also that the delta function $\delta(\rho - a)$ has physical SI units of inverse length (*i.e.* m^{-1}) and, neglecting the sign of the *EMF*, an estimate of the magnitude of the "induced" *EMF* is:

$$\varepsilon_{Cu} = \frac{J_{8AWG} \cdot A_\perp^{loop}}{\sigma_C^{Cu}} \delta(\rho - a) \simeq \frac{J_{8AWG} \cdot L \cdot \cancel{w}}{\sigma_C^{Cu}} \cdot \cancel{\frac{1}{w}} = \left(\frac{J_{8AWG}}{\sigma_C^{Cu}} \right) \cdot L \simeq \left(\frac{4.8 \times 10^6 \left(Amps/m^2 \right)}{6 \times 10^7 \left(Siemens/m \right)} \right) \cdot 1\,m \simeq 80\,mV!!!$$

This size of an *EMF* is ***easily*** measureable with a modern *DVM*…

Using Ohm's Law: $V = I \cdot R$, note that the voltage drop V_{drop} across a $L = 1\,m$ length of 8 *AWG* copper wire with $I = 10\ Amps$ of current flowing thru it is:

$$V_{drop}^{1m} = I \cdot R_{1m} = I \cdot \frac{\rho_C^{Cu} L}{A_\perp^{wire}} = \left(J_{8AWG} \cdot \cancel{A_\perp^{wire}}\right) \cdot \frac{L}{\sigma_C^{Cu}\,\cancel{A_\perp^{wire}}} = \frac{J_{8AWG}}{\sigma_C^{Cu}} \cdot L = \varepsilon_{Cu}\ !!!$$

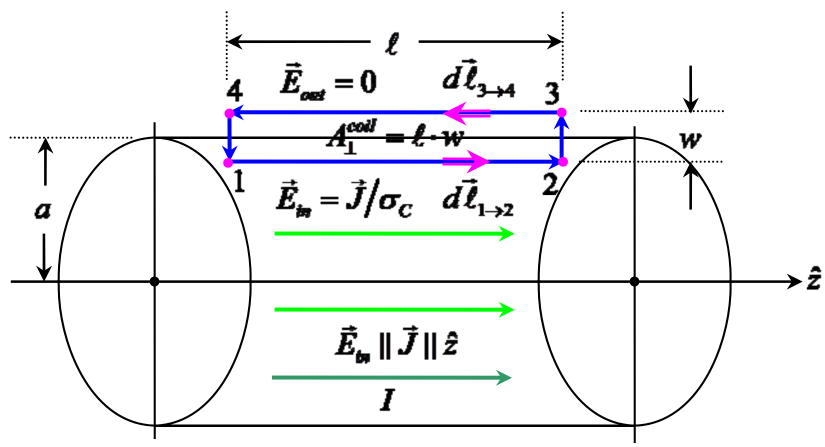
In other words, the "induced" *EMF*, $\varepsilon = \left(\left|\vec{J}_{free}\right| \cdot A_\perp^{loop} \big/ \sigma_C\right)\delta(\rho - a)$ in the one-turn loop coil of length L {oriented as described above} is ***precisely*** equal to the voltage drop $V_{drop} = \left(\left|\vec{J}_{free}\right| \big/ \sigma_C\right) \cdot L$ along a length L of a portion of the long wire with steady current I flowing through it, even though the 1-turn loop coil is completely electrically isolated from the current-carrying wire!!!

This can be easily understood... Using Stoke's theorem, the surface integral of $\vec{\nabla} \times \vec{E}$ can be converted to a line integral of $\vec{E}$ along a closed contour C bounding the surface of integration S; likewise, a surface integral of $\partial\vec{B}/\partial t = \vec{\nabla} \times \partial\vec{A}/\partial t$ can be converted to a line integral of $\partial\vec{A}/\partial t$ along a closed contour C bounding the surface of integration S:

$$\varepsilon = \int_S \left(\vec{\nabla} \times \vec{E}\right) \cdot d\vec{a} = \oint_C \vec{E} \cdot d\vec{\ell} = -\frac{\partial \Phi_m}{\partial t} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} = -\int_S \left(\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t}\right) \cdot d\vec{a} = -\oint_C \frac{\partial \vec{A}}{\partial t} \cdot d\vec{\ell}$$

$$n.b.: \quad \oint_C -\vec{\nabla} V \cdot d\vec{\ell} \equiv 0$$

Then for any closed contour C associated with the surface S that encloses the Faraday law $\vec{\nabla} \times \vec{E}$ δ-function singularity at $\rho = a$, *e.g.* as shown in the figure below:



the "induced" *EMF* ε can thus also be calculated from the line integral $\int_C \vec{E} \cdot d\vec{\ell}$ taken around the closed contour C. From the above discussion(s), the electric field inside (outside) the long current-carrying wire is $\vec{E}_{in} = \vec{J}/\sigma_C$ $\left(\vec{E}_{out} = 0\right)$, respectively {*n.b.* ⇒ tangential $\vec{E}$ is ***discontinuous*** across the boundary of a {volume} current-carrying conductor!}. Then:

$$\varepsilon = \int_C \vec{E} \bullet d\vec{\ell} = \int_1^2 \underbrace{\vec{E}_{in}^{1\to2} \bullet d\vec{\ell}_{1\to2}}_{=J\ell/\sigma_C = E\ell = \Delta V_{1\to2}} + \int_2^3 \underbrace{\vec{E}^{2\to3} \bullet d\vec{\ell}_{2\to3}}_{\equiv 0} + \int_3^4 \underbrace{\vec{E}_{outside}^{3\to4} \bullet d\vec{\ell}_{3\to4}}_{\equiv 0} + \int_4^1 \underbrace{\vec{E}^{4\to1} \bullet d\vec{\ell}_{4\to1}}_{\equiv 0} = E \cdot \ell = \Delta V_{1\to2}$$

The presence of a non-zero Faraday's law $\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t = \left(\left|\vec{J}_{free}\right|/\sigma_C\right)\delta(\rho - a)\hat{\varphi}$ term at the surface of the long current-carrying wire implies that the "induced" *EMF* $\varepsilon = \left(\left|\vec{J}_{free}\right| \cdot A_\perp^{loop}/\sigma_C\right)\delta(\rho - a)$ can also be viewed as arising from the ***mutual*** inductance M $(Henrys)$ associated with the long wire and the coil {oriented as described above}, and a non-zero $\partial I/\partial t$:

$$\boxed{\varepsilon = -\frac{\partial \Phi_m}{\partial t} = -\frac{\partial \vec{B} \cdot A_\perp^{loop}}{\partial t} = -M\frac{\partial I}{\partial t} = \frac{\left|\vec{J}_{free}\right| \cdot A_\perp^{loop}}{\sigma_C}\delta(\rho - a)}$$

We can obtain a relation between $\partial \vec{B}/\partial t$ and $\partial I/\partial t$ using the integral form of Ampere's law: $\oint_C \vec{B} \bullet d\vec{\ell} = \mu_o I_{encl}$. Taking the partial derivative of both sides of this equation with respect to time:

$$\boxed{\frac{\partial}{\partial t}\left(\oint_C \vec{B} \bullet d\vec{\ell}\right) = \oint_C \frac{\partial \vec{B}}{\partial t} \bullet d\vec{\ell} = \mu_o \frac{\partial I_{encl}}{\partial t}}$$

The contour of integration C needs to be taken just outside the surface of the long wire, along the $\hat{\varphi}$-direction, since $\vec{B} \parallel \hat{\varphi}$ at $\rho = a$, *i.e.* $d\vec{\ell} \parallel \hat{\varphi}$ in order to include the non-zero Faraday's law effect at the surface of the long wire.

Then: $\boxed{\frac{\partial B}{\partial t} = \left(\frac{\mu_o}{2\pi a}\right)\frac{\partial I}{\partial t} = -\frac{\left|\vec{J}_{free}\right|}{\sigma_C}\delta(\rho - a)}$ or: $\boxed{\frac{\partial I}{\partial t} = \left(\frac{2\pi a}{\mu_o}\right)\frac{\partial B}{\partial t} = -\left(\frac{2\pi a}{\mu_o}\right)\frac{\left|\vec{J}_{free}\right|}{\sigma_C}\delta(\rho - a)}$

Then: $\boxed{\varepsilon = -\frac{\partial \Phi_m}{\partial t} = -\frac{\partial \vec{B} \cdot A_\perp^{loop}}{\partial t} = -M\frac{\partial I}{\partial t} = \frac{\left|\vec{J}_{free}\right| \cdot A_\perp^{loop}}{\sigma_C}\delta(\rho - a)}$

Solving for the mutual inductance, we obtain a rather simple result: $\boxed{M = \mu_o\left(\frac{A_\perp^{loop}}{2\pi a}\right) (Henrys)}$

Note that the mutual inductance, M involves the magnetic permeability of free space $\mu_o = 4\pi \times 10^{-7}$ $(Henrys/m)$ {*n.b.* which has SI units of inductance/length} and geometrical aspects {only!} of the wire (its radius, *a*) and the cross-sectional area of the loop, $A_\perp^{loop}$.

What is astonishing {and unique} *r.e.* the "induced" Faraday's law *EMF* $\varepsilon = \left(\left|\vec{J}_{free}\right| \cdot A_\perp^{loop}/\sigma_C\right)\delta(\rho - a)$ associated with a long, steady current-carrying wire is that "normal" induced *EMF*'s ***only*** occur in electrical circuits that operate at ***non-zero*** frequencies, *i.e.* $f > 0\ Hz$. However, ***here***, in ***this*** problem, we have an example of a ***DC*** induced *EMF* – *i.e.* an induced *EMF* that occurs at $f \equiv 0\ Hz$, arising from the non-zero Faraday's law effect $\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t = \left(\left|\vec{J}_{free}\right|/\sigma_C\right)\delta(\rho - a)\hat{\varphi}$ due to the longitudinal $\vec{E}$-field discontinuity at the surface $(\rho = a)$ of a long, ***steady*** current-carrying wire!!!

Instead of using a long wire to carry a steady current I to observe this effect, one might instead consider using *e.g.* a long, hollow steady current-carrying ***pipe*** of inner (outer) radius a, (b) respectively. Following the above methodology, one can easily show that for such a long, hollow current-carrying pipe, ***two*** ***opposing*** non-zero Faraday law $\vec{\nabla}\times\vec{E}$ radial δ-function contributions occur – one located at the $\rho=a$ inner surface, and the other located at the $\rho=b$ outer surface of the long hollow current-carrying pipe:

$$\boxed{\vec{\nabla}\times\vec{E}=-\partial\vec{B}/\partial t=-\left(\left|\vec{J}_{free}\right|/\sigma_C\right)\left[\delta(\rho-a)-\delta(\rho-b)\right]\hat{\varphi}}$$

The $\vec{E}$-field is:

$$\boxed{\vec{E}=-\vec{\nabla}V-\partial\vec{A}/\partial t=\left(\left|\vec{J}_{free}\right|/\sigma_C\right)\left[1+\bar{\Theta}(\rho-a)-\Theta(\rho-b)\right]\hat{z}}$$

where: $\bar{\Theta}(\rho-a)\equiv\begin{cases}1 \text{ for } \rho<a\\ 0 \text{ for } \rho\geq a\end{cases}$ is the ***complement*** of the Heaviside step function, such that:

$d\Theta(x)/dx=-\delta(x)$ and: $\bar{\Theta}(x)=-\int_{-\infty}^{x}\delta(t)\,dt$ where: $\delta(x)$ is the Dirac delta-function.

Hence, a 1-turn coil {oriented as described above} enclosing the $\rho=a$ inner surface **.and.** the $\rho=b$ outer surface of a current-carrying hollow pipe will have a "null" induced *EMF*, *i.e.* $\varepsilon=0$ due to the wire loop ***simultaneously*** enclosing the ***two*** ***opposing*** non-zero Faraday law $\vec{\nabla}\times\vec{E}$ radial δ-function contributions, one located at $\rho=a$, the other at $\rho=b$:

$$\boxed{\varepsilon=-\frac{\partial\Phi_m}{\partial t}=-\frac{\partial\vec{B}\cdot A_{\perp}^{loop}}{\partial t}=\frac{\left|\vec{J}_{free}\right|\cdot A_{\perp}^{loop}}{\sigma_C}\left(\delta(\rho-a)-\delta(\rho-b)\right)=0}$$

In general, ***any*** penetration/hole made into the metal conductor of a long, steady current-carrying wire will result in a non-zero Faraday law $\vec{\nabla}\times\vec{E}$ δ-function on the boundary/surface of that penetration/hole! Since the current density $\vec{J}_{free}=0$ in the penetration/hole, $\vec{E}=0$ there and thus a discontinuity in $\vec{E}$ exists on the boundary of the penetration/hole, hence a non-zero Faraday law $\vec{\nabla}\times\vec{E}$ δ-function exists on the boundary of the penetration/hole!

This fact {unfortunately} has ***important*** ramifications for the experimental detection / observation of the predicted non-zero *DC* induced *EMF* in a coil {oriented as described above}, Embedding a portion of a physical wire loop inside the long, steady current-carrying wire requires making a penetration/hole {no matter how small} into the wire, which ***will*** result in a non-zero Faraday law $\vec{\nabla}\times\vec{E}$ δ-function on the boundary/surface of that penetration/hole in the wire! Thus, the wire loop will in fact enclose ***not*** ***only*** the Faraday law $\vec{\nabla}\times\vec{E}$ radial singularity at $\rho=a$ on the surface of the wire, but will ***also*** enclose ***another***, ***opposing*** singularity located on the boundary/surface of the penetration/hole made into the long wire {which was made to embed a portion of the wire loop in a long, steady current-carrying wire in the first place}, thus experimentally, a "null" induced *EMF*, *i.e.* $\varepsilon=0$ is expected/anticipated, because of this…

Hence, in the ***real*** world of experimental physics, it appears that embedding a portion of a ***real*** wire loop in a long, steady current-carrying wire in an attempt to observe this effect is doomed to failure…

