$S = K[x_0, \ldots, x_n]$, $d \geq 1$ $K[x_0 \ldots x_n]_d = S_d$: K-vector space

$\mathbb{P}(S_d)$ "hypersurfaces of deg d in $\mathbb{P}^n_K$"

positive divisors of deg d = linear combinations of irred. hyp. with positive coeff.

$F = F_1^{r_1} \cdots F_s^{r_s}$ $r_1 V_p(F_1) + \cdots + r_s V_p(F_s)$

$\mathbb{P}(S_d) \supseteq X = \{[F] \mid F \text{ reducible}\}$

divisors not of the form $V_p(F)$, F irred.

Prop: X is a projective variety.

Pf: $1 \leq k < d$ $X_k = \{[F] \mid F = GH, \deg G = k\}$

$X = \bigcup_{1 \leq k < d} X_k$ $X_k = X_{d-k}$

Claim $\forall k$ X_k is closed

$\mathbb{P}(S_k) \times \mathbb{P}(S_{d-k}) \longrightarrow \mathbb{P}(S_d)$

Segre product

$([G], [H]) \longrightarrow [GH]$ regular map

because bihomogeneous in the coeff. of G and H

The domain is a proj. var. $\Longrightarrow$ the image is closed: X_k is closed in $\mathbb{P}(S_d)$.

irreducible

$d=2$ $\quad X \subseteq \mathbb{P}(S_2)$ $\quad$ quadrics reducible

Equations of X

Q quadric $\rightsquigarrow$ A matrix of Q
$(n+1)\times(n+1)$

Q reducible $\iff$ rk A is

$n=2$ $\quad \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & - & - \\ a_{20} & - & \end{pmatrix}$ $\quad$ rk 3 irred.

rk 2 $\quad$ $\times$

rk 1 $\quad$ $=$

Q red. $\iff$ rk $A \leq 3$

equation of X is $|A|=0$

$n>2$ $\quad$ rk $n+1-r$ $\implies$ Q is proj-equiv. to $x_0^2+x_1^2+\dots+x_{n-r}^2$: cone with vertex a quadric of max rk in $\mathbb{P}^{n-r}$

Q reducible $\implies$ Q is a cone over 2 pts in $\mathbb{P}^1$
rk ≤ 2 $\quad$ not nec. distinct

Equations of X are the 3×3 minors of A.

Chapter 16

The tangent space and the notion of smoothness

We will always assume K algebraically closed. In this chapter we follow the approach of Šafarevič [S]. We define the tangent space $T_{X,P}$ at a point P of an *affine* variety $X \subset \mathbb{A}^n$ as the union of the lines passing through P and "touching" X at P. It results to be an affine subspace of $\mathbb{A}^n$. Then we will find a "local" characterization of $T_{X,P}$, this time interpreted as a vector space, the direction of $T_{X,P}$, only depending on the local ring $\mathcal{O}_{X,P}$: this will allow to define the tangent space at a point of any quasi–projective variety.

16.1 Tangent space to an affine variety

Assume first that $X \subset \mathbb{A}^n$ is closed and $P = O = (0, \dots, 0)$. Let L be a line through P: if $A(a_1, \dots, a_n)$ is another point of L, then a general point of L has coordinates $(ta_1, \dots, ta_n)$, $t \in K$. If $I(X) = (F_1, \dots, F_m)$, then the intersection $X \cap L$ is determined by the following system of equations in the indeterminate t:

$$F_1(ta_1, \dots, ta_n) = \cdots = F_m(ta_1, \dots, ta_n) = 0.$$

$\iff G(t) = 0$

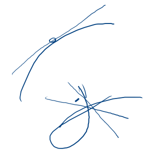
$G = 0$ $L \subseteq X$

$G \neq 0$

The solutions of this system of equations are the roots of the greatest common divisor $G(t)$ of the polynomials $F_1(ta_1, \dots, ta_n), \dots, F_m(ta_1, \dots, ta_n)$ in $K[t]$, i.e. the generator of the ideal they generate. We may factorize $G(t)$ as $G(t) = ct^e(t-\alpha_1)^{e_1} \dots (t-\alpha_s)^{e_s}$, where $c \in K$, $\alpha_1, \dots, \alpha_s \neq 0$, $e, e_1, \dots, e_s$ are non-negative, and $\boxed{e > 0}$ if and only if $P \in X \cap L$. The number e is by definition the **intersection multiplicity at P of X and L.** If $G(t)$ is identically zero, then $L \subset X$ and the intersection multiplicity is, by definition, $+\infty$.

Note that the polynomial $G(t)$ doesn't depend on the choice of the generators $F_1, \dots, F_m$ of $I(X)$, but only on the ideal $I(X)$ and on L.

Definition 16.1.1. The line **L is tangent to the variety X at P** if the intersection multiplicity of L and X at P is at least 2 (in particular, if $L \subset X$). The **tangent space to X at P** is the union of the lines that are tangent to X at P; it is denoted $T_{P,X}$.



We will see now that $T_{P,X}$ is an affine subspace of $\mathbb{A}^n$. Assume that $P \in X$: then the polynomials F_i may be written in the form $F_i = L_i + G_i$, where L_i is a homogeneous linear polynomial (possibly zero) and G_i contains only ~~terms~~ of degree ≥ 2. Then

$$F_i(ta_1, \dots, ta_n) = tL_i(a_1, \dots, a_n) + G_i(ta_1, \dots, ta_n), = tL_i(a_1 \dots a_n) + t^2 G'(t)$$

where the last term is divisible by t^2. Let L be the line $\overline{OA}$, with $A = (a_1, \dots, a_n)$. We note that the intersection multiplicity of X and L at P is the maximal power of t dividing the greatest common divisor, so L is tangent to X at P if and only if $L_i(a_1, \dots, a_n) = 0$ for all $i = 1, \dots, m$.

Therefore the point A belongs to $T_{P,X}$ if and only if

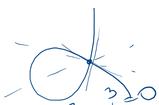
$$L_1(a_1, \dots, a_n) = \dots = L_m(a_1, \dots, a_n) = 0.$$

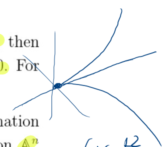
$\langle L_1, .., L_m \rangle$ doesn't depend on the choice of generators of $I(X)$

This shows that $T_{P,X}$ is a linear subspace of $\mathbb{A}^n$, whose equations are the linear components of the equations defining X. affine sub space passing through O

Example 16.1.2. (i) $T_{O,\mathbb{A}^n} = \mathbb{A}^n$, because $I(\mathbb{A}^n) = (0)$.

(ii) If X is a hypersurface, with $I(X) = (F)$, we write as above $F = L + G$; then $T_{O,X} = V(L)$: so $T_{O,X}$ is either a hyperplane if $L \neq 0$, or the whole space $\mathbb{A}^n$ if $L = 0$. For instance, if X is the affine plane cuspidal cubic $V(x^3 - y^2) \subset \mathbb{A}^2$, $T_{O,X} = \mathbb{A}^2$.





$$x^2 - y^2 - y^3 = 0$$

$$\begin{cases} x = t^2 \\ y = t^3 \end{cases} \quad \begin{cases} x' = 2t \\ y' = 3t^2 \end{cases}$$

Assume now that $P \in X$ has coordinates $(y_1, \dots, y_n)$. With a ~~linear~~ affine transformation we may translate P to the origin $(0, \dots, 0)$, taking as new coordinates functions on $\mathbb{A}^n$ $x_1 - y_1, \dots, x_n - y_n$. This corresponds to considering the K-isomorphism $K[x_1, \dots, x_n] \longrightarrow K[x_1 - y_1, \dots, x_n - y_n]$, which takes a polynomial $F(x_1, \dots, x_n)$ to its Taylor expansion

$$G(x_1 - y_1, \dots, x_n - y_n) = F(y_1, \dots, y_n) + d_P F + d_P^{(2)} F + \dots,$$

where $d_P^{(i)} F$ denotes the i^{th} differential of F at P: it is a homogeneous polynomial of degree i in the variables $x_1 - y_1, \dots, x_n - y_n$. In particular the linear term is

$$d_P F = \frac{\partial F}{\partial x_1}(P)(x_1 - y_1) + \dots + \frac{\partial F}{\partial x_n}(P)(x_n - y_n).$$

$y = tx$ $\qquad x = ty$

$$x^2 - t^2x^2 - t^3x^3 = 0$$

$$x^2(\underbrace{1 - t^2 - t^3 x}) = 0$$

$$\begin{cases} x = \dfrac{1-t^2}{t^3} \\ y = \dfrac{1-t^2}{t^2} \end{cases}$$

$$t^2y^2 - y^2 - y^3 = 0$$

$$y^2(t^2 - 1 - y) = 0$$

$$\begin{cases} y = t^2 - 1 \\ x = t(t^2 - 1) \end{cases}$$

$(0,0)$ is obtained from $t = \pm 1$

$T_{P,X} = P + W$

We get that, if $I(X) = (F_1, \ldots, F_m)$, then $T_{P,X}$ is the affine subspace of $\mathbb{A}^n$ defined by the equations

$$d_P F_1 = \cdots = d_P F_m = 0.$$

The affine space $\mathbb{A}^n$, which may identified with K^n, can be given a natural structure of K-vector space with origin P, so in a natural way $T_{P,X}$ is a vector subspace (with origin P). The functions $x_1 - y_1, \ldots, x_n - y_n$ form a basis of the dual space $(K^n)^*$ and their restrictions generate $T^*_{P,X}$. Note moreover that $\dim T^*_{P,X} = k$ if and only if $n - k$ is the maximal number of polynomials linearly independent among $d_P F_1, \ldots, d_P F_m$. If $d_P F_1, \ldots, d_P F_{n-k}$ are these polynomials, then they form a basis of the orthogonal $T^{\perp}_{P,X}$ of the vector space $T_{P,X}$ in $(K^n)^*$, because they vanish on $T_{P,X}$.

$= \dim T_{P,X}$

16.2 Zariski tangent space

$K[X]$

$\|$

Let us define now the *differential of a regular function.* Let $f \in \mathcal{O}(X)$ be a regular function on X. We want to define the differential of f at P. Since X is closed in $\mathbb{A}^n$, f is induced by a polynomial $F \in K[x_1, \ldots, x_n]$ as well as by all polynomials of the form $F + G$ with $G \in I(X)$. Fix $P \in X$: then $d_P(F+G) = d_P F + d_P G$ so the differentials of two polynomials inducing the same function f on X differ by the term $d_P G$ with $G \in I(X)$. By definition, $d_P G$ is zero along $T_{P,X}$, so we may define $d_p f$ as a regular function on $T_{P,X}$, the differential of f at P: it is the function on $T_{P,X}$ induced by $d_P F$. Since $d_P F$ is a linear combination of $x_1 - y_1, \ldots, x_n - y_n$, $d_p f$ can also be seen as an element of $T^*_{P,X}$.

There is a natural map $d_p : \mathcal{O}(X) \to T^*_{P,X}$, which sends f to $d_p f$. Because of the rules of derivation, it is clear that $d_P(f+g) = d_P f + d_P g$ and $d_P(fg) = f(P) d_P g + g(P) d_P f$. In particular, if $c \in K$, $d_p(cf) = c d_P f$. So d_p is a linear map of K-vector spaces. We denote again by d_P the restriction of d_P to $I_X(P)$, the maximal ideal of the regular functions on X which are zero at P. Since clearly $f = f(P) + (f - f(P))$ then $d_P f = d_P(f - f(P))$, so this restriction doesn't modify the image of the map.

vanishes at P : $f - f(P) \in I_X(P)$

Proposition 16.2.1. *The map $d_P : I_X(P) \to T^*_{P,X}$ is surjective and its kernel is $I_X(P)^2$. Therefore $T^*_{P,X} \simeq I_X(P)/I_X(P)^2$ as K-vector spaces.*

Proof. Let $\varphi \in T^*_{P,X}$ be a linear form on $T_{P,X}$. φ is the restriction of a linear form on K^n: $\lambda_1(x_1 - y_1) + \ldots + \lambda_n(x_n - y_n)$, with $\lambda_1, \ldots, \lambda_n \in K$. Let G be the polynomial of degree 1 $\lambda_1(x_1 - y_1) + \ldots + \lambda_n(x_n - y_n)$: the function g induced by G on X is zero at P and coincides with its own differential, so d_p is surjective.

$\varphi = d_P g$

Let now $g \in I_X(P)$ such that $d_p g = 0$, g induced by a polynomial G. Note that $d_P G$ may be interpreted as a linear form on K^n which vanishes on $T_{P,X}$, hence as an element of $T_{P,X}^{\perp}$. So $d_P G = c_1 d_p F_1 + \ldots + c_m d_p F_m$ ($c_1, \ldots, c_m$ suitable elements of K). Let us consider the polynomial $G - c_1 F_1 - \ldots - c_m F_m$: since its differential at P is zero, it doesn't have any term of degree 0 or 1 in $x_1 - y_1, \ldots, x_n - y_n$, so it belongs to $I(P)^2$. Since $G - c_1 F_1 - \ldots - c_m F_m$ defines the function g on X, we conclude that $g \in I_X(P)^2$. □

In $G - c_1F_1 - \cdots - c_mF_m$ expressed as polyn. in $x_1 - y_1, \ldots, x_n - y_n$, the first possible comp. is $\frac{\partial^2 f(\ldots)}{\partial x_1^2}(x_1 - y_1)^2 + \cdots (x_1-y_1)(x_2-y_2)$ — — —

Corollary 16.2.2. *The tangent space $T_{P,X}$ is isomorphic to $(I_X(P)/I_X(P)^2)^*$ as an abstract K-vector space.*

Corollary 16.2.3. *Let $\varphi : X \to Y$ be an isomorphism of affine varieties and $P \in X$, $Q = \varphi(P)$. Then the tangent spaces $T_{P,X}$ and $T_{Q,Y}$ are isomorphic.*

$I_X(P) = (x_1 - y_1, \ldots, x_n - y_n)$

$I_X(P)^2 = ((x_1-y_1)^2, (x_1-y_1)(x_2-y_2), \ldots)$

Proof. φ induces the comorphism $\varphi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, which results to be an isomorphism such that $\varphi^* I_Y(Q) = I_X(P)$ and $\varphi^* I_Y(Q)^2 = I_X(P)^2$. So there is an induced homomorphism

$$I_Y(Q)/I_Y(Q)^2 \to I_X(P)/I_X(P)^2,$$

which is an isomorphism of K-vector spaces. By dualizing we get the claim. □

$\frac{I_X(P)}{I_X(P)^2}$

The above map from $T_{P,X}$ to $T_{Q,Y}$ is called the *differential of φ at P* and is denoted by $d_P\varphi$.

$\cap$

$\frac{\mathcal{O}(X)}{I_X(P)^2}$

Now we would like to find a "more local" characterization of $T_{P,X}$. To this end we consider the local ring of P in X: $\mathcal{O}_{P,X}$. We recall the natural map $\mathcal{O}(X) \to \mathcal{O}_{P,X} = \mathcal{O}(X)_{I_X(P)}$, the last one being the localization. It is natural to extend the map $d_P : \mathcal{O}(X) \to T_{P,X}^*$ to $\mathcal{O}_{P,X}$ setting

$$d_P\left(\frac{f}{g}\right) = \frac{g(P)d_P f - f(P)d_P g}{g(P)^2}.$$

As in the proof of Proposition 16.2.1 one proves that the map $d_P : \mathcal{O}_{P,X} \to T_{P,X}^*$ induces an isomorphism $\mathcal{M}_{P,X}/\mathcal{M}_{P,X}^2 \to T_{P,X}^*$, where $\mathcal{M}_{P,X}$ is the maximal ideal of $\mathcal{O}_{P,X}$. So by duality we have: $T_{P,X} \simeq (\mathcal{M}_{P,X}/\mathcal{M}_{P,X}^2)^*$. This proves that the tangent space $T_{P,X}$ is a *local invariant* of P in X.

Definition 16.2.4. Let X be any quasi-projective variety, $P \in X$. The *Zariski tangent space* of X at P is the vector space $(\mathcal{M}_{P,X}/\mathcal{M}_{P,X}^2)^*$.

It is an abstract vector space, but if $X \subset \mathbb{A}^n$ is closed, taking the dual of the comorphism associated to the inclusion morphism $X \hookrightarrow \mathbb{A}^n$, we have an embedding of $T_{P,X}$ into $T_{P,\mathbb{A}^n} = \mathbb{A}^n$. If $X \subset \mathbb{P}^n$ and $P \in U_i = \mathbb{A}^n$, then $T_{P,X} \subset U_i$: its projective closure $\mathbb{T}_{P,X}$ is called the *embedded tangent space* to X at P.

projective subspace of $\mathbb{P}^n$

16.3 Smoothness

As we have seen the tangent space $T_{P,X}$ is invariant by isomorphism. In particular its dimension is invariant. If $X \subset \mathbb{A}^n$ is closed, $I(X) = (F_1, \ldots, F_m)$, then $\dim T_{P,X} = n - r$, where r is the dimension of the K-vector space generated by $\{d_P F_1, \ldots, d_p F_m\}$.

Since $d_P F_i = \frac{\partial F_i}{\partial x_1}(P)(x_1 - y_1) + \ldots + \frac{\partial F_i}{\partial x_n}(P)(x_n - y_n)$, r is the rank of the following $m \times n$ matrix, the *Jacobian matrix of X at P*:

$$J(P) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(P) & \ldots & \frac{\partial F_1}{\partial x_n}(P) \\ \ldots & \ldots & \ldots \\ \frac{\partial F_m}{\partial x_1}(P) & \ldots & \frac{\partial F_m}{\partial x_n}(P) \end{pmatrix}.$$

The *generic Jacobian matrix of X* is instead the following matrix with entries in $\mathcal{O}(X)$:

$$J = \begin{pmatrix} \frac{\partial \overset{f}{\cancel{F_1}}}{\partial x_1} & \ldots & \frac{\partial \overset{f}{\cancel{F_1}}}{\partial x_n} \\ \ldots & \ldots & \ldots \\ \frac{\partial \overset{f}{\cancel{F_m}}}{\partial x_1} & \ldots & \frac{\partial \overset{f}{\cancel{F_m}}}{\partial x_n} \end{pmatrix}.$$

The rank of J is ρ when all minors of order $\rho + 1$ are functions identically zero on X, while at least one minor of order ρ is different from zero at some point. Hence, for all $P \in X$ rk $J(P) \leq \rho$, and rk $J(P) < \rho$ if and only if all minors of order ρ of J vanish at P. It is then clear that there is a non-empty open subset of X where $\dim T_{P,X}$ is minimal, equal to $n - \rho$, and a proper (possibly empty) closed subset formed by the points P such that $\dim T_{P,X} > n - \rho$.

Definition 16.3.1. The points of an irreducible variety X for which $\dim T_{P,X} = n - \rho$ (the minimal) are called *smooth* or *non-singular* (or *simple) points* of X. The remaining points are called *singular* (or multiple). X is a *smooth* variety if all its points are smooth.

If X is quasi-projective, the same argument may be repeated for any affine open subset.

Example 16.3.2. Let $X \subset \mathbb{A}^n$ be the irreducible hypersurface $V(F)$, with F square-free generator of $I_h(X)$. Then $J = (\frac{\partial F}{\partial x_1} \ldots \frac{\partial F}{\partial x_n})$ is a row matrix. So rk $J = 0$ or 1. If rk $J = 0$, then $\frac{\partial F}{\partial x_i} = 0$ in $\mathcal{O}(X)$ for all i. So $\frac{\partial F}{\partial x_i} \in I(X) = (F)$. Since the degree of $\frac{\partial F}{\partial x_i}$ is $\leq \deg F - 1$, it follows that $\frac{\partial F}{\partial x_i} = 0$ in the polynomial ring. If the characteristic of K is zero this means that F is constant: a contradiction. If char $K = p$, then $F \in K[x_1^p, \ldots, x_n^p]$; since K is algebraically closed, then all coefficients of F are p-th powers, so $F = G^p$ for a suitable polynomial G; but again this is impossible because F is irreducible. So always rk $J = 1 = \rho$. Hence for P general in X, i.e. for P varying in a suitable non-empty open subset