$\varphi: X \to Y$ $\quad y \in Y \quad \varphi^{-1}(y)$ the fibre of φ over y

Fibres of φ are $\{\varphi^{-1}(y) \mid y \in Y\}$

"Finite morphism = morphisms such that the fibres are finite" (and non-empty)

Precise definition requires attention.

Affine case

X, Y affine varieties, $\varphi: X \to Y$ regular map

Ass that φ is dominant: $\overline{\varphi(X)} = Y \Rightarrow$

$\varphi^*: \mathcal{O}(Y) \to \mathcal{O}(X)$ is injective

$\mathcal{O}(Y) \simeq \varphi^*\mathcal{O}(Y) \subseteq \mathcal{O}(X)$ up to φ^* $\mathcal{O}(Y)$ is a K-subalgebra of $\mathcal{O}(X)$

Def. $\varphi: X \to Y$, dominant is a finite morphism if $\mathcal{O}(X)$ is an integral extension of $\mathcal{O}(Y)$.

$\forall f \in \mathcal{O}(X)$ $\exists$ equation of integral dependence with coeff in $\varphi^*\mathcal{O}(Y)$:

$$f^r + \varphi^*(g_1) f^{r-1} + \dots + \varphi^*(g_r) = 0$$

$g_1, \dots, g_r \in \mathcal{O}(Y)$.

Properties

1) Any composition of finite morph. is a finite morp.

$X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ regular, dominant

φ, ψ finite $\Longrightarrow$ $\psi \circ \varphi$ finite

$\mathcal{O}(Z) \xrightarrow{\psi^*} \mathcal{O}(Y) \xrightarrow{\varphi^*} \mathcal{O}(X)$: φ^*, ψ^* inj. $\Longrightarrow$ $\underset{(\psi\circ\varphi)^*}{\underbrace{\varphi^* \circ \psi^*}}$ inj.

$\Longrightarrow$ $\psi \circ \varphi$ is domin.

$\mathcal{O}(X)$ integr. of $\mathcal{O}(Y)$, $\mathcal{O}(Y)$ integ. of $\mathcal{O}(Z)$:
use transitivity of integral extensions
$\mathcal{O}(X)$ is integ. ext. of $\mathcal{O}(Z)$

2) $\varphi : X \to Y$ finite morph., $\forall y \in Y$ $\varphi^{-1}(y)$ is a finite set.

pf. $y \in Y$ $\quad \forall f \in \mathcal{O}(X)$: eq. of integ. dep.

$\mathcal{O}(X) = K[X] = K[t_1, \dots, t_n]$ $\quad X \subseteq \mathbb{A}^n$

$\forall t_i$ $\quad * \ \boxed{t_i^r + \varphi^*(g_1) t_i^{r-1} + \dots + \varphi^*(g_r) = 0}$ in $\mathcal{O}(X)$

Also $\boxed{x \in \varphi^{-1}(y)}$: apply this fn. to x

$t_i(x) = x_i$ $\quad x_i^r + \underbrace{\varphi^*(g_1)(x)}\, x_i^{r-1} + \dots + \varphi^*(g_r)(x) = 0$

$\varphi^*(g_1)(x) = g_1(\varphi(x)) = g_1(y)$ $\quad x_i^r + \underline{g_1(y)}\, x_i^{r-1} + \dots + \underline{g_r(y)} = 0$

$\varphi^*(g_1) = g_1 \circ \varphi$

the coef are constant dep. on y

$\Rightarrow$ x_i satisfies an equat. of deg r in $K[x]$

$\Rightarrow$ for x_i there are only at most r possibilities

$\Rightarrow$ the fibre can have only finite number of points.

3) $\varphi: X \to Y$ finite dominant morphism $\Rightarrow \varphi$ is surjective : $\forall y \in Y \quad \varphi^{-1}(y) \neq \emptyset$

Pf. translation of the LO property : Lying-over

$\varphi^* \mathcal{O}(Y) \subseteq \mathcal{O}(X)$ integr. ext.

LO : $\forall \mathfrak{P}$ prime ideal in $\varphi^*\mathcal{O}(Y)$, $\exists \mathfrak{Q}$ prime in $\mathcal{O}(X)$ st. $\mathfrak{Q} \cap \varphi^*\mathcal{O}(Y) = \mathfrak{P}$.

$y \in Y$: $\varphi^{-1}(y) = \{ x \in X \mid \underline{\varphi(x) = y} \Leftrightarrow \forall i \ \underline{\varphi^*(t_i)(x) = y_i} \}$

where $y = (y_1, \dots, y_m)$, $t_1, \dots, t_m$ coord. on Y.

$\mathcal{O}(Y) = K[t_1, \dots, t_m] \qquad \varphi^*(t_i) = t_i \circ \varphi$

$\varphi^{-1}(y) = \{ x \in X \mid (\underline{\varphi^*(t_i)} - \underset{\text{const}}{y_i})(x) = 0 \} =$

$= V(\varphi^*(t_i) - y_i \mid i = 1, \dots, m)$

$\varphi^{-1}(y) = \emptyset \Leftrightarrow \langle \varphi^*(t_i) - y_i \mid i = 1 \dots m \rangle$ $\boxed{\subseteq \mathcal{O}(X)}$ has no zeros

Nullstellensatz for $\mathcal{O}(X)$ $\Longleftarrow$ $\underline{\langle \varphi^*(t_i) - y_i \mid i = 1, \dots, m \rangle = \mathcal{O}(X)}$

$I_Y(y) \subseteq \mathcal{O}(Y) = K[Y] = \langle t_1 - y_1, \ldots, t_m - y_m \rangle$ maximal

$\overset{\shortparallel}{\alpha}$

$\varphi^* \mathcal{O}(Y) \subseteq \mathcal{O}(\cancel{X})$

$\cup$

$\varphi^*(\alpha) = \langle \varphi^*(t_1 - y_1), \ldots, \varphi^*(t_m - y_m) \rangle =$

$= \langle \varphi^*(t_1) - y_1, \ldots, \varphi^*(t_m) - y_m \rangle$ ideal in $\varphi^* \mathcal{O}(Y)$

$\cap$

$\mathcal{O}(X)$

$\varphi^*(t_1) - y_1, \ldots, \varphi^*(t_m) - y_m$ — $\underline{\varphi^* \mathcal{O}(Y)}$ $\underline{\text{maximal}}$

— $\mathcal{O}(X)$: we want to prove this is proper

LO $\Rightarrow \exists\, P$ prime in $\mathcal{O}(X)$

$P \supseteq \langle \varphi^*(t_1) - y_1, \ldots, \varphi^*(t_m) - y_m \rangle$ ideal generated in $\varphi^* \mathcal{O}(Y)$

$\Rightarrow$ the ideal generated in $\mathcal{O}(X)$ is proper.

$\Rightarrow \varphi^{-1}(y) \neq \emptyset$.

4) $\varphi : X \to Y$ finite morph. $\Longrightarrow$ φ is a closed morphism

The Normalization lemma has an interpretation using ~~of~~ finite morph.