$f : X \longrightarrow Y$, $\overline{f(X)} = Y$, $f^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$

X, Y affine varieties f finite if $\mathcal{O}(X)$ integral ext. of $f^*\mathcal{O}(Y)$.

f finite $\Rightarrow$ 1) $\forall y \in Y$ $f^{-1}(y)$ is finite
2) f is surjective
3) f is closed

3) $Z \subseteq X$ closed $f|_Z : Z \longrightarrow \overline{f(Z)}$ results

to be a fin. morphism

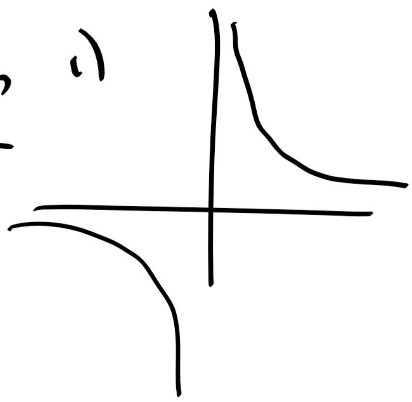
$$f|_Z^* : \mathcal{O}(\overline{f(Z)}) \longrightarrow \underset{\varphi}{\mathcal{O}(Z)} = \mathcal{O}(X) / I_X(Z) \qquad \varphi = [\overline{\varphi}]$$

$\overline{\varphi} \in \mathcal{O}(X)$ $\overline{\varphi}$ satisfies an eq. of integral dependence over $f^*\mathcal{O}(Y)$: we reduce this equation

So because of 2) $f|_Z$ is surjective.

$\Rightarrow f(Z) = \overline{f(Z)}$ is closed.

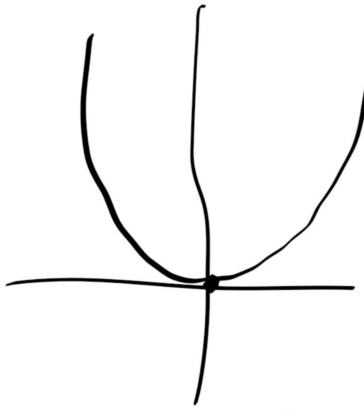
<u>Examples</u> 1)



$X = V(xy - 1)$
$\downarrow$ is not finite : dominant but not surj.
$\mathbb{A}^1$

2) $X = V(y - x^2) \xrightarrow{f} \mathbb{A}^1$ over $\mathbb{C}$

$(x,y) \longrightarrow y$ is finite



$\leftarrow$ $f^*: \mathcal{O}(\mathbb{A}^1) \longrightarrow \mathcal{O}(X) = K[t, t^2]$

$t \longrightarrow t$

t^2 is integral over $f^* K[t]$:

$x^2 - f^*(t) = 0$: the generators are integral

$\Rightarrow \mathcal{O}(X)$ results to be integral over $f^*\mathcal{O}(\mathbb{A}^1)$.

Geometrical interpretation of Norm. Lemma.

A K- algebra, integral dom., $A = K[y_1, \ldots, y_n]$

$Q(A)$ $\quad$ tr. d. $Q(A)/K = r$

$\exists z_1, \ldots, z_r \in A$ alg. indep. over K s.t.

A is integral extension of $K[z_1, \ldots, z_r]$

$A = K[X]$ $\quad$, $\quad X \subseteq \mathbb{A}^n$ $\quad \varphi: K[x_1 \ldots x_n] \twoheadrightarrow A$

$= \mathcal{O}(X)$ $\qquad F(x_1 \ldots x_n) \longrightarrow F(y_1 \ldots y_n)$

$\operatorname{Ker} \varphi = I \subseteq K[x_1 \ldots x_n]$ prime

$V(I) = X$ s.t. $K[X] \simeq A$

$j: K[z_1, \ldots, z_r] \hookrightarrow A$ $\qquad$ A integr. ext of $K[z_1 \ldots z_r]$

$\quad \| \qquad\qquad\qquad \wr\wr$

$\mathcal{O}(\mathbb{A}^r) \qquad\qquad \mathcal{O}(X)$

$j^\# = \varphi$ $\qquad \varphi: X \longrightarrow \mathbb{A}^r$ finite morphism

$r = \mathrm{tr.\,d.}\; K(X)/K = \dim X$

$\forall X$ irred. aff. var. $\exists$ a fin. morph.

$\varphi : X \to \mathbb{A}^r$, where $r = \dim X$

From the proof of Norm. Lemma : $z_1, \dots, z_r$
can be taken linear combinations of $y_1, \dots, y_n$
$\Rightarrow \varphi$ is represented by linear polynom. so
is a projection.

$\forall \varphi$ finite there is a degree:

$\forall y \in Y$ general $\quad \varphi^{-1}(y) = \{x_1, \dots, x_d\}$ d distinct pts

Being a finite morphism is a local property:

if $f : X \to Y$ reg. map, X, Y affine

Ass. $y \in Y$ $\exists$ U_y affine open nbhd s.t.
$f^{-1}(U_y)$ is affine and $\neq \emptyset$ and
$f|_{f^{-1}(U_y)} : f^{-1}(U_y) \longrightarrow U_y$ is a finite morph.

$\Rightarrow f$ is finite.

$\Rightarrow$ definition of finite morph. for
$f : X \to Y$, X, Y quasi-proj.

Thm. $X \subseteq \mathbb{P}^n$ proj. variety
$\Lambda \subseteq \mathbb{P}^n$ linear subvariety

$\pi_\Lambda : \mathbb{P}^n \dashrightarrow \mathbb{P}^r$

If $X \cap \Lambda = \emptyset$, $\pi_\Lambda : X \longrightarrow \mathbb{P}^r$ is regular.
Then $\pi_\Lambda : X \to \mathbb{P}^r$ is finite to the image.

Thm. $X \subseteq \mathbb{P}^n$

$\varphi : X \longrightarrow Y \subseteq \mathbb{P}^m$ defined by $[F_0, \dots, F_m]$, the same polynomials on the whole X

$V_p(F_0, \dots, F_m) \cap X = \emptyset$

$\Rightarrow \varphi$ is a finite morph. $X \to \varphi(X)$.

This follows from the previous thm using Veronese map.

$$\begin{tikzcd} X \arrow[rr, "{[F_0 \dots F_m]}"] \arrow[dr, "v_{n,d}"'] & & Y \\ & v_{n,d}(X) \subseteq \mathbb{P}^N \arrow[ur] & \end{tikzcd}$$

the components are linear projection

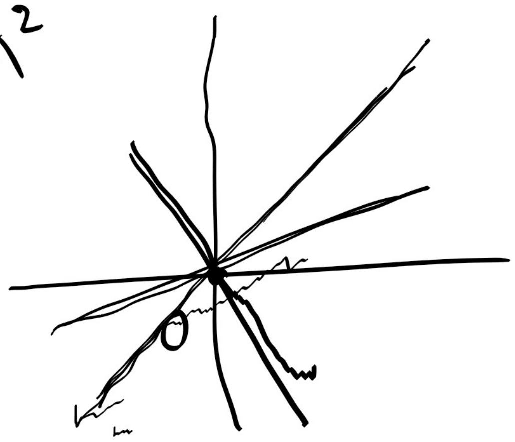
Blow-ups: class of morphisms, which are birational but not finite.

$\varphi : X \longrightarrow Y \qquad \exists\ \varphi^{-1} : Y \dashrightarrow X$

some fibers of φ are not finite

Blow-up of $\mathbb{A}^n$ at the origin

$\mathbb{A}^2$



$T_{0,\mathbb{A}^n}$ Zariski tang. space

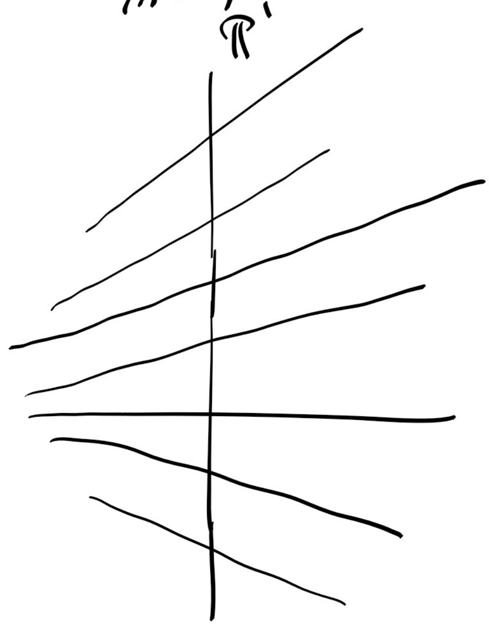
$\shortparallel$

k^n

$\mathbb{P}(T_{0,\mathbb{A}^n}) = \{$ subspaces of dim 1 of $T_{0,\mathbb{A}^n} \}$

= "directions thru O in $\mathbb{A}^n$"

$\mathbb{P}(T_{0,\mathbb{A}^2}) = \mathbb{P}^1$



$\mathbb{A}^n \times \mathbb{P}^{n-1}$ $\qquad$ $(x_1, \dots, x_n)$ coord. on $\mathbb{A}^n$

$[y_1, \dots, y_n]$ homog. coord. on $\mathbb{P}^{n-1}$

Def. $X \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ is by def. the subvariety def. by the equations $\mathrm{rk}\begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} < 2$

$\{ x_i y_j - x_j y_i = 0 \quad \forall i,j = 1,\dots,n$

$X = \{ ((x_1 \dots x_n), [y_1 \dots y_n]) \mid (x_1 \dots x_n)$ is proport to $(y_1 \dots y_n)$

$$\begin{array}{ccc} X & \xrightarrow{p_1} & \mathbb{A}^n \\ {\scriptstyle p_2}\downarrow & & \\ \mathbb{P}^{n-1} & & \end{array}$$

regular

<u>Def.</u> blow-up of $\mathbb{A}^n$ at O is

$\sigma : X \longrightarrow \mathbb{A}^n$ where $\sigma = p_1$

Fibres of σ: $P \in \mathbb{A}^n$ $P(a_1 \dots a_n)$

$\sigma^{-1}(P)$
- $P = O$ $\mathrm{rk}\begin{pmatrix} 0 & \dots & 0 \\ y_1 & \dots & y_n \end{pmatrix} < 2$ $\sigma^{-1}(P) = O \times \mathbb{P}^{n-1}$
- $P \neq O$ $\sigma^{-1}(P)$ is one point $= \{((a_1 \dots a_n), [a_1, \dots, a_n])\}$

$\sigma^{-1}(O) = O \times \mathbb{P}^{n-1} = E$: exceptional division of the blow-up

$\sigma|: X \setminus E \longrightarrow \mathbb{A}^n \setminus \{0\}$ is isomorphism:

$\sigma^{-1}: \mathbb{A}^n \setminus \{0\} \longrightarrow X \setminus E$ regular

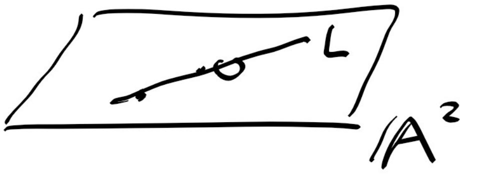
$(a_1 \dots a_n) \longrightarrow ((a_1 \dots a_n), [a_1 \dots a_n])$

$L \subseteq \mathbb{A}^n$ line through O:

$p(a_1 \dots a_n) \neq 0$

$\uparrow$

L



$L\{(ta_1, \dots, ta_n) \mid t \in K\}$

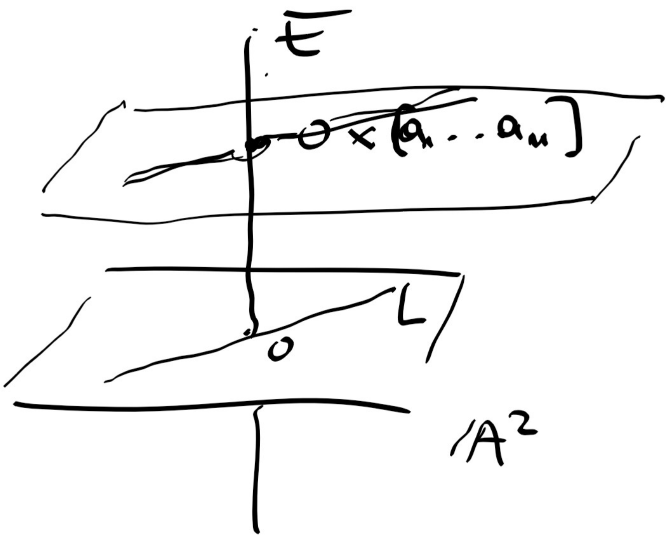
$\sigma^{-1}(L) = E \cup \tilde{L}$

$\sigma^{-1}(0)$ $\quad$ $\overline{\sigma^{-1}(L \setminus \{0\})}$

$\tilde{L} = \overline{\sigma^{-1}(L \setminus \{0\})}$

$\tilde{L}: \quad \sigma^{-1}(L \setminus \{0\}) = \{((ta_1, \dots, ta_n), [ta_1 \dots ta_n]) \mid t \neq 0\}$

$= \{((ta_1, \dots, ta_n), [a_1 \dots a_n]) \mid t \neq 0\} =$



$\tilde{L} = \overline{\{((ta_1, \dots ta_n), [a_1 \dots a_n])\}}$

$0 \times [a_1 \dots a_n] \in \tilde{L} = \overline{\sigma^{-1}(L \setminus 0)}$

$\tilde{L}$ strict transform of L

$X = \bigcup \tilde{L}$

L line through 0

Prop. X is irreducible.

Pf. $X = E \cup (X \setminus E)$

$\forall$ point in E $\quad 0 \times [a_1 \dots a_n] \in \tilde{L}$ where $\tilde{L} = \overline{\sigma^{-1}(L \setminus \{0\})}$

$L = \{(ta_1 \dots ta_n), t \in K\}$

$0 \times [a_1 \dots a_n]$ belongs to the closure of

$\sigma^{-1}(L \setminus \{0\}) \subseteq X \setminus E \implies$ it belongs to $\overline{X \setminus E}$ $\Longrightarrow$ $X = \overline{X \setminus E}$

$X \setminus E \simeq \mathbb{A}^n \setminus \{0\}$ irred. (open in $\mathbb{A}^n$)

$\Rightarrow X \setminus E$ irred. $\Rightarrow$ $\underset{\overset{\shortparallel}{X}}{\overline{X \setminus E}}$ irred

<u>Cor</u> X is birational to $\mathbb{A}^n$: contain isomorphic open subsets.

dim $X = n$ E has codim 1 in X
"divisor".

$p_2 : X \longrightarrow \mathbb{P}^{n-1}$: is the univ. bundle

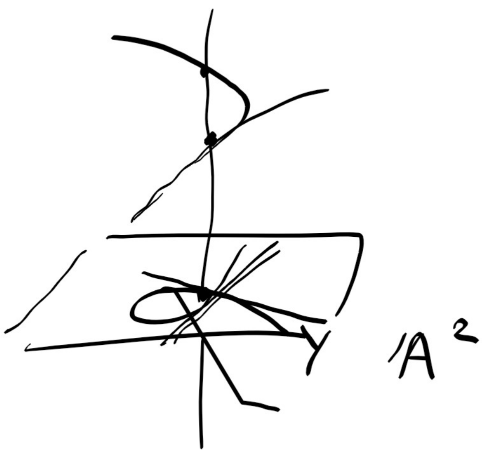
$[a_1 \dots a_n] = A$

$p_2^{-1}(A) = \{ (ta_1 \dots ta_n), (a_1 \dots a_n) \mid t \in K \}$

$\cong$

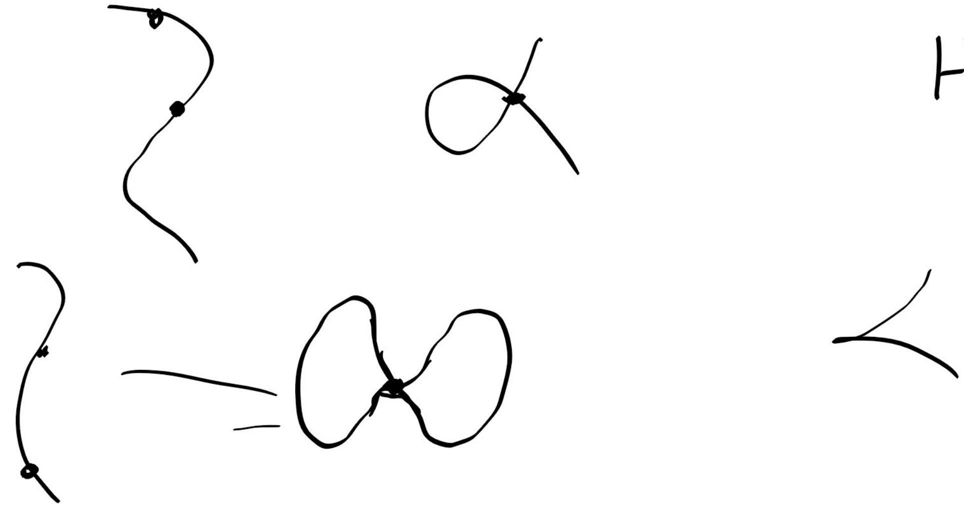
line represented by A

$Y \subseteq \mathbb{A}^n$ $\sigma^{-1}(Y) \simeq Y$ if $0 \notin Y$



$0 \in Y$ $\sigma^{-1}(Y) \supseteq E$

$\overline{\sigma^{-1}(Y \setminus \{0\})} = \tilde{Y}$ strict or proper transform of Y



Hironaka