Therefore X is birational to $\mathbb{A}^n$: they are both irreducible and contain the isomorphic open subsets $X \setminus \sigma^{-1}(O)$ and $\mathbb{A}^n \setminus O$. In particular $\dim X = n$, and $\sigma^{-1}(O) = E \simeq \mathbb{P}^{n-1}$ has codimension 1 in X. The tangent space $T_{O,\mathbb{A}^n}$ coincides with $\mathbb{A}^n = K^n$, and the set of the lines through O can be interpreted as the projective space $\mathbb{P}(T_{O,\mathbb{A}^n})$. So there is a bijection between the exceptional divisor E and $\mathbb{P}(T_{O,\mathbb{A}^n})$.

Figure 17.2, taken from the book [S], illustrates the case of the plane.

If we consider the second projection $p_2 : X \to \mathbb{P}^{n-1}$, for any $[a] = [a_1, \dots, a_n] \in \mathbb{P}^{n-1}$, $p_2^{-1}[a]$ is the line L' of (17.4). X with the map p_2 is an example of non-trivial line bundle, called the universal bundle over $\mathbb{P}^{n-1}$.

If Y is a closed subvariety of $\mathbb{A}^n$ passing through O, it is clear that $\sigma^{-1}(Y)$ contains the exceptional divisor $E = \sigma^{-1}(O)$. It is called the total trasform of Y in the blow-up. We define the *strict transform of Y* in the blow-up of $\mathbb{A}^n$ as the closure $\widetilde{Y} := \overline{\sigma^{-1}(Y \setminus O)}$. It is interesting to consider the intersection $\widetilde{Y} \cap E$, it depends on the behaviour of Y in a neighborhood of O, and allows to analyse its singularities at O.

proper transform

Example 17.2.2.

1. Let $Y \subset \mathbb{A}^2$ be the plane cubic curve of equation $y^2 - x^2 = x^3$. The origin is a singular point of Y, with multiplicity 2, and the tangent cone $TC_{O,Y}$ is the union of the two lines of equations $x - y = 0$, $x + y = 0$, respectively. We consider the blow-up $X \subset \mathbb{A}^2 \times \mathbb{P}^1$ of $\mathbb{A}^2$ with centre O. Using coordinates t_0, t_1 in $\mathbb{P}^1$, X is defined by the unique equation $xt_1 = t_0y$. Then $\sigma^{-1}(Y)$ is defined by the system

X hypersurface, $P \in X$

$TC_{P,X}$: defined by $d^m_P F$: first differential which is not 0

$m =$ multiplicity of P on X

$$\begin{vmatrix} x & y \\ t_0 & t_1 \end{vmatrix} = 0$$

$\mathbb{P}^1 = U_0 \cup U_1$

$\mathbb{A}^2 \times \mathbb{P}^1 = (\mathbb{A}^2 \times U_0) \cup (\mathbb{A}^2 \times U_1)$

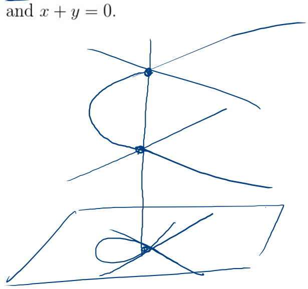
$$\begin{cases} y^2 - x^2 = x^3 \\ xt_1 = t_0y \end{cases}$$

As usual $\mathbb{P}^1$ is covered by the two open subsets $U_0 : t_0 \neq 0$ and $U_1 : t_1 \neq 0$, so $\mathbb{A}^2 \times \mathbb{P}^1 = (\mathbb{A}^2 \times U_0) \cup (\mathbb{A}^2 \times U_1)$, the union of two copies of $\mathbb{A}^3$, and we can study X considering its intersection X_0, X_1 with each of them. If $t_0 \neq 0$, we use $t = t_1/t_0$ as affine coordinate; if $t_1 \neq 0$ we use $u = t_0/t_1$. X_0 has equation $y = tx$ and X_1 has equation $x = uy$. For $\sigma^{-1}(Y) \cap X_0$ we get the equations $y^2 - x^2 - x^3 = 0$ and $y = tx$ in $\mathbb{A}^3$ with coordinates x, y, t. Substituting we get $t^2x^2 - x^2 - x^3 = x^2(t^2 - 1 - x) = 0$. So there are two components: one is defined by $x = y = 0$, which is $E \cap X_0$; the other is defined by $\begin{cases} x = t^2 - 1 \\ y = t(t^2 - 1) \end{cases}$, it is $\widetilde{Y} \cap X_0$. Note that it meets E at the two points $P(0, 0, 1)$, $Q(0, 0, -1)$. They correspond on E to the two tangent lines to Y at O: $y - x = 0$ and $x + y = 0$.

$xt_1 = t_0y$

$x\frac{t_1}{t_0} = y$

$$\begin{cases} x = t^2 - 1 \\ y = t(t^2-1) \\ x = 0 \\ y = 0 \end{cases}$$



$\widetilde{Y} - \{P, Q\}$

$\updownarrow ?$

$Y - \{O\}$

$x - y = 0$

(t, t) $\quad \mathbb{P}(T_{\mathbb{A}^2, O})$

$\updownarrow$

$[1,1] \longrightarrow 1 \in U_0$ $\qquad (0,0,1) \in E \cap X_0$

If we work on the other open set $\mathbb{A}^2 \times U_1$, $\sigma^{-1}(Y)$ is defined by $x = uy$ and $y^2 - u^2y^2 - u^3y^3 = y^2(1-u^2-u^3y) = 0$. So $\widetilde{Y} \cap X_1$ is defined by $\begin{cases} x = uy \\ 1-u^2-u^3y = 0 \end{cases}$. We find the same two points of intersection with E: $(0,0,1), (0,0,-1)$.

$[1,1]$

$[1,-1]$

The restriction of the projection $\sigma : \widetilde{Y} \to Y$ is an isomorphism outside the points P, Q on $\widetilde{Y}$ and O on Y. The result is that the two branches of the singularity O have been separated, and the singularity has been resolved.

$\widetilde{Y}$ is smooth

2. Let $Y \subset \mathbb{A}^2$ be the cuspidal cubic curve of equation $y^2 - x^3 = 0$. The total transform is defined by

$$\begin{cases} y^2 - x^3 = 0 \\ xt_1 = t_0 y. \end{cases}$$

$TC_{O,Y}$: $y^2 = 0$: O is a double pt. cusp

$y = 0$

$((0,0),[1,0])$

On the first open subset it becomes $y^2 - x^3 = 0$ together with $y = tx$; replacing and simplifying t, which corresponds to E, we get the equations for $\widetilde{Y}$:

$$\begin{cases} x = t^2 \\ y = t^3 \end{cases}.$$

$\begin{cases} y^2 - x^3 = 0 \\ y = tx \end{cases}$

(x,y,t)

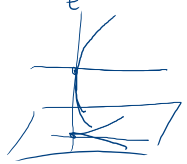
$\begin{cases} x - t^2 = 0 \\ y - t^3 = 0 \end{cases}$ tg line at $(0,0,0)$

$t^2x^2 - x^3 = 0$

$x^2(t^2 - x) = 0$

This is the affine skew cubic, that meets E at the unique point $(0,0,0)$, corresponding to the tangent line to Y at O: $y = 0$. By the way, we can check that E is the tangent line to $\widetilde{Y}$ at $(0,0,0)$. On the second open subset, we have the equations $y^2 - x^3 = 0$ together with $x = uy$; the strict transform is defined by $1 - u^3y = 0$ and $x = uy$. There is no point of intersection with E in this affine chart. The map $\sigma : \widetilde{Y} \to Y$ is therefore regular, birational, bijective, but not biregular; Y and $\widetilde{Y}$ cannot be isomorphic, because one is smooth and the other is not smooth.

$\widetilde{Y}$ is smooth



3. Let $Y = V(x^2 - x^4 - y^4) \subset \mathbb{A}^2$. O is a singular point of multiplicity 2 with tangent cone the line $x = 0$ counted twice. Let $\widetilde{Y}$ be the strict transform of Y in the blow-up of the plane in the origin. Proceeding as in the previous example we find that $\widetilde{Y}$ meets the exceptional divisor $E = O \times \mathbb{P}^1$ at the point $O' = ((0,0),[0,1])$, which belongs only to the second open subset $\mathbb{A}^2 \times U_1$. In coordinates $x, y, u = t_0/t_1$, $\widetilde{Y}$ is defined by the equations

$$\begin{cases} x = uy \\ u^2 - u^4y^2 - y^2 = 0 \end{cases},$$

$u^2y^2 - u^4y^4 - y^4 = 0$

$(u^2 - y^2) + u^4y^2 = 0$

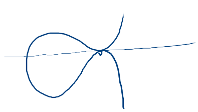
$x^2 - x^4 - y^4 = 0$

$x^2 = 0$ $\quad TC_{O,Y}$

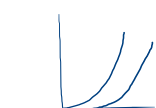
$\begin{cases} x = 0 \\ y^4 = 0 \end{cases}$

and $O' = (0,0,0)$. We compute the equation of the tangent space $T_{O',\widetilde{Y}}$, it is $x = 0$: it is a

O is singular for Y : $\forall L$ through O $L \cap Y$: the mult. of inters. is ≥ 2; if L is a tg line = line of $TC_{O,Y}$ $\implies$ mult. of inters. is > 2



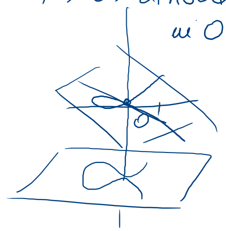
$x^2 - y^5 = 0$



2-plane in $\mathbb{A}^3$, so $\widetilde{Y}$ is singular at O'. The tangent cone $TC_{O',\widetilde{Y}}$ is $x = 0, u^2 - y^2 = 0$, the union of two lines in the tangent plane.

$\dim T_{O',\widetilde{Y}} = 2 \Rightarrow O'$ is singular for $\widetilde{Y}$

$\widetilde{Y}$ has a node in O'



Let us consider a second blow-up σ', of $\mathbb{A}^3$ in O'. It is contained in $\mathbb{A}^3 \times \mathbb{P}^2$; using coordinates z_0, z_1, z_2 in $\mathbb{P}^2$, it is defined by

$$rk \begin{pmatrix} x & y & u \\ z_0 & z_1 & z_2 \end{pmatrix} < 2.$$

$x z_1 - y z_0 = 0$
$x z_2 - u z_0 = 0$
$y z_2 - u z_1 = 0$

We first work on the open subset $\mathbb{A}^3 \times U_0 \simeq \mathbb{A}^5$; we put $z_0 = 1$ and we work with affine coordinates x, y, u, z_1, z_2; the exceptional divisor E' is defined by $x = y = u = 0$, and the total transform $\sigma'^{-1}(\widetilde{Y})$ of $\widetilde{Y}$ by

$$\begin{cases} x = uy \\ y = z_1 x \\ u = z_2 x \\ x^2(z_2^2 - z_1^2 - u^4 z_1^2) = 0 \end{cases}$$

Replacing $x = uy$ in the second and third equation we get the equivalent system

$$\begin{cases} x = uy \\ y(1 - z_1 u) = 0 \\ u(1 - z_2 y) = 0 \\ x^2(z_2^2 - z_1^2 - u^4 z_1^2) = 0 \end{cases}.$$

Combining the factors of the four equations in all possible ways, we find that, on $\mathbb{A}^3 \times U_0$, $\sigma'^{-1}(\widetilde{Y})$ is union of E' and of the strict transform $\widetilde{Y}'$ defined by

$$\begin{cases} x = uy \\ 1 - z_1 u = 0 \\ 1 - z_2 y = 0 \\ z_2^2 - z_1^2 - u^4 z_1^2 = 0 \end{cases}.$$

The intersection $\widetilde{Y}' \cap E' \cap (\mathbb{A}^3 \times U_0)$ results to be empty.

We then work on the open subset $\mathbb{A}^3 \times U_1 \simeq \mathbb{A}^5$; we put $z_1 = 1$ and we work with affine coordinates x, y, u, z_0, z_2. Proceeding as in the first case, we find the equations of the total transform

$$\begin{cases} x = uy \\ y(z_0 - u) = 0 \\ u = z_2 y \\ y^2(z_2^2 - 1 - z_2^4 y^4) = 0 \end{cases}$$

The strict transform results to be defined by

$$\begin{cases} x = uy \\ z_0 - u = 0 \\ u = z_2 y \\ z_2^2 - 1 - z_2^4 y^4 = 0 \end{cases},$$

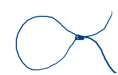
$$\begin{cases} x = 0 \\ y = 0 \\ u = 0 \\ z_0 = 0 \\ z_2^2 - 1 = 0 \end{cases}$$

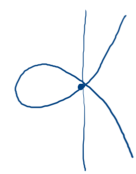
2 points

and its intersection with the exceptional divisor $x = y = u = 0$ is the union of the two points P, Q of coordinates $((0,0,0),[0,1,\pm 1]) \in \mathbb{A}^3 \times \mathbb{P}^2$. Considering the third open subset $\mathbb{A}^3 \times U_2 \simeq \mathbb{A}^5$ one finds the same two points.

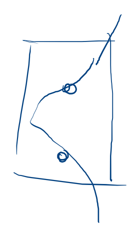
In conclusion, we consider the composition of the two blow-ups $\widetilde{Y}' \xrightarrow{\sigma'} \widetilde{Y} \xrightarrow{\sigma} Y$, which is birational. In the first blow-up σ, we pass from Y, with a singularity at the blown-up point O with one tangent line, to $\widetilde{Y}$ with a node in O', its point of intersection with E. In the second blow-up σ', O' is replaced by two points on the second exceptional divisor E'. To verify if $\widetilde{Y}'$ is smooth, it is enough to check if P, Q are smooth, and this can be checked easily.

$\widetilde{Y}'$ is smooth





The singularity of Y is called a *tacnode*. We have just checked that to resolve it two blow-ups are needed. What allows to distinguish the singularity of the curve of Example 2 from the present example, is the multiplicity of intersection at the point O of the tangent line at the singular point O with the curve: it is 3 in Example 2 and 4 in Example 3.



The general problem of the *resolution of singularities* is, given a variety Y, to find a birational morphism $f : Y' \to Y$ with Y' non-singular. It is possible to prove that, if Y is a curve, the problem can be solved with a finite sequence of blow-ups. If $\dim Y > 1$, the problem is much more difficult, and is presently completely solved only in characteristic 0 (see for instance [rH], Ch. V, 3).

To conclude this chapter, we will see a different way to introduce the blow-up of $\mathbb{A}^n$ at O. Let $p : \mathbb{A}^n \setminus O \to \mathbb{P}^{n-1}$ be the natural projection $(a_1, \ldots, a_n) \to [a_1, \ldots, a_n]$. Let Γ be the graph of p, $\Gamma \subset (\mathbb{A}^n \setminus O) \times \mathbb{P}^{n-1} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$. We immediately have that the closure of

$\Gamma \subseteq \mathbb{A}^n \setminus \{0\} \times \mathbb{P}^{n-1}$

$\{((a_1, \ldots, a_n), [a_1, \ldots, a_n]) \mid (a_1 \ldots a_n) \in \mathbb{A}^n \setminus \{0\}\}$

$\overline{\Gamma} = X \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$

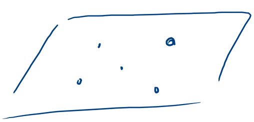
Γ in $\mathbb{A}^n \times \mathbb{P}^{n-1}$ is precisely the blow-up X of $\mathbb{A}^n$ at O. This interpretation suggests how to extend Definition 17.2.1 and define the blow up of a variety X along a subvariety Y.

Suppose that X is an affine variety and $I = I_X(Y) \subset K[X]$ is the ideal of a subvariety Y of X. Suppose that $I = (f_0, \dots, f_r)$. Let λ be the rational map $X \dashrightarrow \mathbb{P}^r$ defined by $\lambda = [f_0, \dots, f_r]$. The blow-up of X along Y is the closure of the graph of λ, together with the projection map to X. Similarly one can define the blow-up of a projective variety along a subvariety, provided that its ideal is generated by homogeneous polynomials all of the same degree. For details, see for instance [C].

$$\begin{array}{c} \overline{\Gamma} \subseteq X \times \mathbb{P}^r \\ \downarrow \sigma \\ X \end{array}$$

Exercises 17.2.3. Let $Y \subset \mathbb{P}^2$ be a smooth plane projective curve of degree $d > 1$, defined by the equation $f(x, y, z) = 0$. Let $C(Y) \subset \mathbb{A}^3$ be the affine variety defined by the same polynomial f: $C(Y)$ is the affine cone of Y. Let $O(0, 0, 0) \in \mathbb{A}^3$ be the origin, vertex of $C(Y)$. Let $\sigma : X \to \mathbb{A}^3$ be the blow-up in O.

1. Show that $C(Y)$ has only one singular point, the vertex O;
2. show that $\widetilde{C(Y)}$, the strict transform of $C(Y)$, is nonsingular (cover it with open affine subsets);
3. let E be the exceptional divisor; show that $\widetilde{C(Y)} \cap E$ is isomorphic to Y.



$\{P_1 - - P_r\}$ $\qquad I_{\mathbb{A}^n}(P_1, \dots, P_r) = \langle f_0 - f_r \rangle$

$\mathbb{P}^n \supseteq X \supseteq Y \qquad I_{\mathbb{P}^n}(Y) \subset K[x_0 - x_n]$ homog.

$\qquad \| $

$\qquad \langle F_0, F_1 - - , F_r \rangle$

$\langle F_0, F_1 \rangle \qquad \deg F_0 = d_0 < d_1 = \deg F_1$

$x_0^{d_1-d_0} F_0, \ x_1^{d_1-d_0} F_0, \ -, MF_0, \quad \forall M$ monomial of deg $d_1 - d_0$

$\langle x_0^{d_1-d_0} F_0, \ - \ - \ , F_1 \rangle \subseteq \langle F_0, F_1 \rangle$

this is saturated

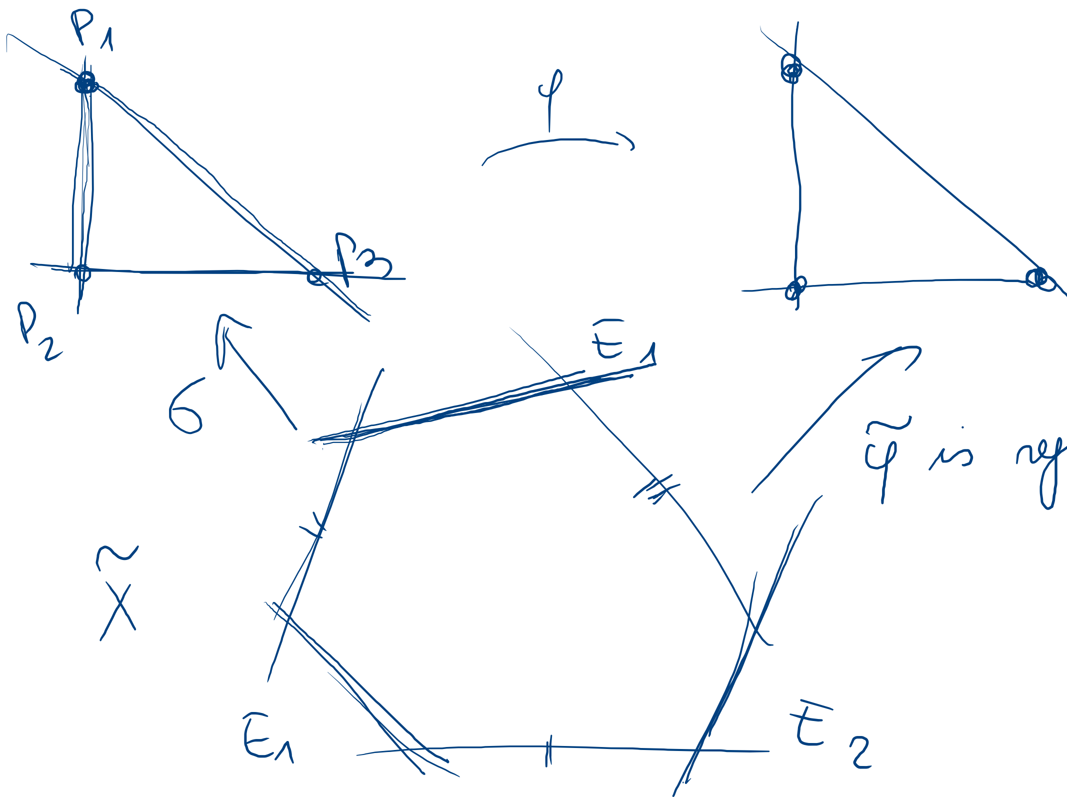
$\varphi : X \dashrightarrow \mathbb{P}^n$ rational

$Z \subseteq X$ indeterminacy locus of φ

$\tilde{X}$ = blow-up of X along Z

$$\begin{tikzcd} \tilde{X} \arrow[d, "\sigma"'] \arrow[dr, "\tilde{\varphi}"] & \\ X \arrow[r, dashed, "\varphi"'] & \mathbb{P}^n \end{tikzcd}$$

$\tilde{\varphi}$ results to be regular



$\tilde{\varphi}$ is regular

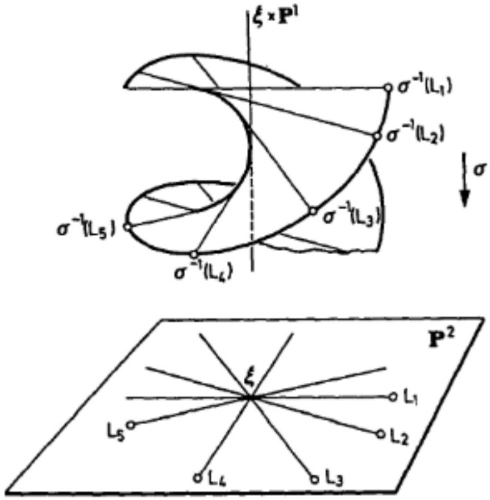


Figure 17.1: Blow-up of the plane