# Chapter 18

# Grassmannians

$\dim V = n$

$1 \le k \le n$

$G(k, V)$

In this chapter we will see how the antisymmetric tensors play an important role in algebraic geometry, providing an ambient space in which naturally embeds the Grassmannian of subspaces of fixed dimension of a vector space, or, equivalently, of a projective space.

## 18.1 Exterior powers of a vector space

To define the exterior powers of the vector space $V$, one proceeds in a way which is similar to the one used to define its symmetric powers. We define the *d-th exterior power* $\wedge^d V$ as the quotient $V^{\otimes d}/\Lambda$, where $\Lambda$ is generated by the tensors of the form $v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_d$, with $v_i = v_j$ for some $i \neq j$. The following notation is used: $[v_1 \otimes \cdots \otimes v_d] = v_1 \wedge \cdots \wedge v_d$.

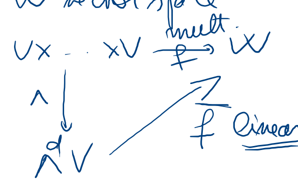
wedge

$\wedge^d V \ni \sum v_{\tau_1} \wedge \ldots \wedge v_{\tau_d}$

$(v_1 \ldots v_d) \to v_1 \wedge \ldots \wedge v_d$

There is a natural multilinear alternating map $V \times \cdots \times V = V^d \to \wedge^d V$, that enjoys the universal property. Given a basis $\mathcal{B} = (e_1, \ldots, e_n)$ of $V$, a basis of $\wedge^d V$ is formed by the tensors $e_{i_1} \wedge \ldots \wedge e_{i_d}$, with $1 \le i_1 < \ldots < i_d \le n$. Therefore $\dim \wedge^d V = \binom{n}{d}$. The exterior algebra of $V$ is the following direct sum: $\wedge V = \oplus_{d \ge 0} \wedge^d V = K \oplus V \oplus \wedge^2 V \oplus \ldots$. To define an inner product that gives it the structure of algebra we can proceed as follows.

$W$ vector space



**Step 1.** Fixed $v_1, \ldots, v_q \in V$, define $f : V^d \to \wedge^{d+q} V$ posing $f(x_1, \ldots, x_d) = x_1 \wedge \ldots \wedge x_d \wedge v_1 \wedge \ldots \wedge v_q$. Since $f$ results to be multilinear and alternating, by the universal property we get a factorization of $f$ through $\wedge^d V$, which gives a linear map $\bar{f} : \wedge^d V \to \wedge^{d+q} V$, extending $f$. For any $\omega \in \wedge^d V$, we denote $\bar{f}(\omega)$ by $\omega \wedge v_1 \wedge \ldots \wedge v_d$.

**Step 2.** Fixed $\omega \in \wedge^d V$, consider the map $g : V^p \to \wedge^{d+p} V$ such that $g(y_1, \ldots, y_p) = \omega \wedge y_1 \wedge \ldots \wedge y_p$: it is multilinear and alternating, therefore it factorizes through $\wedge^p V$ and we get a linear map $\bar{g} : \wedge^p V \to \wedge^{d+p} V$, extending $g$. We denote $\bar{g}(\sigma) := \omega \wedge \sigma$.

**Step 3.** For any $d, p \ge 0$ we have got a map $\wedge : \wedge^d V \times \wedge^p V \to \wedge^{d+p} V$, that results to

$\binom{n}{d} = \binom{n}{n-d}$



$v_1 \wedge \ldots \wedge v_d = (-1)^{\sigma} v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(d)}$

$\sigma \in S_d$

be bilinear, and extends to an inner product $\wedge : (\wedge V) \times (\wedge V) \to \wedge V$, which gives $\wedge V$ the required structure of algebra. It is a graded algebra, the non-zero homogeneous components are those of degree from 0 to $n = \dim V$.

**Proposition 18.1.1.** *Let $V$ be a vector space of dimension $n$.*

*(i) Vectors $v_1, \dots, v_p \in V$ are linearly dependent if and only if $v_1 \wedge \dots \wedge v_p = 0$.*

*(ii) Let $v \in V$ be a non-zero vector, and $\omega \in \wedge^p V$. Then $\omega \wedge v = 0$ if and only if there exists $\Phi \in \wedge^{p-1} V$ such that $\omega = \Phi \wedge v$. In this case we say that $v$ divides $\omega$.*

*Proof.* The proof of (i) is standard. If $\omega = \Phi \wedge v$, then $\omega \wedge v = (\Phi \wedge v) \wedge v = \Phi \wedge (v \wedge v) = 0$. Conversely, if $\omega \wedge v = 0$, $v \neq 0$, we choose a basis of $V$, $\mathcal{B} = (e_1, \dots, e_n)$ with $e_1 = v$. Write $\omega = \Sigma_{i_1 < \dots < i_p} a_{i_1 \dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p}$. Then $0 = \omega \wedge e_1 = \Sigma_{i_1 < \dots < i_p} (\pm) a_{i_1 \dots i_p} e_1 \wedge e_{i_1} \wedge \dots \wedge e_{i_p}$. If $i_1 = 1$, the corresponding summand does not appear in this sum, so it remains a linear combination of linearly independent tensors, which implies $a_{i_1 \dots i_p} = 0$ every time $i_1 > 1$. Therefore $\omega = e_1 \wedge \Phi$ for a suitable $\Phi$.

$\wedge^{p+1} V$

$\omega = v_1 \wedge \dots \wedge v_p$

**Proposition 18.1.2.** *Let $\omega \neq 0$ be an element of $\wedge^p V$. Then $\omega$ is totally decomposable if and only if the subspace of $V$: $W = \{v \in V \mid v \text{ divides } \omega\}$ has dimension $p$.*

*Proof.* If $\omega = x_1 \wedge \dots \wedge x_p \neq 0$, then $x_1, \dots, x_p$ are linearly independent and belong to $W$. So we can extend them to a basis of $V$ adding vectors $x_{p+1}, \dots, x_n$. If $v \in W$, $v = \alpha_1 x_1 + \dots + \alpha_n x_n$, and $v$ divides $\omega$, then $\omega \wedge v = 0$, i.e. $x_1 \wedge \dots \wedge x_p \wedge (\alpha_1 x_1 + \dots + \alpha_n x_n) = 0$. This implies $\alpha_{p+1} x_1 \wedge \dots \wedge x_p \wedge x_{p+1} + \dots + \alpha_n x_1 \wedge \dots \wedge x_p \wedge x_n = 0$, therefore $\alpha_{p+1} = \dots = \alpha_n = 0$, so $v \in \langle x_1, \dots, x_p \rangle$. $\Longrightarrow W = \langle x_1, \dots, x_p \rangle$.

$\Longrightarrow \dim W \geq p$

Conversely, if $(x_1, \dots, x_p)$ is a basis of $W$, we can complete it to a basis of $V$ and write $\omega = \Sigma a_{i_1 \dots i_p} x_{i_1} \wedge \dots \wedge x_{i_p}$. But $x_1$ divides $\omega$, so $\omega \wedge x_1 = 0$. Replacing $\omega$ with its explicit expression, we obtain that $a_{i_1 \dots i_p} = 0$ if $1 \notin \{i_1, \dots, i_p\}$. Repeating this argument for $x_2, \dots, x_p$, it remains $\omega = a_{1 \dots p} x_1 \wedge \dots \wedge x_p$. □

With explicit computations, one can prove the following proposition.

**Proposition 18.1.3.** *Let $V$ be a vector space with $\dim V = n$. Let $\mathcal{B} = (e_1, \dots, e_n)$ be a basis of $V$ and $v_1, \dots, v_n$ be any vectors. Then $v_1 \wedge \dots \wedge v_n = \det(A) e_1 \wedge \dots \wedge e_n$, where $A$ is the matrix of the coordinates of the vectors $v_1, \dots, v_n$ with respect to $\mathcal{B}$.*

**Corollary 18.1.4.** *Let $v_1, \dots, v_p \in V$, with $v_i = \Sigma a_{ij} e_j$, $i = 1, \dots, p$. Then $v_1 \wedge \dots \wedge v_p = \Sigma_{i_1 < \dots < i_p} a_{i_1 \dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p}$, with $a_{i_1 \dots i_p} = \det(A_{i_1 \dots i_p})$, the determinant of the $p \times p$ submatrix of $A$ containing the columns of indices $i_1, \dots, i_p$.*

$A = (a_{ij})$
$p \times n$ matrix

## 18.2 The Plücker embedding

$W \subseteq V$

$\mathbb{P}(V)$
$\cup\!\!\mid$
$\mathbb{P}(W)$

We are now ready to introduce the Grassmannian and to give it an interpretation as projective variety via the Plücker map. Let $V$ be a vector space of dimension $n$, and $r$ be a positive integer, $1 \le r \le n$. The Grassmannian $G(r, V)$ is the set whose elements are the subspaces of $V$ of dimension $r$. It is usual also to denote it by $G(r, n)$.

vector

There is a natural bijection between $G(r, V)$ and the set of the projective subspaces of $\mathbb{P}(V)$ of dimension $r-1$, denoted by $\mathbb{G}(r-1, \mathbb{P}(V))$ or $\mathbb{G}(r-1, n-1)$. Let $W \in G(r, V)$; if $(w_1, \ldots, w_r)$ and $(x_1, \ldots, x_r)$ are two bases of $W$, then $w_1 \wedge \cdots \wedge w_r = \lambda x_1 \wedge \cdots \wedge x_r$, where $\lambda \in K$ is the determinant of the matrix of the change of basis. Therefore $W$ uniquely determines an element of $\wedge^r V$ up to proportionality. This allows to define a map, called the Plücker map, $\psi : G(r, V) \to \mathbb{P}(\wedge^r V)$, such that $\psi(W) = [w_1 \wedge \cdots \wedge w_r]$.

**Proposition 18.2.1.** *The Plücker map is injective.*

*Proof.* Assume $\psi(W) = \psi(W')$, where $W, W'$ are subspaces of $V$ of dimension $r$ with bases $(x_1, \ldots, x_r)$ and $(y_1, \ldots, y_r)$. So there exists $\lambda \neq 0$ in $K$ such that $x_1 \wedge \cdots \wedge x_r = \lambda y_1 \wedge \cdots \wedge y_r$. This implies $x_1 \wedge \cdots \wedge x_r \wedge y_i = 0$ for any $i$, so $y_i$ is linearly dependent from $x_1, \ldots, x_r$, so $y_i \in W$. Therefore $W' \subset W$. The reverse inclusion is similar. $\square$

In coordinates with respect to the basis of $\wedge^r V$ $\{e_{i_1} \wedge \ldots \wedge e_{i_r}, 1 \le i_1 < \ldots < i_r \le n\}$, $[w_1 \wedge \ldots \wedge w_r] =$ $\psi(W)$ is given by the minors of maximal order $r$ of the matrix of the coordinates of the vectors of a basis of $W$, with respect to $e_1, \ldots, e_n$.

**Example 18.2.2.**

$\{$vector subspaces of $V$ of dim $n-1\}$ $= V^*$

$\dim V^* = \dim V = n$

(i) $r = n-1$: $\wedge^{n-1} V$ has dimension $n$. It results to be isomorphic to the dual vector space $V^*$, and an explicit isomorphism is obtained associating to $e_1 \wedge \cdots \wedge \hat{e}_k \wedge \cdots \wedge e_n$ the linear form $e_k^*$ of the dual basis. In this case the Plücker map is surjective, so $\psi(G(n-1, n)) \simeq \mathbb{P}(V^*)$.

(ii) $n = 4, r = 2$: $G(2,4)$ or $\mathbb{G}(1,3)$, the Grassmannian of lines in $\mathbb{P}^3$. In this case $\binom{4}{2} = 6$ $\psi : \mathbb{G}(1,3) \to \mathbb{P}(\wedge^2 V) \simeq \mathbb{P}^5$. Let $(e_0, e_1, e_2, e_3)$ be a basis of $V$. Let $\ell = \mathbb{P}(L)$ be the line of $\mathbb{P}^3$ obtained by projectivisation of the vector subspace $L \subset V$ of dimension 2, let $L = \langle x, y \rangle$; then $\psi(\ell) = [x \wedge y]$. Its Plücker coordinates are traditionally denoted by $p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}$, with $p_{ij} = x_i y_j - x_j y_i$, the $2 \times 2$ minors of the matrix

$e_0 \wedge e_1, e_0 \wedge e_2, \ldots$

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}.$$

This time $\psi$ is not surjective; its image is the subset of $\wedge^2 V$ of the totally decomposable tensors. Assume $char(K) \neq 2$. They satisfy the equation of degree 2: $p_{01}p_{23} - p_{02}p_{13} +$



$W = \langle w_1, \ldots, w_{n-1} \rangle$ $\quad A = \begin{pmatrix} a_{11} & \ldots & \ldots & a_{1n} \\ \vdots & & & \\ a_{n-1,1} & \ldots & & a_{n-1,n} \end{pmatrix}$ $\quad$ homog. linear system $\boxed{AX = 0}$

$rk A = n-1$: The space of solutions has dim 1:

one solution is $[A_1, \ldots, A_n]$ minors of order $n-1$ of $A$ with the correct sign

$\psi: W \longrightarrow [A_1 \ldots A_n]$ $\quad \mathbb{P}(\wedge^{n-1} V)$

$A_1x_1 + \ldots + A_nx_n = 0$ is the equation of $W$, because it is satisfied by $w_1, \ldots, w_{n-1}$

$\psi$ is the map $W \longrightarrow$ linear form vanishing on $W$ up to prop.

$\psi: G(n-1, n) \longrightarrow V^*$

$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$

$p_{03}p_{12} = 0$, which represents a quadric of maximal rank in $\mathbb{P}^5$, called the Klein quadric. The fact that this equation is satisfied can be seen by considering the $4 \times 4$ matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}:$$

its determinant is precisely the above equation (consider the development of the determinant according to the first two rows).

$\langle e_0, e_1 \rangle$

For instance the line of equations $x_2 = x_3 = 0$, obtained projectivising the subspace $\langle e_0, e_1 \rangle$, has Plücker coordinates $[1, 0, 0, 0, 0, 0]$.

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

In general we can prove the following theorem.

**Theorem 18.2.3.** *The image of the Plücker map is a closed subset in $\mathbb{P}(\wedge^r V)$.*

$\operatorname{Im} \varphi = r$
$= \{[\omega] \mid \omega \in \Lambda^r V$
$\dim W = r$
$\Longleftrightarrow \operatorname{rk} \Phi_\omega = n - r$
$\operatorname{rk} \Phi_\omega = n - r$
$\Updownarrow$
minors of order $n-r+1$ are $0$

*Proof.* The image of the Plücker map is the set of the proportionality classes of totally decomposable tensors. By Proposition 18.1.2, a tensor $\omega \in \wedge^r V$ is totally decomposable if and only if the subspace $W = \{v \in V \mid v \text{ divides } \omega\}$ has dimension $r$. We consider the linear map $\underset{\omega}{\Phi} : V \to \wedge^{r+1} V$, such that $\Phi(v) = \omega \wedge v$. The kernel of $\Phi$ is equal to $W$. So $\omega$ is totally decomposable if and only if the rank of $\Phi$ is $n - r$. Fixed a basis $\mathcal{B} = (e_1, \ldots, e_n)$ of $V$, we write $\omega = \Sigma_{i_1 < \cdots < i_r} a_{i_1 \ldots i_r} e_{i_1} \wedge \ldots \wedge e_{i_r}$. We then consider the basis of $\wedge^{r+1} V$ associated to $\mathcal{B}$ and we construct the matrix $A$ of $\Phi$ with respect to these bases: its minors of order $n - r + 1$ are equations of the image of $\psi$, and they are polynomials in the coordinates $a_{i_1 \ldots i_r}$ of $\omega$. $\square$

From now on we shall identify the Grassmannian with the projective algebraic set that is its image in the Plücker map. The equations obtained in Theorem 18.2.3 are nevertheless not generators for the ideal of the Grassmannian. For instance, in the case $n = 4, r = 2$, let

$\overset{2}{\Lambda} V \Rightarrow$ $\omega = p_{01} e_0 \wedge e_1 + p_{02} e_0 \wedge e_2 + \ldots$. Then:

$\Phi(e_0) = \omega \wedge e_0 = p_{12} e_0 \wedge e_1 \wedge e_2 + p_{13} e_0 \wedge e_1 \wedge e_3 + p_{23} e_0 \wedge e_2 \wedge e_3;$

$\Phi(e_1) = \omega \wedge e_1 = -p_{02} e_0 \wedge e_1 \wedge e_2 - p_{03} e_0 \wedge e_1 \wedge e_3 + p_{23} e_1 \wedge e_2 \wedge e_3;$

$\Phi(e_2) = \omega \wedge e_2 = p_{01} e_0 \wedge e_1 \wedge e_2 - p_{03} e_0 \wedge e_2 \wedge e_3 + p_{13} e_1 \wedge e_2 \wedge e_3;$

$\Phi(e_3) = \omega \wedge e_3 = p_{01} e_0 \wedge e_1 \wedge e_3 + p_{02} e_0 \wedge e_2 \wedge e_3 + p_{12} e_1 \wedge e_2 \wedge e_3.$

So the matrix is

$$\Phi_\omega \qquad \begin{pmatrix} p_{12} & -p_{02} & p_{01} & 0 \\ p_{13} & -p_{03} & 0 & p_{01} \\ p_{23} & 0 & -p_{03} & p_{02} \\ 0 & p_{23} & p_{13} & p_{12} \end{pmatrix}. \qquad n - r + 1 = 4 - 2 + 1 = 3$$



$$\det A = \det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ & & \end{pmatrix} = \sum_{1 \le i_1 < i_2 \le n} (-1)^{i_1 + i_2 + 1 + 2} \left| A^{i_1 i_2}_{12} \right| \left| A^{1 \cdots \hat{i}_1 \cdots \hat{i}_2 \cdots n}_{3 \cdots n} \right|$$

Its $3 \times 3$ minors are equations defining $\mathbb{G}(1,3)$, but the radical of the ideal generated by these minors is in fact $(p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12})$.

To find equations for the Grassmannian and to prove that it is irreducible, it is convenient to give an explicit open covering with affine open subsets. In $\mathbb{P}(\wedge^r V)$, let $U_{i_1 \dots i_r}$ be the affine open subset where the Plücker coordinate $p_{i_1 \dots i_r} \neq 0$. To simplify notation we assume $i_1 = 1, i_2 = 2, \dots, i_r = r$, and we put $U = U_{1 \dots r}$. If $W \in G(r,n) \cap U$, and $w_1, \dots, w_r$ is a basis of $W$, then the first minor of the matrix $M$ of the coordinates of $w_1, \dots, w_r$ is non-degenerate. So we can choose a new basis of $W$ such that $M$ is of the form

$$M = \begin{pmatrix} 1 & 0 & \dots & 0 & \alpha_{1,r+1} & \dots & \alpha_{1,n} \\ 0 & 1 & \dots & 0 & \alpha_{2,r+1} & \dots & \alpha_{2,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \alpha_{r,r+1} & \dots & \alpha_{r,n} \end{pmatrix}.$$

Conversely, any matrix of this form defines a subspace $W \in G(r,n) \cap U$. So there is a bijection between $G(r,n) \cap U$ and $K^{r(n-r)}$, i.e. the affine space of dimension $r(n-r)$. The coordinates of $W$ result to be equal to 1 and all minors of all orders of the submatrix of the last $n-r$ columns of $M$. Therefore they are expressed as polynomials in the $r(n-r)$ elements of the last $n-r$ columns of $M$. This shows that $G(r,n) \cap U$ is an affine subvariety of $U$ isomorphic to $\mathbb{A}^{r(n-r)}$. By homogenising the equations obtained in this way, one gets equations for $G(r,n)$.

For instance, in the case $n = 4, r = 2$, the matrix $M$ becomes

$$M = \begin{pmatrix} 1 & 0 & \alpha_{13} & \alpha_{14} \\ 0 & 1 & \alpha_{23} & \alpha_{24} \end{pmatrix}.$$

One gets $1 = p_{01}, \alpha_{23} = p_{02}, \alpha_{24} = p_{03}, -\alpha_{13} = p_{12}, -\alpha_{14} = p_{13}, \alpha_{13}\alpha_{24} - \alpha_{23}\alpha_{14} = p_{23}$. If we make the substitutions and homogenise the last equation with respect to $p_{01}$, we find the equation of the Klein quadric $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$.

**Theorem 18.2.4.** *$G(r,n)$ is an irreducible projective variety of dimension $r(n-r)$, and it is rational.*

*Proof.* We remark that $G(r,n) \cap U_{i_1 \dots i_r}$ is the set of the subspaces $W$ which are complementar to the subspace of equations $x_{i_1} = \dots = x_{i_r} = 0$. It is clear that they have two by two non-empty intersection. Therefore, the projective algebraic set $G(r,n)$ has an affine open covering with irreducible varieties isomorphic to $\mathbb{A}^{r(n-r)}$. Using Exercise 5 of Chapter 6, we conclude that $G(r,n)$ is irreducible. Its dimension is equal to the dimension of any open subset of the



$p_{i_1 \dots i_r}$

$$\begin{pmatrix} w_{11} & \dots & w_{1n} \\ \vdots & & \\ w_{r1} & \dots & w_{rn} \end{pmatrix}$$

$\mathbb{G}(r,n) \cap U_{i_1 \dots i_r}$ has some equations

$p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Klein quadric

$\mathbb{A}^{r(n-r)}$

dim $\mathbb{G}(1,3) = 4$

open covering, $r(n-r)$. Since it is irreducible and contains open subsets isomorphic to the affine space, it is rational. $\square$

Assume $char(K) \neq 2$. In the special case $r = 2$ with $n \geq 4$, using the Plücker coordinates $[\dots, p_{ij}, \dots]$, the equations of the Grassmannian $G(2,n)$ are of the form $p_{ij}p_{hk} - p_{ih}p_{jk} + p_{ik}p_{jh} = 0$, for any $i < j < h < k$.

Also in the case of $G(2,n)$, as for $\mathbb{P}^n \times \mathbb{P}^m$ and $V_{n,2}$, there is an interpretation in terms of matrices, that I expose here without entering in all the details. Given a tensor in $\wedge^2 V$ with coordinates $[p_{ij}]$, we can consider the skew-symmetric $n \times n$ matrix whose term of position $i,j$ is $p_{ij}$, with the conditions $p_{ii} = 0$ and $p_{ji} = -p_{ij}$. In this way we can construct an isomorphism between $\wedge^2 V$ and the vector space of skew-symmetric matrices of order $n$.

From ${}^tA = -A$, it follows $\det(A) = (-1)^n \det(A)$. If $n$ is odd, this implies $\det(A) = 0$. If $n$ is even, one can prove that $\det(A)$ is a square. For instance if $n = 2$, and $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, then $\det(A) = a^2$.

If $n = 4$, and $P = \begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} \\ -p_{12} & 0 & p_{23} & p_{24} \\ -p_{13} & -p_{23} & 0 & p_{34} \\ -p_{14} & -p_{24} & -p_{34} & 0 \end{pmatrix}$, then $\det(P) = (p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})^2$.

In general, for a skew-symmetric matrix $A$ of even order $2n$, one defines the **pfaffian of** $A$, $pf(A)$, in one of the following equivalent ways:

(i) by recursion: if $n = 1$, $pf \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a$; if $n > 1$, one defines

$$pf(A) = \Sigma_{i=2}^{2n} (-1)^i a_{1i} Pf(A_{1i}),$$

where $A_{1i}$ is the matrix obtained from $A$ by removing the rows and the columns of indices 1 and $i$. Then one verifies that $pf(A)^2 = \det(A)$;

(ii) (in characteristic 0) given the matrix $A$, one considers the tensor $\omega = \Sigma_{i,j=1}^{2n} a_{ij} e_i \wedge e_j \in \wedge^2 K^{2n}$. Then one defines the pfaffian of $A$ as the unique constant such that $pf(A) e_1 \wedge \dots \wedge e_{2n} = \frac{1}{n!} \omega \wedge \dots \wedge \omega$.

For a skew-symmetric matrix of odd order, one defines the pfaffian to be 0.

**Proposition 18.2.5.** *A 2-tensor $\omega \in \wedge^2 V$ is totally decomposable if and only if $\omega \wedge \omega = 0$.*

$\omega \in \wedge^2 V$

$\omega = \sum_{i<j} p_{ij} e_i \wedge e_j$

$(p_{ij})$ $\quad p_{ii} = 0$

$p_{ji} = -p_{ij}$

$\wedge^2 V \longleftrightarrow \{$matrices $n \times n$ antisymm.$\}$

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ -a_{12} & 0 & a_{23} & \cdots & \\ -a_{13} & & & & \\ \vdots & & & \ddots & \\ -a_{1n} & & & & 0 \end{pmatrix}$$

$\omega = \sum a_{ij} e_i \wedge e_j$

$\underbrace{\omega \wedge \omega \wedge \dots \wedge \omega}_{n} = \omega^{\wedge (2n)}$ $\quad (2n)$-tensor

$\omega \longleftrightarrow (a_{ij})$

$\omega \wedge \omega = \left(\sum a_{ij} e_i \wedge e_j\right) \wedge \left(\sum a_{ij} e_i \wedge e_j\right) =$
$= \sum \boxed{\phantom{x}}\, e_i \wedge e_j \wedge e_h \wedge e_k$

$a_{ij}a_{hk} - a_{ih}a_{jk} + a_{ik}a_{jh} + \ldots$

2 pf

$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix}$

$a_{12} \neq 0 \quad a_{12}e_2 + a_{13}e_3 + a_{14}e_4 = v$
$-a_{12}e_1 + a_{23}e_3 + a_{24}e_4 = w \qquad \omega = v \wedge w$

$\omega = v_1 \wedge v_2$

$\omega \wedge \omega = v_1 \wedge v_2 \wedge v_1 \wedge v_2 = 0$

*Proof.* If $\omega$ is decomposable, the conclusion easily follows. Conversely, if $\omega = \Sigma_{i,j=1}^{2n} a_{ij} e_i \wedge e_j$ and $\omega \wedge \omega = 0$, then the pfaffians of the principal minors of order 4 of the matrix $A$ corresponding to $\omega$ are all 0, therefore from definition (ii) it follows that the pfaffians of the principal minors of all orders are 0, and also $\det(A) = 0$. In conclusion $A$ has rank 2. Then one checks that $\omega$ is the $\wedge$ product of two vectors corresponding to two linearly independent rows of $A$. For instance, if $a_{12} \neq 0$, then $\omega = (a_{12}e_2 + \ldots + a_{1n}e_n) \wedge (-a_{12}e_1 + a_{23}e_3 + \ldots + a_{2n}e_n)$. □

The equations of $G(2,n)$ are the pfaffians of the principal minors of order 4 of the matrix $P$. They are all zero if and only if the rank of $P$ is 2. Therefore the points of the Grassmannian $G(2,n)$, for any $n$, can be interpreted as (proportionality classes of) skew-symmetric matrices of order $n$ and rank 2.

$\bigwedge^2 V$

The subvarieties of the Grassmannian $\mathbb{G}(r,n)$ correspond to subvarieties of $\mathbb{P}^n$ covered by linear spaces of dimension $r$. Conversely, any subvariety of $\mathbb{P}^n$ covered by linear spaces of dimension $r$ gives rise to a subvariety of the Grassmannian.

$X \subseteq \mathbb{G}(r,n)$
family of $r$-dim'l subspaces of $\mathbb{P}^n$

$\bigcup_{\substack{\Lambda \in \mathbb{P}^n \\ \Lambda \in X}} \Lambda = Z \subseteq \mathbb{P}^n$ subvariety

**Example 18.2.6.** $\mathbb{G}(1,n)$

1. **Pencils of lines.** A pencil of lines in $\mathbb{P}^n$ is the set of lines passing through a fixed point $O$ and contained in a 2-plane $\pi$ such that $O \in \pi$. Assume that $O$ has coordinates $[y_0, \ldots, y_n]$, and fix two points $A, B \in \pi$, different from $O$. Let $A = [a_0, \ldots, a_n]$, $B[b_0, \ldots, b_n]$. Then a general line of the pencil is generated by $O$ and by a point of coordinates $[\ldots, \lambda a_i + \mu b_i, \ldots]$. Therefore the Plücker coordinates of a general line of the pencil are $p_{ij} = y_i(\lambda a_j + \mu b_j) - y_j(\lambda a_i + \mu b_i) = \lambda q_{ij} + \mu q'_{ij}$, where $q_{ij}, q'_{ij}$ are the Plücker coordinates of the lines $OA$ and $OB$ respectively. So the lines of the pencil are represented in the Grassmannian by the points of a line. Conversely one can check that any line contained in a Grassmannian of lines represents the lines of a pencil.

2. **Lines in a smooth quadric surface.** Let $\Sigma : x_0x_3 - x_1x_2 = \det\begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} = 0$ be the Segre quadric in $\mathbb{P}^3$. A line of the first ruling of $\Sigma$ is characterised by a constant ratio of the rows of the matrix $\begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix}$. Therefore it can be generated by two points with coordinates $[x_0, x_1, 0, 0]$, $[0, 0, x_0, x_1]$. The Plücker coordinates of such a line are $[x_0^2, 0, x_0x_2, -x_0x_2, 0, x_2^2]$. This parametrizes a conic contained in $\mathbb{G}(1,3)$. Similarly, the lines of the second ruling describe the points of another conic, indeed the coordinates are $[0, x_0^2, x_0x_1, x_0x_1, x_1^2, 0]$. These two conics are disjoint and contained in disjoint planes.

$[x_0, x_1]$

$\begin{pmatrix} x_0 & x_1 & 0 & 0 \\ 0 & 0 & x_0 & x_1 \end{pmatrix}$

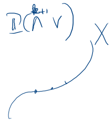
$[0, x_0^2, x_0x_1, x_0x_1, x_1^2, 0]$

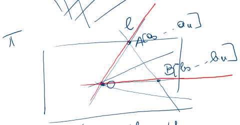
$\begin{cases} p_{01} = 0 \\ p_{23} = 0 \\ p_{03} - p_{12} = 0 \end{cases}$ 2-plane in $\mathbb{P}^5$

a conic in the 2-plane $p_{01} = p_{23} = p_{03} - p_{12} = 0$



$\mathbb{P}^n = \mathbb{P}(V) \longleftrightarrow \mathbb{P}(\bigwedge^{k+1} V)$




$P \in \ell \iff P[\lambda a_0 + \mu b_0, \ldots, \lambda a_n + \mu b_n]$

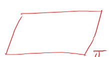
$\begin{pmatrix} y_0 & y_1 & \ldots & y_n \\ \lambda a_0 + \mu b_0 & \lambda a_1 + \mu b_1 & & \lambda a_n + \mu b_n \end{pmatrix}$

Plücker coord of $\ell$: $p_{ij} = \begin{vmatrix} y_i & y_j \\ \lambda a_i + \mu b_i & \lambda a_j + \mu b_j \end{vmatrix} =$

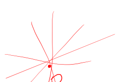
$= \lambda \underbrace{(y_i a_j - y_j a_i)}_{OA} + \mu \underbrace{(y_i b_j - y_j b_i)}_{OB}$

$\Rightarrow$ line contained in $\mathbb{G}(1,n)$

pencils of lines $\longleftrightarrow$ lines contained in $\mathbb{G}(1,n)$

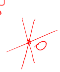



all lines $\subseteq \pi$ $\longleftrightarrow$ plane contained in $\mathbb{G}(1,n)$




all lines through O
star of lines $\longleftrightarrow$ a $\mathbb{P}^{n-1}$ contained in $\mathbb{G}(1,n)$

$n = 3$




$\Rightarrow$ a $\mathbb{P}^2$ contained in $\mathbb{G}(1,3)$
Klein quadric

In $\mathbb{G}(1,3)$: two types of planes $\begin{cases} \alpha\text{-planes} \longleftrightarrow \text{lines of planes in } \mathbb{P}^3 \\ \beta\text{-planes} \updownarrow \text{stars of lines} \end{cases}$

$\{\text{planes in } \mathbb{P}^3\} = (\mathbb{P}^3)^\vee \longrightarrow \alpha$-planes

$P \in \mathbb{P}^3 \longrightarrow \beta$-planes

$\Rightarrow$ 2 families of 2-planes both of dim 3. no other 2-planes

$\nexists\ \mathbb{P}^3$ contained in $\mathbb{G}(1,3)$

$\begin{pmatrix} x_0 & 0 & x_2 & 0 \\ 0 & x_0 & 0 & x_2 \end{pmatrix}$

$[x_0, 0, x_0x_2, -x_0x_2, 0, x_2^2]$

$\begin{cases} p_{02} = 0 \\ p_{13} = 0 \\ p_{03} + p_{12} = 0 \end{cases}$ conic

2 disjoint conics

3. **Planes in** $\mathbb{G}(1,3)$. One can prove that $\mathbb{G}(1,3)$ contains two families of planes, and no linear space of dimension $> 2$. The planes of one family correspond to stars of lines in $\mathbb{P}^3$ (lines in $\mathbb{P}^3$ through a fixed point), while the planes of the second family correspond to the lines contained in the planes of $\mathbb{P}^3$. The geometry of the lines in $\mathbb{P}^3$ translates to give a decription of the geometry of the planes contained in $\mathbb{G}(1,3)$. Since on an algebraically closed field of characteristic $\neq 2$ two quadric hypersurfaces are projectively equivalent if and only if they have the same rank, one obtains a description of the geometry of all quadrics of maximal rank in $\mathbb{P}^5$.

**Exercises 18.2.7.** 1. Let $\ell, \ell'$ two distinct lines in $\mathbb{P}^3$. Let $[p_{ij}]$ be the Plücker coordinates of $\ell$ and $[q_{ij}]$ those of $\ell'$, $0 \leq i < j \leq 3$. Prove that $\ell \cap \ell' \neq \emptyset$ if and only if

$$p_{01}q_{23} - p_{02}q_{13} + p_{03}q_{12} + p_{12}q_{03} - p_{13}q_{02} + p_{23}q_{01} = 0.$$

(Hint: fix points on the two lines to get the Plücker coordinates.)

# Chapter 19

# Fibres of a morphism and lines on hypersurfaces

In this last chapter we will state the Theorem on the dimension of the fibres of a morphism, and we will see an application, involving Grassmannians, about the existence of lines on a hypersurface of given degree in a projective space.

## 19.1 Fibres of a morphism

Let us recall that the *fibres of a morphism* are the inverse images of the points of the codomain. More precisely, if $f : X \to Y$ is a morphism, for any $y \in Y$, the fibre of $f$ over $y$ is $f^{-1}(y)$. Since in the Zariski topology every point is closed, the fibre $f^{-1}(y)$ is closed in $X$, and we want to study the dimensions of its irreducible components. We have seen in Chapter 17 that finite morphisms have the property that all the fibres are finite and non-empty, so all irreducible components have dimension 0.

The following theorem gives informations about the behaviour of the fibres of general morphisms.

**Theorem 19.1.1** (Theorem on the dimension of the fibres.). *Let $f : X \to Y$ be a dominant morphism of algebraic sets. Then:*

*1.* $\dim(X) \geq \dim(Y)$*;*

*2. for any $y \in Y$, and for any irreducible component $F$ of $f^{-1}(y)$,* $\dim F \geq \dim(X) - \dim(Y)$*;*

*3. there exists a non-empty open subset $U \subset Y$, such that* $\dim f^{-1}(y) = \dim(X) - \dim(Y)$ *for any $y \in U$;*

*4. the sets $Y_k = \{y \in Y \mid \dim f^{-1}(y) \geq k\}$ are closed in $Y$ (upper semicontinuity of the dimension of the fibres).*

Before giving a sketch of the proof, let us see an example.

**Example 19.1.2.** Let $V$ be an affine variety and consider $W \subset V \times \mathbb{A}^r$ defined by $s$ linear equations with coefficients in $K[V]$:

$$\left\{\sum_{j=1}^r a_{ij}x_j = 0,\ a_{ij} \in K[V],\ i = 1, \dots, s.\right.$$

Let $\varphi : W \to V$ be the projection. For $P \in V$, $\varphi^{-1}(P)$ is the set of solutions of the system of linear equations with constant coefficients

$$\sum_{j=1}^{r} a_{ij}(P)x_j,\ a_{ij}(P) \in K,\ i = 1, \dots, s,$$

so its dimension is $r - rk(a_{ij}(P))$. For any $k \in \mathbb{N}$ the set $\{P \in V \mid rk(a_{ij}(P)) \leq k\}$ is closed in $V$, defined by the vanishing of the minors of order $k + 1$, and it is precisely $V_{r-k}$, the subset of $V$ where the dimension of the fibre is $\geq r - k$.

The meaning of this example is that we have a family of subspaces of $\mathbb{A}^r$ defined by a system of linear equations with coefficients parametrized by $V$. A "general" space of the family has maximal dimension $r - rkA$, where $A = (a_ij)$ is the matrix of the coefficients of the system. General spaces correspond to the points of an open non-empty subset of $V$. There are closed subsets in $V$ corresponding to spaces of lower dimension.

*Proof of Theorem 19.1.1.* 1. Since $f$ is dominant, there is the $K$-homomorphism $f^* : K(Y) \hookrightarrow K(X)$, and $tr.d.K(Y)/K \leq tr.d.K(X)/K$, because algebraically independent elements of $K(Y)$ remain algebraically independent in $K(X)$. So $\dim(Y) \leq \dim(X)$.

2. Fix $y \in Y$. We observe that we can replace $Y$ with an affine open neighborhood $U$ of $y$ and $X$ with $f^{-1}(U)$. So we can assume that $Y$ is closed in an affine space $\mathbb{A}^N$. Let $n = \dim(X), m = \dim(Y)$. We observe that we can find a polynomial $G$ in $N$ variables which does not vanish identically on any irreducible component of $Y$. For instance, we can fix a point on any irreducible component and choose a hyperplane not passing through any of these points. Then all irreducible components of $Y^{(1)} := Y \cap V(G)$ have dimension $m - 1$. Repeating this argument, we can find a chain of subvarieties of $Y$ of the form $Y \supset Y^{(1)} \supset \cdots \supset Y^{(m)} \supset Y^{(m+1)}$, where all irreducible components of $Y^{(i)}$ have dimension $m - i$. In particular the irreducible components of $Y^{(m)}$ are points, among which there is $y$, and $Y^{(m)}$ is defined by $m$ equations of the form $g_1 = \cdots = g_m = 0$, with $g_1, \dots, g_m \in K[Y]$. Possibly restricting the open set $U$, we can assume that $Y^{(m)} \cap U = \{y\}$. Hence, the fibre

$f^{-1}(y)$ is defined by the system of $m$ equations $f^*(g_1) = \cdots = f^*(g_m) = 0$. The conclusion follows from the Theorem of the intersection 14.1.1.

3. See [S].

4. By induction on the dimension of $Y$. It is obviously true if $\dim Y = 0$. We know from 3. that there is an open subset $U$ of $Y$ such that $\dim f^{-1}(y) = n - m$ if and only if $y \in U$. Let $Z$ be the complement of $U$ in $Y$; thus $Z = Y_{n-m+1}$. Let $Z_1, \dots, Z_r$ be the irreducible components of $Z$. We can now apply the induction to the restrictions of $f$, $f^{-1}(Z_j) \to Z_j$ for each $j$ , and we obtain the result. $\square$

As a consequence of Theorem 19.1.1, we are able to prove the following very useful proposition.

**Proposition 19.1.3.** *Let $f : X \to Y$ be a surjective morphism of projective algebraic sets. Assume that $Y$ is irreducible and that all fibres of $f$ are irreducible and of the same dimension $r$, then $X$ is irreducible of dimension* $\dim(Y) + r$.

*Proof.* Note first of all that $r = \dim(X) - \dim(Y)$. Let $Z$ be an irreducible closed subset of $X$, and consider the restriction $f|_Z : Z \to Y$ ; its fibres are $f|_Z^{-1}(y) = f^{-1}(y) \cap Z$. There are three possibilities:

(a) $f(Z) \neq Y$. Then $f(Z)$ is a proper closed subset of $Y$;

(b) $f(Z) = Y$ and $\dim(Z) < r + \dim(Y)$. Then 2. of Theorem 19.1.1 shows that there is a nonempty open subset $U$ of $Y$ such that for $y \in U$, $\dim(f^{-1}(y) \cap Z) = \dim(Z) - \dim(Y) < r = \dim(X) - \dim(Y)$. Thus, for $y \in U$, the fibre is not contained in $Z$.

(c) $f(Z) = Y$ and $\dim(Z) \geq r + \dim(Y)$. Then again 2. of Theorem 19.1.1 shows that $\dim(f^{-1}(y) \cap Z) \geq \dim(Z) - \dim(Y) \geq r$ for all $y$; thus $f^{-1}(y) \subset Z$ for all $y \in Y$, so $Z = X$.

Now let $Z_1, \dots, Z_r$ be the irreducible components of $X$ . We claim that (c) holds for at least one of the $Z_i$. Otherwise, there will be an open subset $U$ in $Y$, such that for $y \in U$, $f^{-1}(y)$ is contained in none of the $Z_i$; but $f^{-1}(y)$ is irreducible and $f^{-1}(y) = \bigcup_i (f^{-1}(y) \cap Z_i)$ so this is impossible. We conclude that $X$ is irreducible. $\square$

## 19.2 Lines on hypersurfaces

As an important application, we will study the existence of lines on hypersurfaces of fixed degree. Let $S = K[x_0, \dots, x_n]$, let $d \geq 1$ be an integer number, then $\mathbb{P}(S_d)$ is a projective space of dimension $N = \binom{n+d}{d} - 1$, parametrizing the hypersurfaces of degree $d$ in $\mathbb{P}^n$. Among them there are reducible and even non-reduced hypersurfaces (i.e. those corresponding to non square-free polynomials). Let us introduce the *incidence correspondence* line-hypersurface

$V_p(F)$ hypersurface of deg $d$ in $\mathbb{P}^n$

$F \in K[x_0 \dots x_n]_d$



$G(1,n)$ $\{$lines contained in $V_p(F)\}$

$x_n = \lambda_0 p_{0n} + \lambda_1 p_{1n} - - + \lambda_{n-1} p_{n-1,n}$

as follows. Let $\mathbb{G}(1, n)$ be the Grassmannian parametrising the lines in $\mathbb{P}^n$. We consider the product variety $\mathbb{G}(1, n) \times \mathbb{P}(S_d)$, whose points are the pairs $(\ell, [F])$, where $\ell$ is a line in $\mathbb{P}^n$ and $F \in S_d$, that we can identify with the hypersurface $V_P(F)$. By definition the incidence variety (or correspondence) is $\Gamma_d := \{(\ell, [F]) \mid \ell \subset V_P(F)\} \subset \mathbb{G}(1, n) \times \mathbb{P}(S_d)$.

$x_0 = \lambda_1 p_{10} + - + \lambda_n p_{n0}$
$= -\lambda_1 p_{01} - - - -\lambda_n p_{0n}$

**Proposition 19.2.1.** *$\Gamma_d$ is a projective algebraic set, i.e. it is the set of zeros of a set of bihomogeneous polynomials in two series of variables: the Plücker coordinates $p_{ij}$ on the Grasmannian and the coefficients $a_{i_0 \dots i_n}$ of $F$.*

*Proof.* Let $P = (p_{ij})$ be the skew-symmetric matrix, whose elements are the coordinates of a line $\ell$: it has rank two and from Proposition 18.2.5, it follows that each non-zero row of $P$ contains the coordinates of a point of $\ell$. So the rows of $P$ are a system of generators of a vector subspace $W$ of dimension 2, such that $\ell = \mathbb{P}(W)$. Hence the coordinates of any point of $\ell$ are linear combinations of the rows of $P$, of the form $(x_0 = \Sigma_i \lambda_i p_{0i}, \dots, x_n = \Sigma_i \lambda_i p_{ni})$. A line $\ell$ is contained in $V_P(F)$ if and only if the equation $F(\Sigma_i \lambda_i p_{0i}, \dots, \Sigma_i \lambda_i p_{ni}) = 0$ is an identity in $\lambda_0, \dots, \lambda_n$. Therefore, $\Gamma_d$ is the set of common zeros of the coefficients of the monomials of degree $d$ in $\lambda_0, \dots, \lambda_n$: they are homogeneous of degree 1 in the coefficients of $F$ and of degree $d$ in the $p_{ij}$'s. $\square$

$\lambda_0, -, \lambda_n$ : linear combinations of the rows of P w. coeff. $\lambda_0 -, \lambda_n$

$$\begin{pmatrix} 0 & p_{01} & p_{02} & - & - & p_{0n} \\ -p_{01} = p_{10} & 0 & p_{12} & & - & \\ \vdots & & & & & \\ p_{n0} & & & & & \\ -p_{0n} & & & & & \end{pmatrix}$$

**Example 19.2.2.**

Let $n = d = 3$, $F = x_0^3 - x_1 x_2 x_3 \in S_3$. We put

$$\begin{cases} x_0 = \lambda_1 p_{01} + \lambda_2 p_{02} + \lambda_3 p_{03} \\ x_1 = -\lambda_0 p_{01} + \lambda_2 p_{12} + \lambda_3 p_{13} \\ x_2 = -\lambda_0 p_{02} - \lambda_1 p_{12} + \lambda_3 p_{23} \\ x_3 = -\lambda_0 p_{03} - \lambda_1 p_{13} - \lambda_2 p_{23} \end{cases}$$

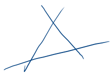
then we replace in $F$, and we get the identity $(\lambda_1 p_{01} + \lambda_2 p_{02} + \lambda_3 p_{03})^3 - (-\lambda_0 p_{01} + \lambda_2 p_{12} + \lambda_3 p_{13})(-\lambda_0 p_{02} - \lambda_1 p_{12} + \lambda_3 p_{23})(-\lambda_0 p_{03} - \lambda_1 p_{13} - \lambda_2 p_{23}) = 0$. By equating to zero the coefficients of the 20 monomials of degree 3 in $\lambda_0, \dots, \lambda_3$ we get the equations representing the lines contained in $V_P(F)$.

$\Gamma_3$

$$\binom{3+3}{3} = \binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{6} = 20$$

$V_P(F) \cap V_P(x_0)$

As a matter of fact, for this particular surface finding the lines contained in it is particularly simple. Indeed, we can distinguish the lines contained in the hyperplane "at infinity" from the lines which are projective closure of a line in $\mathbb{A}^3$. The first ones are contained in $x_0 = 0$, and it is clear that there are only three of them: $x_0 = x_1 = 0, x_0 = x_2 = 0, x_0 = x_3 = 0$. To find the others we dehomogenize $F$ and get the equation $x_1 x_2 x_3 - 1 = 0$, and consider the parametrization of a general line in $\mathbb{A}^3$: $x_i = a_i t + b_i$, $i = 1, 2, 3$. By substituting, we



$$\begin{tikzcd} \Gamma_3 \subset \mathbb{G}(1,3)\times\mathbb{P}(S_3) \arrow[d, "p_2"'] \\ \mathbb{P}(S_3) \end{tikzcd}$$

$p_2(\Gamma_3)$ proj. subvar. of $\mathbb{P}(S_3)$

$\parallel$

$\{$hypersurfaces containing at least 1 line$\}$

immediately see that there are no solutions. We conclude that the surface contains only three lines.

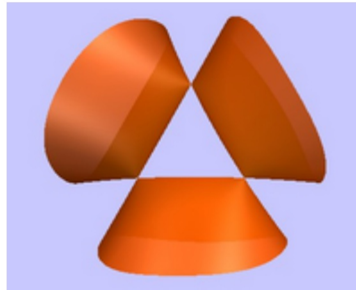


Figure 19.1: The cubic surface of Example 19.2.2

We consider now the restrictions to $\Gamma_d$ of the two projections, and we get $\varphi_1 : \Gamma_d \to \mathbb{G}(1,n)$, $\varphi_2 : \Gamma_d \to \mathbb{P}(S_d)$. We will see now that the fibres of $\varphi_1$ are all irreducible and of the same dimension; this will allow to compute the dimension of $\Gamma_d$ and get informations on the fibres of $\varphi_2$.

1. $\varphi_1(\Gamma_d) = \mathbb{G}(1,n)$, because any line $\ell$ is contained in some hypersurface of degree $d$. Indeed, up to a change of coordinates, we can assume that $\ell : x_0 = x_1 = \cdots = x_{n-2} = 0$. So $\ell \subset V_P(F)$ if and only if $F(0,\ldots,0,x_{n-1},x_n) \equiv 0$, if and only if the coefficients of the monomials containing only $x_{n-1}, x_n$ vanish, i.e. $F$ is of the form $x_0G_0 + \cdots + x_{n-2}G_{n-2}$. So $\varphi_1^{-1}(\ell)$ is a linear subspace of dimension $N-(d+1)$, because the $d+1$ monomials $x_{n-1}^d, x_{n-1}^{d-1}x_n, \ldots, x_n^d$ don't appear in $F$. In particular we have that the fibres of $\varphi_1$ are all irreducible and of the same dimension. By applying Proposition 19.1.3, we obtain that $\Gamma_d$ is irreducible of dimension $\dim \mathbb{G}(1,n) + \dim \varphi_1^{-1}(\ell) = 2(n-1) + N - (d+1)$.

2. Consider now $\varphi_2 : \Gamma_d \to \mathbb{P}(S_d) = \mathbb{P}^N$. If $\dim \Gamma_d < N$, then $\varphi_2$ cannot be surjective. This happens if

$$\dim(\Gamma_d) = 2(n-1) + N - (d+1) < N \text{ if and only if } d > 2n-3.$$

We have proved the following theorem.

**Theorem 19.2.3.** *If $d > 2n - 3$, there is an open non-empty subset $U \subset \mathbb{P}(S_d)$, such that if $[F] \in U$ then the hypersurface $V_P(F)$ does not contain any line; shortly, a "general" hypersurface of degree $d > 2n - 3$ in $\mathbb{P}^n$ does not contain any line. The hypersurfaces containing a line form a proper closed subset in $\mathbb{P}(S_d)$.*

**Example 19.2.4.** Let $n = 3$, the case of surfaces in $\mathbb{P}^3$. Theorem 19.2.3 says that a general surface of degree $\geq 4$ does not contain any line. Let us analyse the cases $d = 1, 2, 3$.

• $d = 1$: the surface is a plane, the lines contained in a plane form a $\mathbb{P}^2$.

• $d = 2$: the surface is a quadric, any quadric contains lines, and precisely, if its rank is 4, it contains two families of dimension 1 parametrised by two conics in $\mathbb{G}(1, 3)$; if the rank is 3, the quadric is a cone, and it contains a family of dimension 1 of lines, parametrised by a conic in $\mathbb{G}(1, 3)$. In both cases of rank $3, 4$ the fibres of $\varphi_2$ have dimension 1. If the rank is 2 or 1, the quadric is a pair of distinct planes or one plane with multiplicity 2, and the fibres of $\varphi_2$ have dimension 2.

• $d = 3$: in this case $N = 19 = \dim \Gamma_d$. Two cases can occur: either $\varphi_2$ is surjective, and a general fibre has dimension 0, or it is not surjective. In the second case, $\varphi_2(\Gamma_3)$, the variety of the cubic surfaces containing at least one line, has dimension $< 19$, so the fibres of $\Gamma_3 \to \varphi_2(\Gamma_3)$ have all dimension $> 0$. Hence, if a cubic surface contains a line, it contains by consequence infinitely many lines. But in Example 19.2.2 we have seen an explicit example of a cubic surface containing finitely many lines; this shows that the first possibility occurs, i.e. a "general" cubic surface contains finitely many lines. Theorem 19.1.1 explains the meaning of the adjective "general": it means that the property holds true in an open dense subset of $\mathbb{P}^{19}$.

It is a classical fact that any smooth cubic surface contains exactly 27 lines, whose configuration is completely described (see for instance [rH]). Figure 19.2 shows the Clebsch cubic surface, the only one having 27 real lines. In particular, among these 27 lines there are many pairs of skew lines.

It is a nice application of the theory we have developed so far to prove that such a cubic surface is rational.

**Theorem 19.2.5.** *Let $S \subset \mathbb{P}^3$ be a cubic surface containing two skew lines. Then $S$ is rational.*

*Proof.* Let $\ell, \ell'$ be two skew lines contained in $S$. For any point $P \in \mathbb{P}^3$, $P \notin \ell \cup \ell'$, there is exactly one line $r_P$ passing through $P$ and meeting both $\ell$ and $\ell'$: $r_P$ is the intersection of the two planes passing through $P$ and containing $\ell$ and $\ell'$ respectively. So we can consider the rational map $f : \mathbb{P}^3 \dashrightarrow \ell \times \ell' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, such that $f(P) = (r_P \cap \ell, r_p \cap \ell')$, the pair of points of intersection of $r_P$ with $\ell$ and $\ell'$. We consider now the restriction $\bar{f}$ of $f$ to $S$, and we get a birational map. Indeed, for any pair of points $x \in \ell$ and $x' \in \ell'$, the line joining $x$ and $x'$, if not contained in $S$, meets $S$ in a third point. Since not all lines meeting $\ell$ and $\ell'$ can be contained in $S$, this defines the rational inverse of $\bar{f}$. Therefore $S$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, that is birational to $\mathbb{P}^2$. By transitivity we conclude that $S$ is rational. $\square$

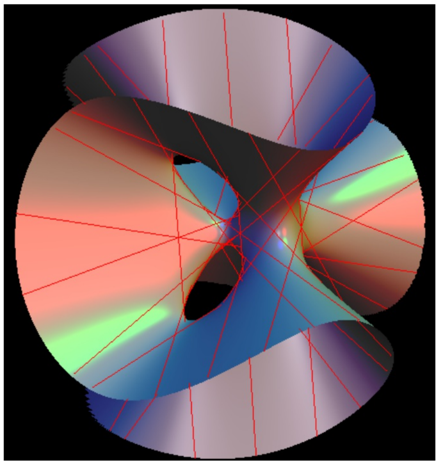


Figure 19.2: The Clebsch cubic surface

Possible equations for the Clebsch cubic surface, for different choices of coordinates, are

$$x^2y + y^2z + z^2w + w^2x = 0$$

or

$$x_0 + x_1 + x_2 + x_3 + x_4 = x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0.$$

The following equation represents the Cayley cubic surface with 4 singular points of multiplicity 2, containing 9 lines

$$xyz + yzw + zwx + wxy = 0.$$

Figure 19.2 is the image of such a surface.

A list of all possible types of singularities of cubic surfaces, with figures, can be found in the following web page: https://singsurf.org/parade/Cubics.php

$\mathbb{G}(k,n) = \{ \mathbb{P}^k \subseteq \mathbb{P}^n \}$

$X \subseteq \mathbb{P}^n$ proj. $\quad \{ L \subseteq X \mid L = \mathbb{P}^k \}$ is a subvariety of $\mathbb{G}(k,n)$

Fano variety of $k$-spaces $\subseteq X$

$k = 1$ $\quad$ $X$ surface of deg 3 $\subseteq \mathbb{P}^3$

$X$ smooth $\Rightarrow X \supseteq 27$ lines

$X$ singular $\begin{cases} X \text{ is ruled, } X \text{ has a sing. line} \\ X \text{ has isolated sing. points} \end{cases}$

$X \subseteq \mathbb{P}^3$ $\quad \deg X > 3$ : a general $X \not\supset$ lines

$X \subset \mathbb{P}^n$ hypersurface: $\quad \deg X > n$, $X$ general $\Rightarrow X \not\supset$ line

$\deg X = n$, $X$ general : $X \supseteq$ finitely many lines

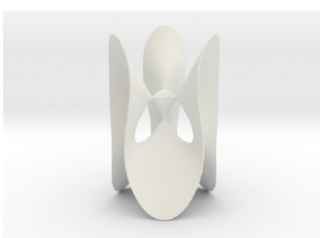


Figure 19.3: The Cayley cubic surface

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