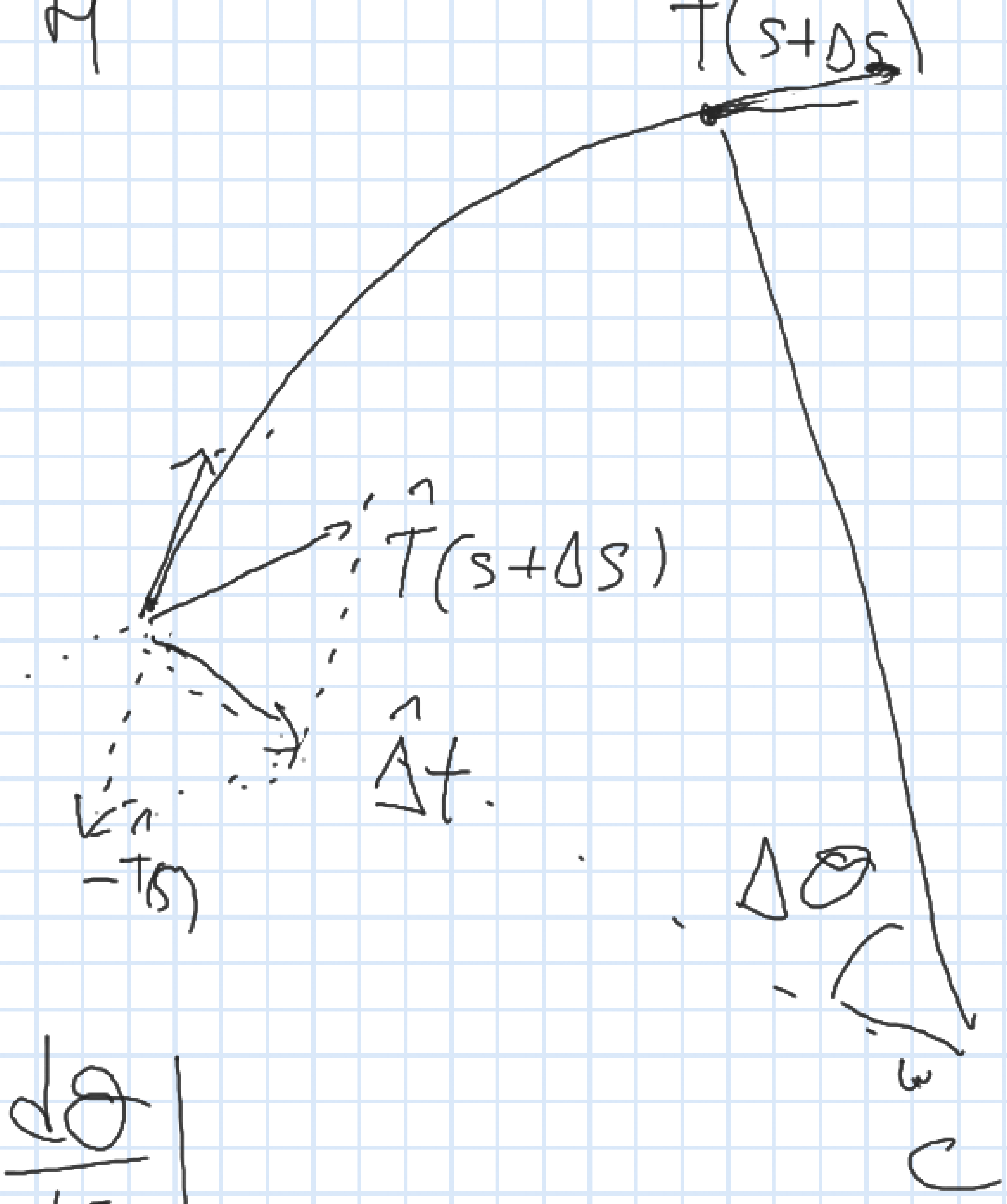
$S^2 - H^2$

$\bar{x} \quad (x_1(t), x_2(t))$

s

$\hat{t}(s)$



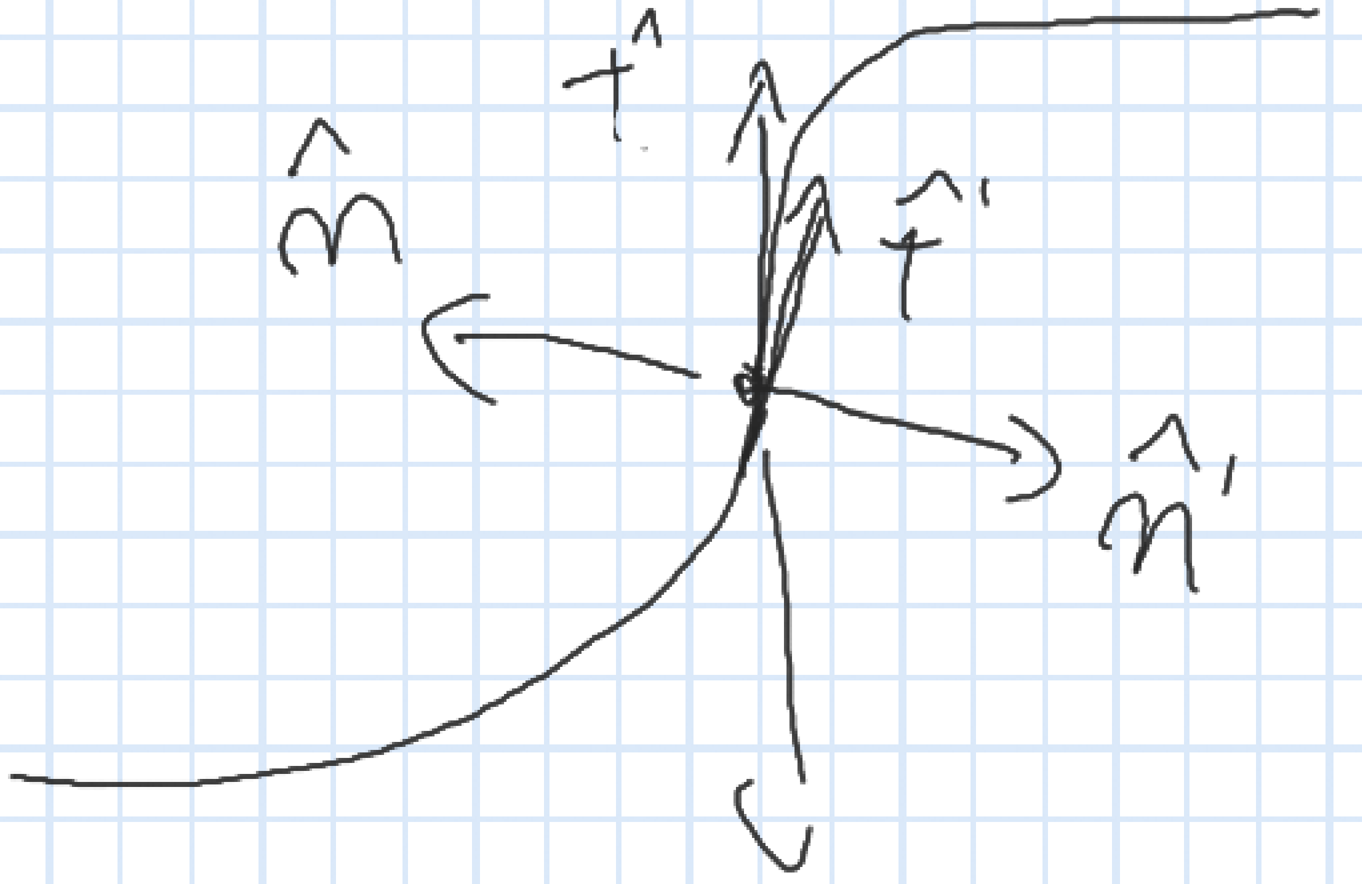
$\hat{\Delta t}$

$$\frac{d\hat{t}}{ds} = \kappa \hat{n}$$

κ = CURVATURA

$\frac{1}{\kappa} \equiv \rho =$ RAGGIO DI CURVATURA

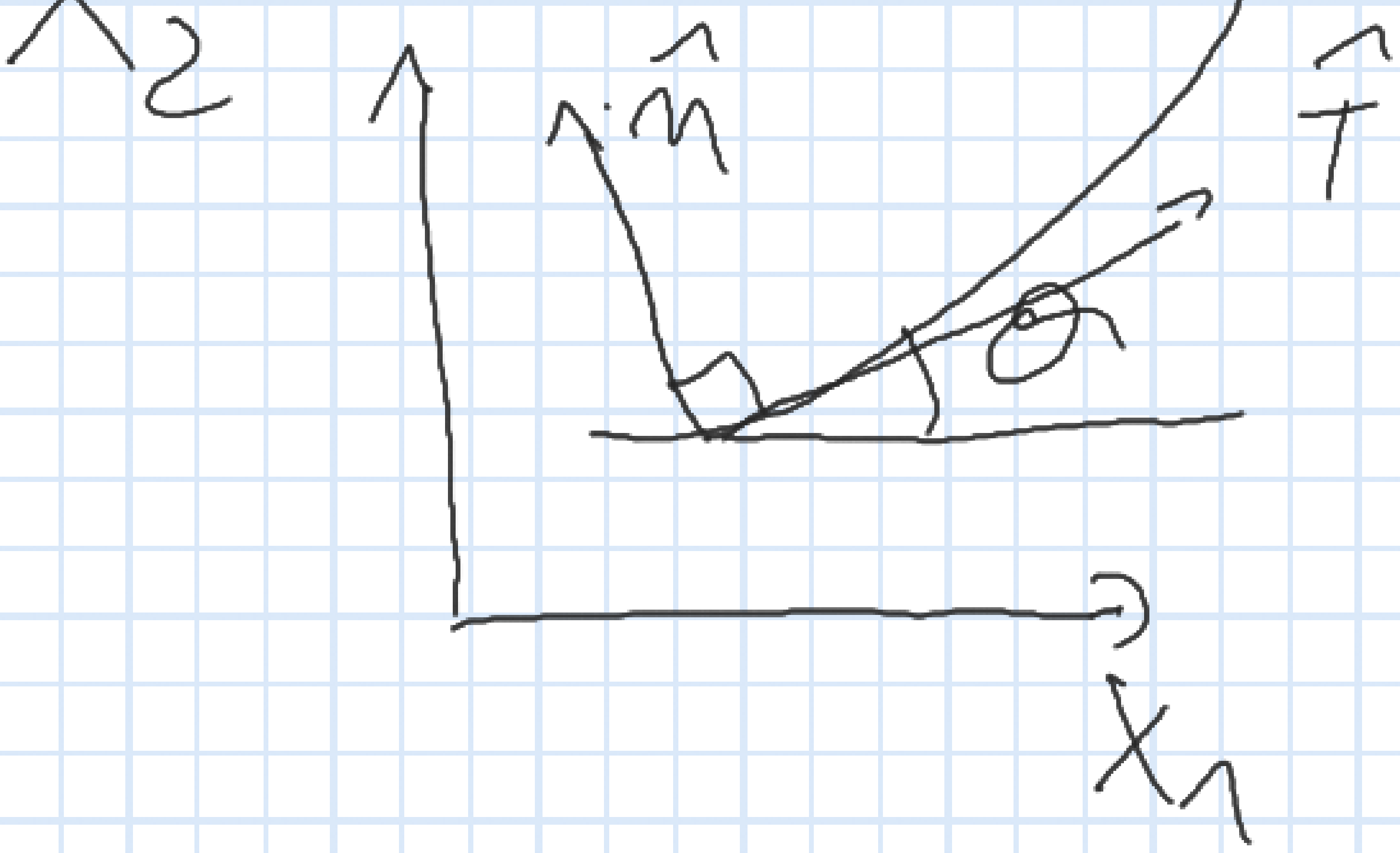
$$\kappa = \left| \frac{d\theta}{ds} \right|$$



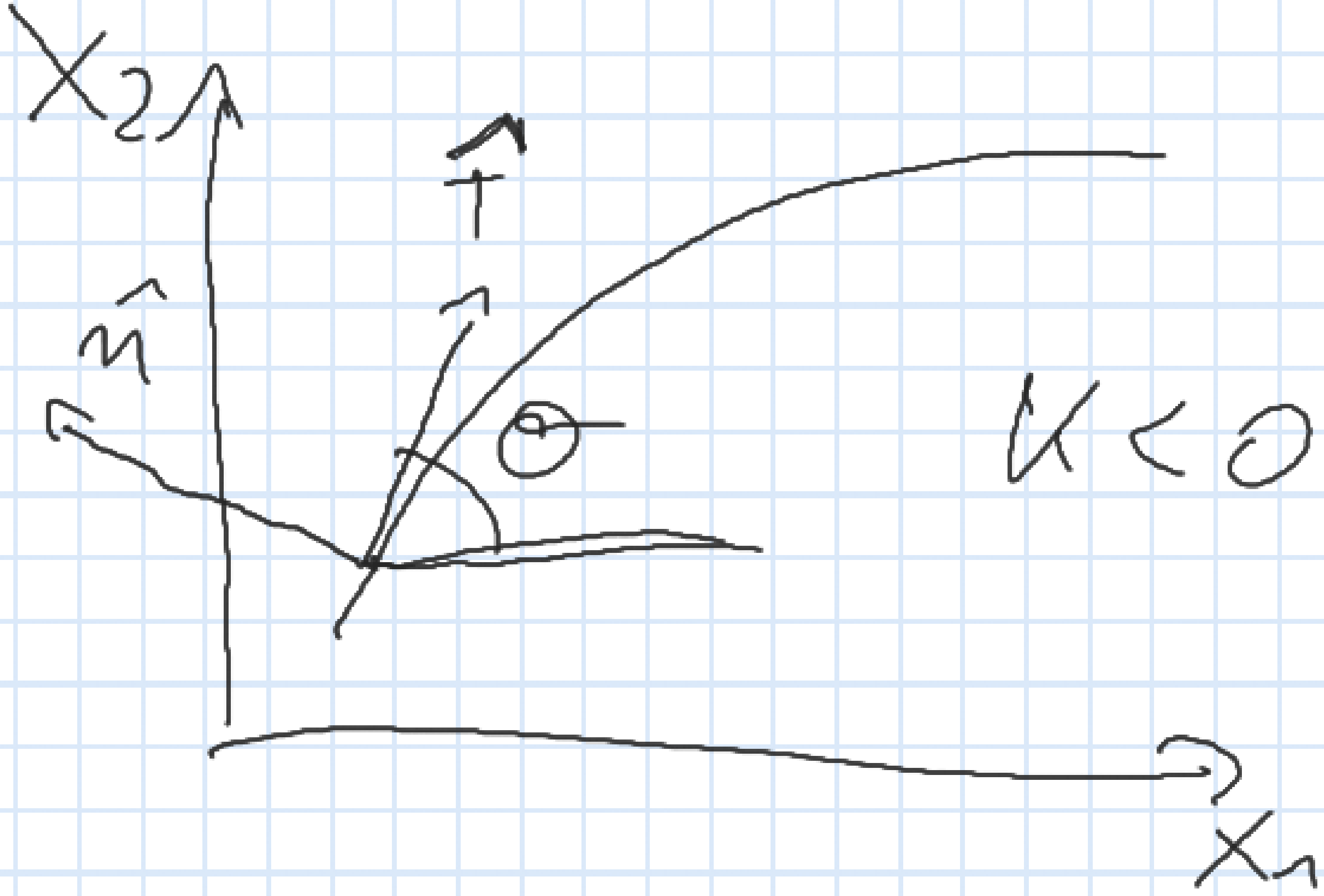
flesso

$$k \geq 0 \equiv \left| \frac{d\theta}{ds} \right|$$

$$K = \frac{d\theta(s)}{ds}$$



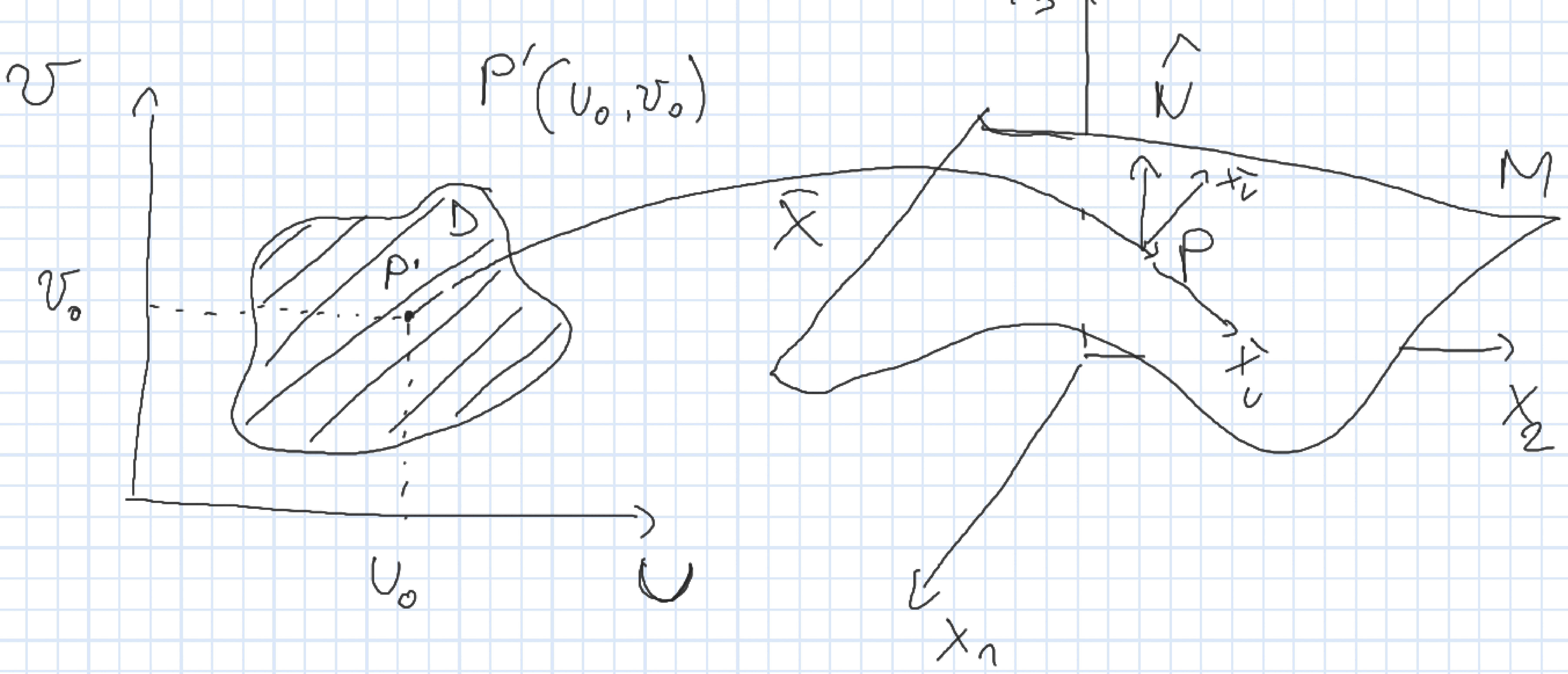
$K > 0$



$K < 0$

SUPERFICI (E^3)

$$\bar{x} : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$



$$\bar{x}(u,v) = \left(x_1(u,v),\ x_2(u,v),\ x_3(u,v) \right)$$

$$z = f(x,y) \Rightarrow x_3 = f(x_1, x_2) \qquad \bar{x}(u,v) = \left(u, v, f(u,v) \right)$$

$$\bar{x}_u(u,v) = \frac{\partial \bar{x}}{\partial u} = \left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u} \right)$$

$$\bar{x}_v(u,v) = \frac{\partial \bar{x}}{\partial v} = \left(\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v} \right)$$

M È SUPERFICIE REGOLARE (SMOOTH)

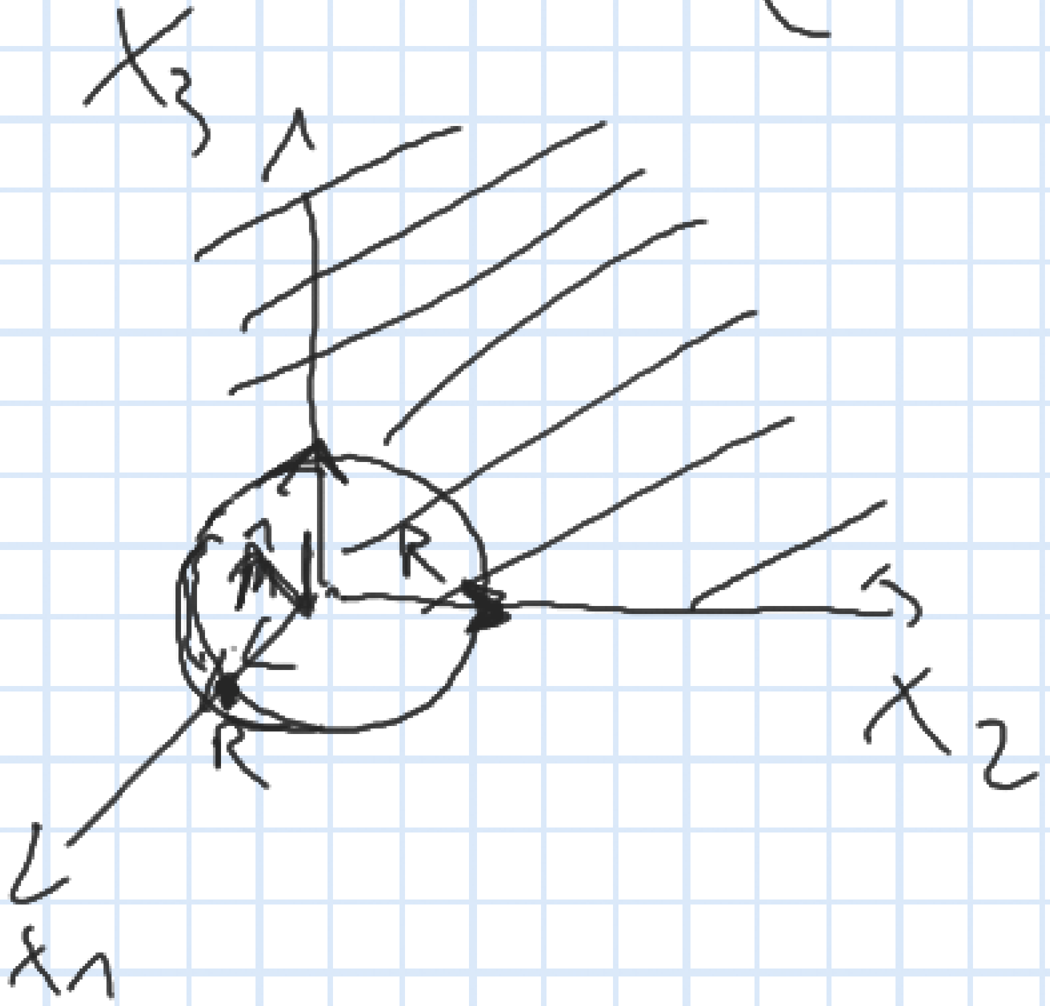
$$\bar{X}_u \times \bar{X}_v \neq 0$$

$$\hat{N} = \frac{\bar{X}_u \times \bar{X}_v}{|\bar{X}_u \times \bar{X}_v|} \qquad \left(\bar{X}_u, \bar{X}_v, \hat{N}\right) \equiv \text{TRIEDRO}$$

ES: SUPERFICIE SFERA

$u \in [-\pi, \pi] \qquad v \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\bar{x}(u,v) = \left(R \cos u \cdot \cos v,\ R \sin u \cdot \cos v,\ R \sin v\right)$



$u = v = 0$

$x_1 = R \qquad =$

$x_2 = 0,$

$x_3 = 0$

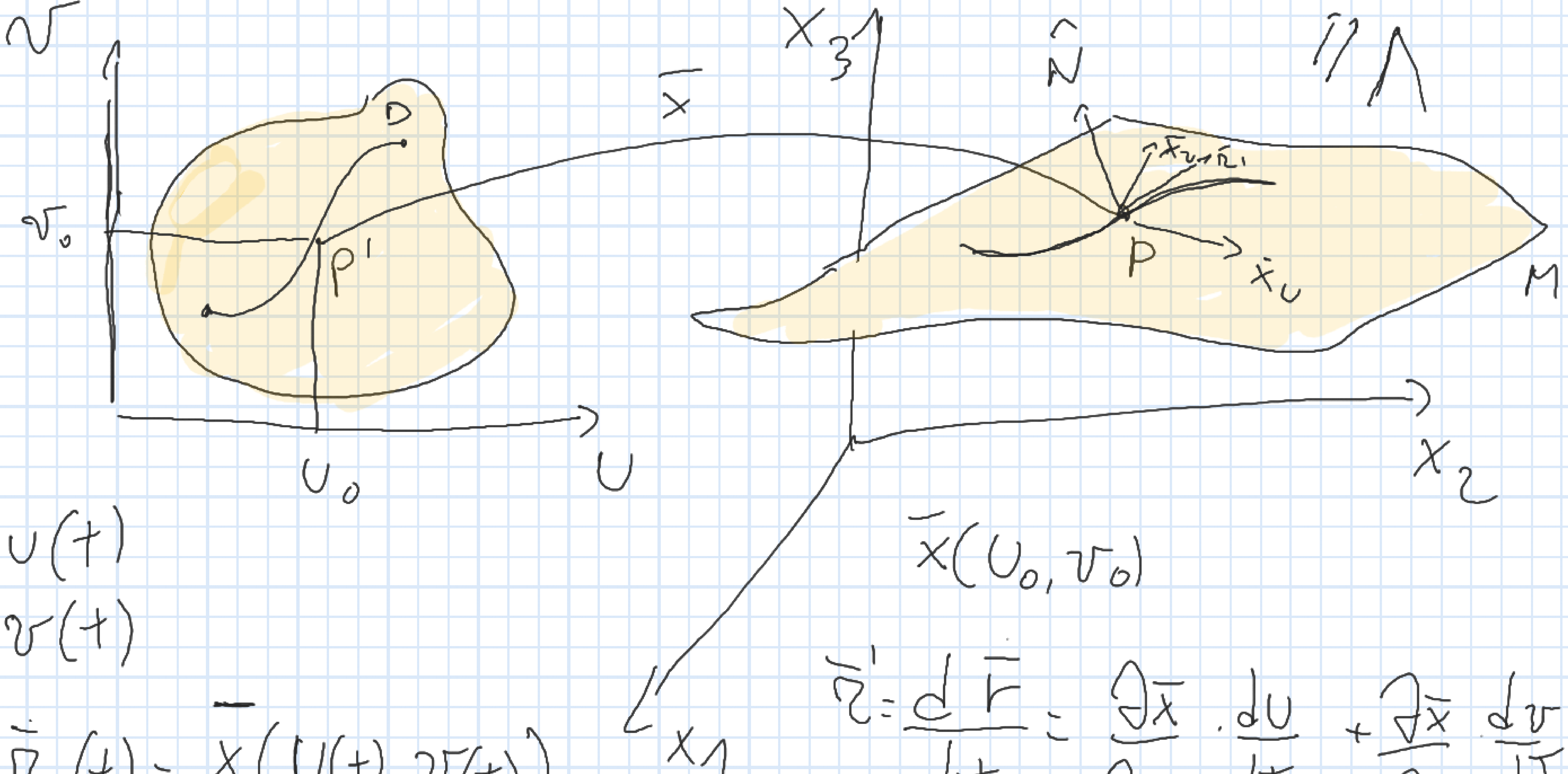
$$\bar{X}_u = \frac{\partial \bar{X}}{\partial u} = (-R \sin u \cos v, R \cos u \cos v, 0)$$

$$\bar{X}_v = \frac{\partial \bar{X}}{\partial v} = (-R \cos u \sin v, -R \sin u \sin v, R \cos v)$$

$u = 0$
$v = 0$

$$\bar{X}_u(0,0) = 0, R, 0$$

$$\bar{X}_v(0,0) = 0, 0, R$$



$u(t)$

$v(t)$

$$\bar{r}(t) = \bar{x}(u(t), v(t))$$

$$\bar{r}' = \frac{d\bar{r}}{dt} = \frac{\partial \bar{x}}{\partial u} \cdot \frac{du}{dt} + \frac{\partial \bar{x}}{\partial v} \frac{dv}{dt}$$

$$\Rightarrow \bar{x}_u \frac{du}{dt} + \bar{x}_v \frac{dv}{dt}$$

$\overline{X}_u$ e $\overline{X}_v$ $\equiv$ BASE PER IL PIANO TANGENTE

$a \leq t \leq b$ $\qquad s = S(t)$

$$L = \int_a^b ds = \int_a^b \frac{ds}{dt}\, dt = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt$$

$$\dot{r} = \overline{X}_u \cdot \frac{du}{dt} + \overline{X}_v \frac{dv}{dt} \Rightarrow$$

$$\left(\frac{ds}{dt}\right)^2 = \left|\frac{d\bar{r}}{dt}\right|^2 = \dot{\vec{r}}\cdot\dot{\vec{r}} = (\bar{x}_u\dot{u} + \bar{x}_v\dot{v})\cdot(\bar{x}_u\dot{u} + \bar{x}_v\dot{v})$$

$$= \dot{u}^2\underbrace{(\bar{X}_u\cdot\vec{X}_u)}_{E} + 2\dot{u}\dot{v}\underbrace{(\bar{x}_u\cdot\bar{X}_v)}_{F} + \dot{v}^2\underbrace{(\bar{x}_v\cdot\bar{X}_v)}_{G}$$

$$\left(\frac{ds}{dt}\right)^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

$$L = \int_a^b \left[E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2 \right]^{1/2} dt$$

$$= \int_{\vec{r}} ds = \int_{\vec{r}} \left[E\,du^2 + 2F\,du\,dv + G\,dv^2 \right]^{1/2}$$

$$\boxed{ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

I FORMA FONDAMENTALE O FORMA METRICA

FORMA METRICA DETERMINA COMPLETAMENTE LA GEOMETRIA "INTRINSECA" DI UNA SUPERFICIE (INCLUSA LA CURVATURA)

ES IL PIANO

$$\bar{x}(u,v) = (u,v,0) \to \text{in } E^3$$

$x=u$ & $y=v$ $\quad \bar{x}_u = (1,0,0) \quad \bar{x}_v = (0,1,0)$

$$E = \bar{x}_u \cdot \bar{x}_u = 1 \qquad F = \bar{x}_u \cdot \bar{x}_v = 0$$

$$G = \bar{x}_v \cdot \bar{x}_v = 1$$

$$ds^2 = du^2 + dv^2 = dx^2 + dy^2$$

$$L = \int_A^B \left[\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2 \right]^{1/2} dt$$

SFERA IN COORDINATE SFERICHE

$$\bar{x}(u,v) = \left(R \cos u \cos v, \; R \sin u \cos v, \; R \sin v \right)$$

$$\bar{X}_u = \frac{\partial \bar{X}}{\partial u} = \left(-R\sin u\cos v,\ R\cos u\cos v,\ 0\right)$$

$$\bar{X}_v = \frac{\partial \bar{X}}{\partial v} = \left(-R\cos u\sin v,\ -R\sin u\sin v,\ R\cos v\right)$$

$$\underline{E} = \bar{X}_u \cdot \bar{X}_u = R^2\sin^2 u\cos^2 v + R^2\cos^2 u\cos^2 v + 0$$

$$\overset{!}{=} \underline{R^2\cos^2 v}\left[\sin^2 u + \cos^2 u\right] \longrightarrow 1$$

$$G = R^2\sin^2 v\cos^2 u + R^2\sin^2 u\sin^2 v + R^2\cos^2 v$$

$$\overset{!}{=} R^2\sin^2 v\left[\cos^2 u + \sin^2 u\right] + R^2\cos^2 v = R^2\sin^2 v + R^2\cos^2 v$$

$G = R^2$

$F = \bar{x}_u \cdot \bar{x}_v = \underline{R^2 \cos v \sin v} \cancel{\sin u} \cos u - \underline{R^2 \cos u} \cancel{\sin u} \cos v$ $\sin v$

$= 0$

$$ds^2 = R^2 \cos^2 v \, du^2 + R^2 dv^2$$

$\bar{v} = a\bar{x}_u + b\bar{x}_v$ $\qquad a, b, c, d \in \mathbb{R}$

$\bar{w} = c\bar{x}_u + d\bar{x}_v$

$\boxed{\bar{v} \cdot \bar{w}} = (a\bar{x}_u + b\bar{x}_v) \cdot (c\bar{x}_u + d\bar{x}_v)$

$$= acE + bdG + adF + bcF$$

$$\begin{pmatrix} a & b \end{pmatrix} \boxed{\begin{pmatrix} E & F \\ F & G \end{pmatrix}} \begin{pmatrix} c \\ d \end{pmatrix}$$

MATRICE DELLA
PRIMA FORMA
FONDAMENTALE

IDENTITÀ DI LAGRANGE

$\overline{X}_u$

$\overline{X}_v$

$$\hat{N} = \frac{\overline{X}_u \times \overline{X}_v}{|\overline{X}_u \times \overline{X}_v|}$$

$$\overline{X}_u \times \overline{X}_v \neq 0$$

$$\boxed{|\overline{X}_u \times \overline{X}_v|^2 = \det\begin{pmatrix} E & F \\ F & G \end{pmatrix}}$$

$$|\bar{X}_u \times \bar{X}_v|^2 = \left(|\bar{X}_u| \cdot |\bar{X}_v| \sin\theta\right)^2$$

$$\sin^2\theta = (1 - \cos^2\theta)$$

$$\bar{X}_u \cdot \bar{X}_v = |\bar{X}_u|\,|\bar{X}_v| \cdot \cos\theta$$

$$= |\bar{X}_u|^2\,|\bar{X}_v|^2 \sin^2\theta = |\bar{X}_u^2|\,|\bar{X}_v|^2 (1 - \cos^2\theta)$$

$$= \underbrace{(\bar{X}_u \cdot \bar{X}_u)}_{E} \cdot \underbrace{(\bar{X}_v \cdot \bar{X}_v)}_{G} - \underbrace{(\bar{X}_u \cdot \bar{X}_v)}_{F}{}^2 =$$

$$|\bar{X}_u \times \bar{X}_v|^2 = EG - F^2 = \det\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

$$|\bar{x}_u \times \bar{x}_v| \neq 0 \qquad \Downarrow \qquad \det\begin{pmatrix} E & F \\ F & G \end{pmatrix} \neq 0$$

$$g_{11} \equiv E \qquad g_{22} \equiv G \qquad g_{12} = g_{21} \equiv F$$

$$\bar{X}_1 \equiv \bar{X}_u \qquad \bar{X}_2 \equiv \bar{X}_v \qquad (u, v) \to u^1, u^2$$

$$g_{ij} = \bar{x}_i \cdot \bar{x}_j \qquad (i, j = 1, 2)$$

$$\begin{pmatrix} g_{12} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \qquad \underline{\underline{g}} \equiv \det(g_{ij})$$

$$\left| \bar{x}_1 \times \bar{x}_2 \right|^2 = g$$

$$\vec{v} = \sum_i v^i \bar{x}_i \qquad \bar{w} = \sum_j w^j \bar{x}_j$$

$$\bar{v} \cdot \bar{w} = \sum_{ij} (v^i \bar{x}_i) \cdot (w^j \bar{x}_j) = \sum_{ij} v^i w^j \bar{x}_i \cdot \bar{x}_j = \sum_{ij} v^i w^j g_{ij}$$