$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \Rightarrow \det g_{ij} = g \qquad |\bar{x}_u \times \bar{x}_v|^2 = g$$

$$ds^2 = g_{11}(du^1)^2 + 2g_{12}\,du^1 du^2 + g_{22}(du^2)^2$$

$$\overset{!}{=} \sum_{i,j} g_{ij}\,du^i du^j$$

$$\underbrace{\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}}_{g_{ij}} \cdot \underbrace{\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}}_{g^{ij}} = \underline{I}$$

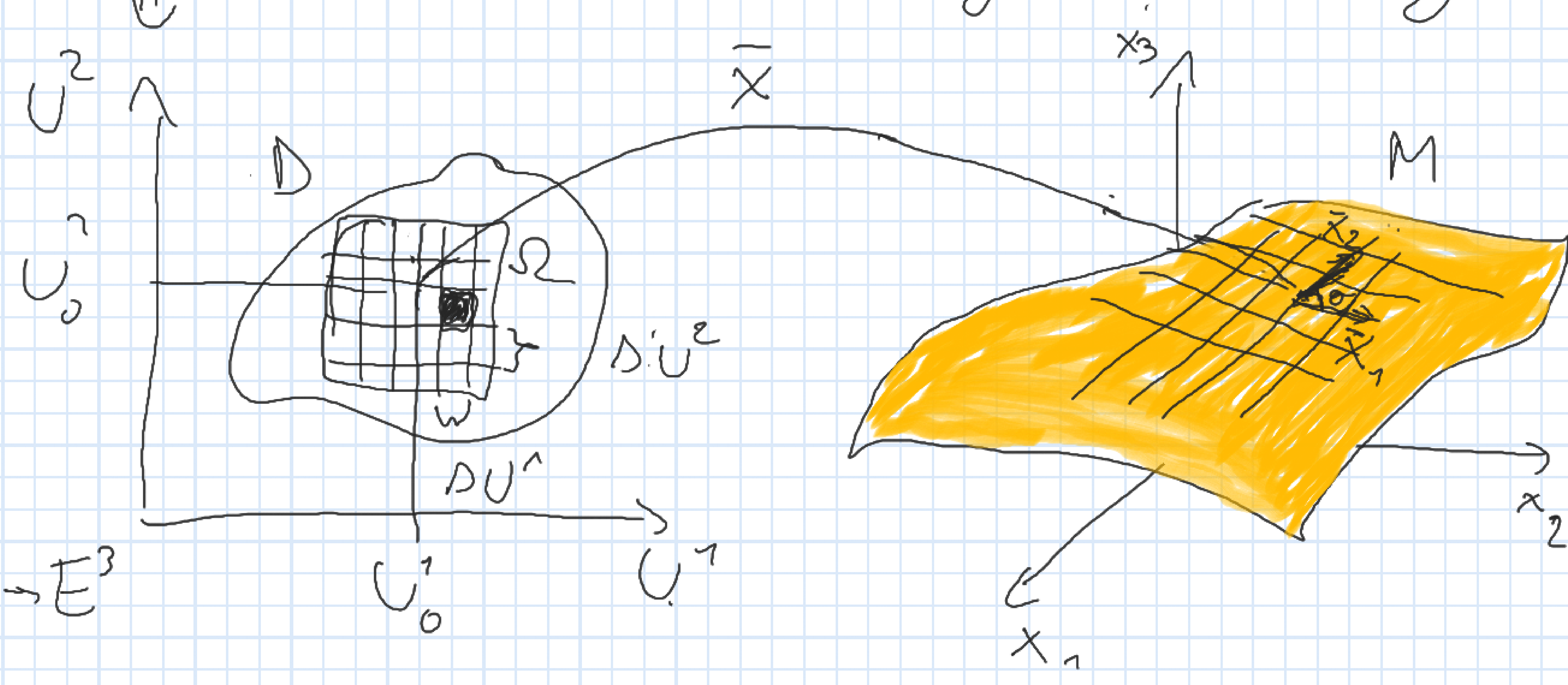
$$\underline{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \sum_{j} g_{ij} \cdot g^{jk} = \delta_i^{k}$$

DELTA DI KRONECKER

$$g^{11} = \frac{g_{22}}{g} \qquad g^{12} = g^{21} = -\frac{g_{12}}{g} \qquad g^{22} = \frac{g_{11}}{g}$$

$\Downarrow$

A

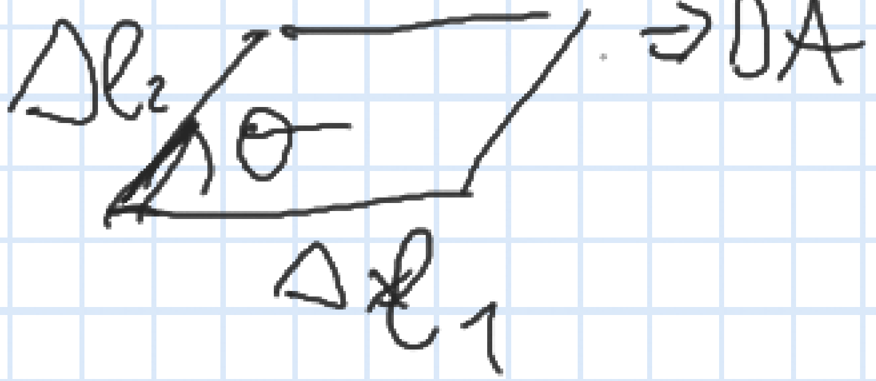


$\Omega \subset D$

$\bar{x} : D \to E^3$

$$\Delta\Omega = \Delta u^1 \cdot \Delta u^2 \qquad \Delta l_1 \simeq |\bar{x}_1|\,\Delta u^1 \qquad \Delta l_2 \simeq |\bar{x}_2|\,\Delta u^2$$

$$\bar{x}_1 = \frac{\partial \bar{x}}{\partial u^1} \qquad \Delta \bar{x}_1 = \frac{\partial \bar{x}}{\partial u^1} \cdot \Delta u^1$$



$$\Delta A = |\bar{x}_1|\,\Delta u^1 \cdot |\bar{x}_2|\,\Delta u^2 \cdot \sin\theta$$

$$\overset{!}{=} |\bar{x}_1 \times \bar{x}_2| \cdot \Delta u^1 \cdot \Delta u^2 = \sqrt{g}\,\Delta u^1 \Delta u^2$$

$$\Delta u^i \to 0 \quad \Rightarrow \quad A = \iint_\Omega \sqrt{g}\, du^1 du^2 \qquad \boxed{}$$

LA MISURA SI OTTIENE INTEGRANDO $\sqrt{g}$

RIEMANNIANI : (SPAZI / MANIFOLD)

$$ds^2 > 0$$

PSEUDO-RIEMANNIANI $\quad ds^2 \left(\begin{matrix} > \\ < \\ = \end{matrix} \right) 0$

$$\sqrt{g} \Rightarrow \sqrt{|g|}$$

E.G. MINKOWSKI

ES. 1 SFERA IN COORDINATE GEOGRAFICHE

$$ds^2 = \underbrace{R^2 \cos^2 v}\, du^2 + R^2 dv^2 \qquad -\pi < u < \pi$$

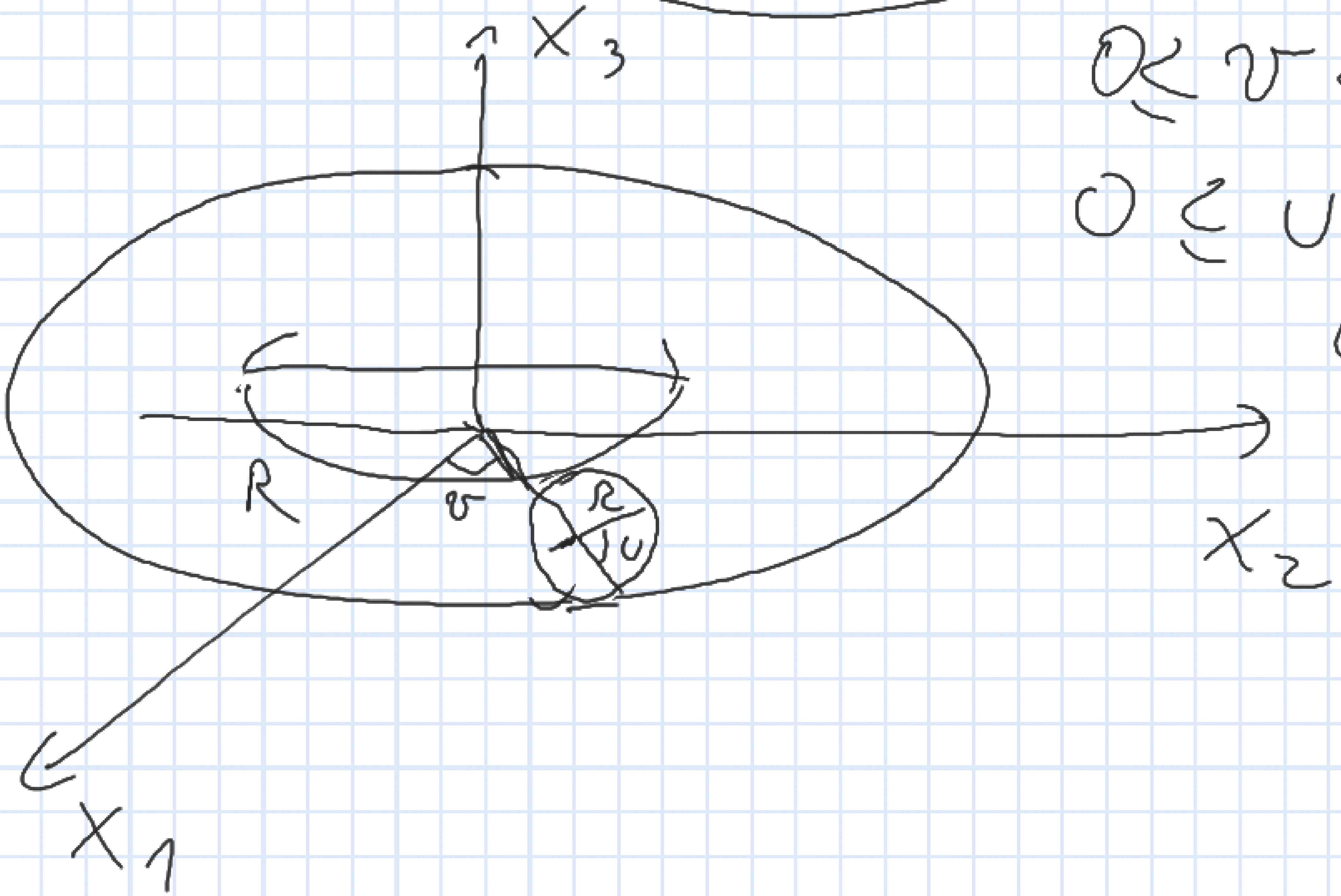
$$-\frac{\pi}{2} < v < \frac{\pi}{2}$$

$$g_{ij} = \begin{pmatrix} R^2 \cos^2 v & 0 \\ 0 & R^2 \end{pmatrix} \Rightarrow g = R^4 \cos^2 v$$

$$\sqrt{g} = R^2 \cos v \Rightarrow A = \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} \sqrt{g}\, du\, dv = \iint R^2 \cos v\, du\, dv$$

$$= 2\pi R^2 \int_{-\pi/2}^{\pi/2} \cos v \, dv = 4\pi R^2$$

Es (2)



$0 \leq v < 2\pi$

$0 \leq u \leq 2\pi$

$0 < \varrho < R$

$$\bar{X}(u, v) = \left[(R + r\cos u)\cos v,\ (R + r\cos u)\sin v,\ r\sin u\right]$$

$$\bar{X}_u \quad , \quad \bar{X}_v$$

$$\hookrightarrow \frac{\partial \bar{X}}{\partial u} = \left[-r\sin u\cos v,\ -r\sin u\sin v,\ r\cos u\right]$$

$$\frac{\partial \bar{X}}{\partial v} = \bar{X}_v = \left[-(R + r\cos u)\sin v,\ (R + r\cos u)\cos v,\ 0\right]$$

$$\begin{pmatrix} \overline{X}_u \cdot \overline{X}_u & \overline{X}_u \cdot \overline{X}_v \\ \overline{X}_v \cdot \overline{X}_u & \overline{X}_v \cdot \overline{X}_v \end{pmatrix} \Rightarrow$$

$$g_{11} = r^2 \sin^2 u \cos^2 v + r^2 \sin^2 u \sin^2 v + r^2 \cos^2 u$$

$$\stackrel{!}{=} r^2 \sin^2 u \overset{\to 1}{\left(\cos^2 v + \sin^2 v\right)} + r^2 \cos^2 u$$

$$\stackrel{!}{=} r^2$$

$$g_{12} = g_{21} = r^2 \sin^2 u \cos v \left(R + r \cos v\right) - r \sin v \sin u \cdot$$

$$\cdot \cos v \left[R + r \cos u\right] =$$

$$\Rightarrow r \sin u \,(R + r\cos v)\left[\sin u \cos v - \sin u \cos v\right] = 0$$

$$g_{22} = (R + r\cos u)^2$$

$$\begin{pmatrix} r^2 & 0 \\ 0 & (R + r\cos u)^2 \end{pmatrix} \Rightarrow g = r^2\left[R + r\cos u\right]^2$$

$$\Rightarrow \sqrt{g} = r\left[R + r\cos u\right]$$

$$\Rightarrow A = \int_0^{2\pi}\left[\int_0^{2\pi} r(R + r\cos u)\,du\right]dv = 2\pi\, r\left[\int_0^{2\pi} R\,du + \int_0^{2\pi} r\cos u\, d\right.$$

$$= 2\pi r \left[2\pi R + r \int_0^{2\pi} \cos u \, du \right] = \underline{\underline{4\pi^2 R r}}$$