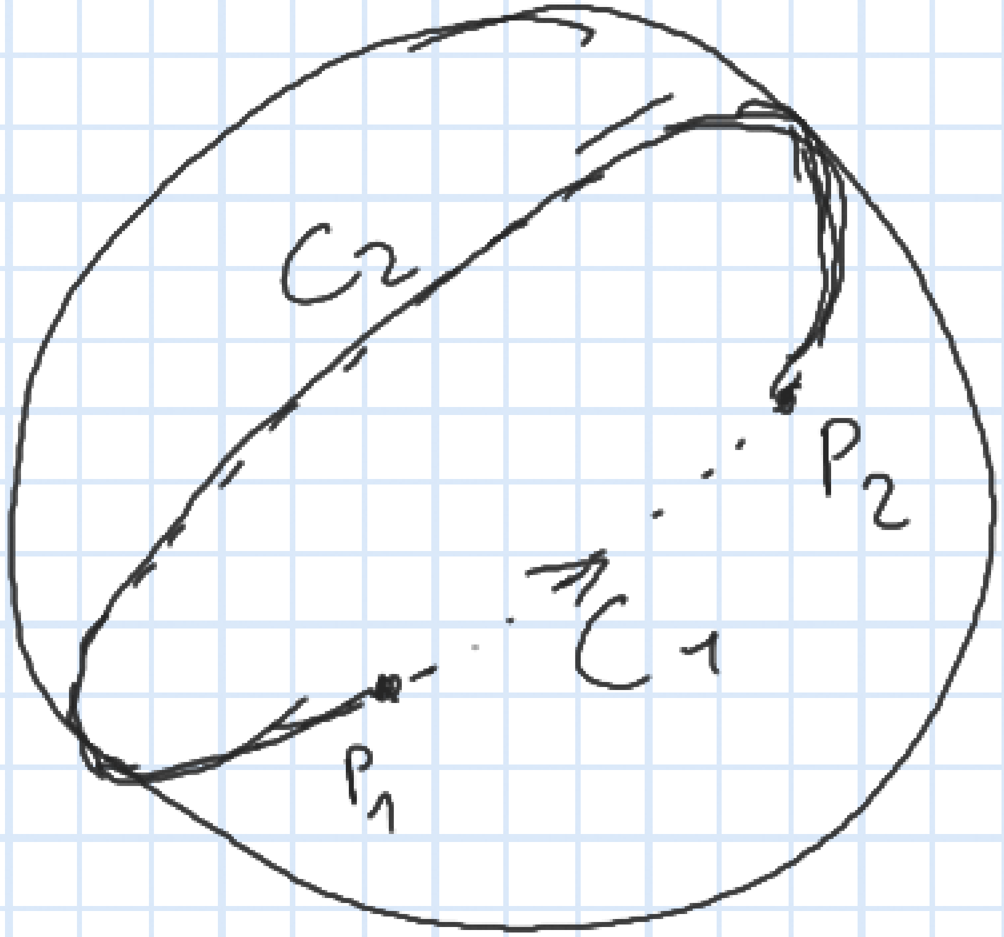
- GEODETICHE

$$\bar{r}(s) = \left(u^i(s)\right) \qquad a \leq s \leq b$$

$$P_1 = \bar{r}(a) \quad e \quad P_2 = \bar{r}(b)$$

DEF LA CURVA $\bar{r}$ È UNA GEODETICA TRA P_1 E P_2 QUANDO LA SUA LUNGHEZZA È STAZIONARIA PER PICCOLE VARIAZIONI DELLA CURVA CHE SI ANNULLANO AGLI ESTREMI



C_1 E C_2

$$ds^2 = g_{jk}\, du^j du^k$$

$$u^i = u^i(t)$$

$$ds^2 = \left(g_{jk} \underbrace{\frac{du^j}{dt}}_{\dot{u}^j} \frac{du^k}{dt} \right) dt^2$$

$$\Rightarrow L(u^i, \underbrace{\dot{u}^i}_{\frac{du^i}{dt}}, t) = \left(g_{jk} \dot{u}^j \dot{u}^k \right)^{1/2}$$

S = LUNGHEZZA CURVA TRA P_1 E P_2

$$\stackrel{!}{=} \int_{P_1}^{P_2} ds = \int_{P_1}^{P_2} L\, dt$$

EQ. DI EULERO-LAGRANGE

$$\boxed{\frac{\partial L}{\partial u^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^i}\right) = 0}$$

$$\frac{\partial L}{\partial u^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^i}\right)$$

$$L = \sqrt{g_{jk}\,\dot{u}^j \dot{u}^k} \equiv \sqrt{F}$$

$$\boxed{L = \sqrt{g_{jk}\dot{u}^j\dot{u}^j} = \sqrt{F}}$$

$$\frac{\partial L}{\partial u^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^i}\right) = \frac{1}{2\sqrt{F}}\frac{\partial g_{ij}}{\partial u^i}\dot{u}^j\dot{u}^k - \frac{d}{dt}\left[\frac{1}{2\sqrt{F}}\left(g_{ik}\dot{u}^k + g_{ji}\dot{u}^j\right)\right]$$

$$\sum_j \frac{\partial\left(g_{jk}\dot{u}^j\dot{u}^k\right)}{\partial \dot{u}^i} = \frac{\partial\left(\sum_j g_{jk}\dot{u}^j\dot{u}^k\right)}{\partial \dot{u}^i} = g_{ik}\dot{u}^k \stackrel{!}{=} 0$$

$$\Rightarrow \frac{1}{2\sqrt{F}}\frac{\partial g_{jk}}{\partial u^i}\dot{u}^j\dot{u}^j - \frac{d}{dt}\left[\frac{1}{2\sqrt{F}}\left(\underbrace{g_{ik}\dot{u}^k + g_{ji}\dot{u}^j}\right)\right] = 0$$

$$g_{ij} = g_{ji} \Rightarrow g_{ik}\dot{u}^k = \sum_k g_{ik}\dot{u}^k = \sum_j g_{ij}\dot{u}^j = g_{ij}\dot{u}^j = g_{ji}\dot{u}^j \rightarrow 2g_{ji}\dot{u}^j$$

$$\frac{1}{2\sqrt{F}} \frac{\partial g_{jk}}{\partial u^i} \dot{u}^j \dot{u}^k - \frac{d}{dt}\left(\frac{1}{\sqrt{F}}\, g_{ji}\, \dot{u}^j\right) = 0 = \qquad \frac{d g_{ji}}{dt} = \frac{\partial g_{ji}}{\partial u^l}\frac{du^l}{dt}$$

$$= \frac{1}{2\sqrt{F}} \frac{\partial g_{jk}}{\partial u^i} \dot{u}^j \dot{u}^k - \left\{ -\frac{1}{2F^{3/2}}\frac{dF}{dt}\, g_{ji}\, \dot{u}^j + \frac{1}{\sqrt{F}}\left(\frac{\partial g_{ji}}{\partial u^l}\dot{u}^l \dot{u}^j + g_{ji}\,\ddot{u}^j\right)\right\}$$

$$= 0$$

$$t \propto s \qquad T = s \qquad \frac{dF}{dt} = 0$$

$$ds = L\,dt \Rightarrow ds^2 = L^2 (dt)^2 = F\,dt^2 \Rightarrow F = \left(\frac{ds}{dt}\right)^2 \Rightarrow \frac{dF}{dt} = 2\frac{ds}{dt}\underbrace{\frac{d^2 s}{dt^2}}_{=0}$$

$$s = \alpha T + \beta \Rightarrow \frac{ds}{dt} = \alpha \Rightarrow \frac{d^2 s}{dt^2} = 0$$

$$g_{ji}\,\ddot{u}^j + \boxed{\frac{\partial g_{ji}}{\partial u^l}\,\dot{u}^l\,\dot{u}^j} - \frac{1}{2}\,\frac{\partial g_{jk}}{\partial u^i}\,\dot{u}^j\,\dot{u}^k = 0$$

$$\frac{d g_{ji}}{dt} = \frac{\partial g_{ji}}{\partial u^l}\,\frac{du^l}{dt}$$

$$\to \frac{1}{2}\left[\frac{\partial g_{ji}}{\partial u^l} + \frac{\partial g_{li}}{\partial u^j}\right]\dot{u}^l\,\dot{u}^j$$

$$\sum_{lj}\frac{\partial g_{ji}}{\partial u^l}\,\dot{u}^l\,\dot{u}^j \qquad l \to k$$

$$g_{ji}\,\ddot{u}^j + \frac{1}{2}\left[\frac{\partial g_{ji}}{\partial u^k} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i}\right]\dot{u}^j\,\dot{u}^k = 0 \quad \left[\cdot\, g^{il}\right.$$

$$\delta^l_j \ddot{u}^j + \frac{1}{2} g^{il}\left[\frac{\partial g_{ji}}{\partial u^k} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i}\right]\dot{u}^j \dot{u}^k = 0$$

$$\frac{d^2 u^l}{ds^2} + \frac{1}{2} g^{il}\left[\frac{\partial g_{ji}}{\partial u^k} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^i}\right]\frac{du^j}{ds}\frac{du^k}{ds} = 0$$

$$\frac{d^2 u^l}{ds^2} + \Gamma^l_{jk}\frac{du^j}{ds}\frac{du^k}{ds} = 0$$

SIMBOLO DI CHRISTOFFEL

$$\boxed{\Gamma^i_{jk} = \frac{1}{2} g^{il}\left(\frac{\partial g_{lj}}{\partial u^k} + \frac{\partial g_{lk}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^l}\right)} \doteq$$

CONNESSIONE AFFINE

$$\Gamma^{i}_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial u^{k}} + \frac{\partial g_{lk}}{\partial u^{j}} - \frac{\partial g_{jk}}{\partial u^{l}} \right) =$$

$$\Gamma^{i}_{kj} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lk}}{\partial u^{j}} + \frac{\partial g_{lj}}{\partial u^{k}} - \frac{\partial g_{kj}}{\partial u^{l}} \right)$$

$$\boxed{\Gamma^{i}_{kj} = \Gamma^{i}_{jk}}$$

$\Gamma^{i}_{jk} \neq$ NON È UN TENSORE

$$\boxed{\Gamma'^{l}_{m n} \neq \frac{\partial u'^{l}}{\partial u^{i}} \frac{\partial u^{j}}{\partial u'^{m}} \frac{\partial u^{k}}{\partial u'^{n}} \Gamma^{i}_{jk}}$$

$$\boxed{\frac{d^2 u^l}{ds^2} + \Gamma^l_{jk} \frac{du^j}{ds} \frac{du^k}{ds} = 0}$$

NON È UN TENSORE

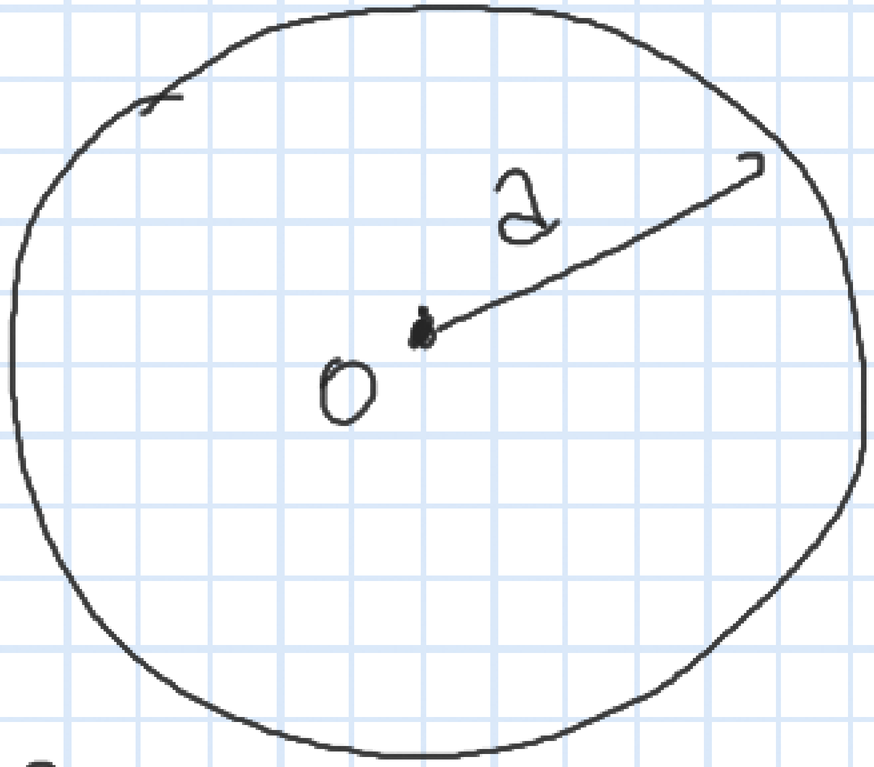
$$ds^2 = g_{ij}\, du^i du^j \qquad u \qquad v \qquad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \sum_{ij} g_{ij}\, du^i du^j = du^2 + dv^2$$

$$\frac{d^2 u}{ds^2} = 0 \quad \& \quad \frac{d^2 v}{ds^2} = 0 \Rightarrow \begin{matrix} \frac{du}{ds} = a \\ \frac{dv}{ds} = c \end{matrix} \Rightarrow \begin{cases} u = as + b \\ v = cs + d \end{cases} \Rightarrow s = \frac{u - b}{a}$$

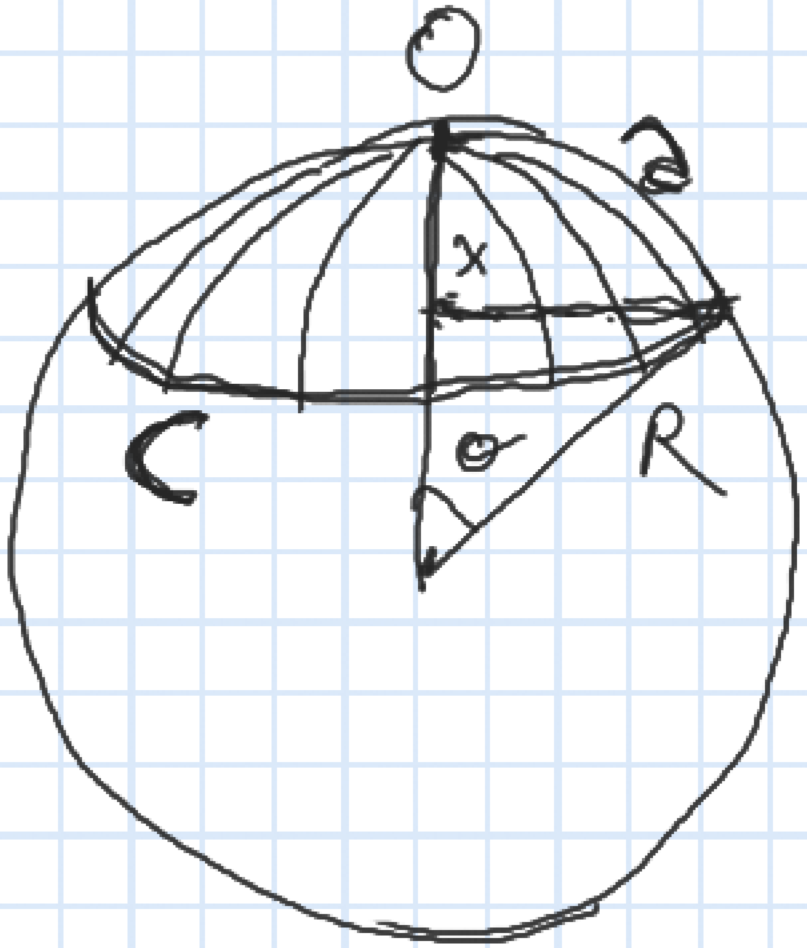
$$v = c \cdot \frac{u-b}{a} + d \Rightarrow \boxed{y = mx + q} \Rightarrow \text{EQ DI RETTA}$$

$$C = 2\pi a$$



E^3

raggio a



$$C = 2\pi x = 2\pi R \sin\theta$$

$$\theta = \frac{a}{R}$$

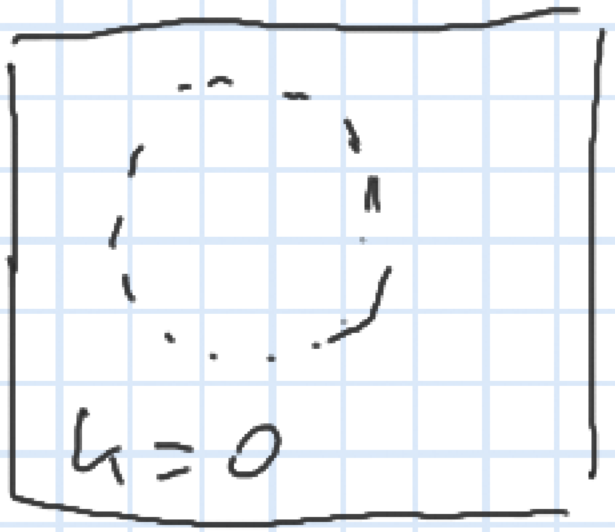
$$C = 2\pi R \sin\left(\frac{a}{R}\right) \simeq 2\pi R \left[\frac{a}{R} - \frac{1}{\underset{=6}{3!}}\frac{a^3}{R^3} + \dots\right] = 2\pi a - \frac{\pi}{3}\frac{a^3}{R^2} + O(a^5)$$

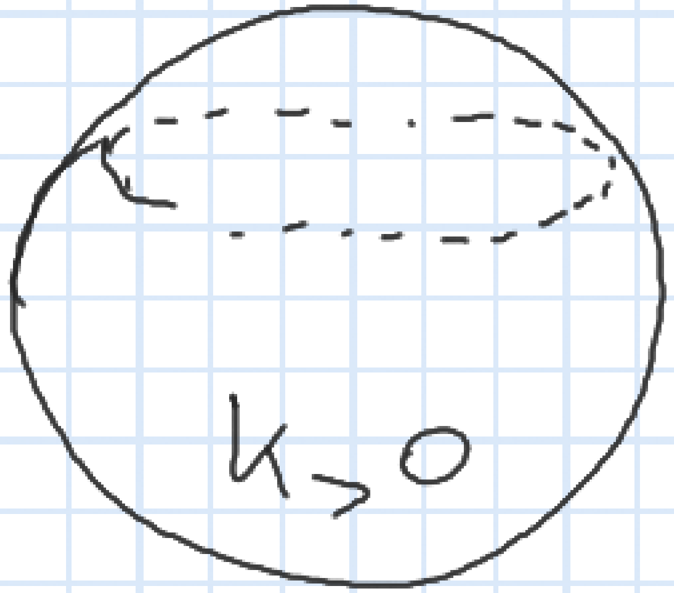
$$< 2\pi a$$

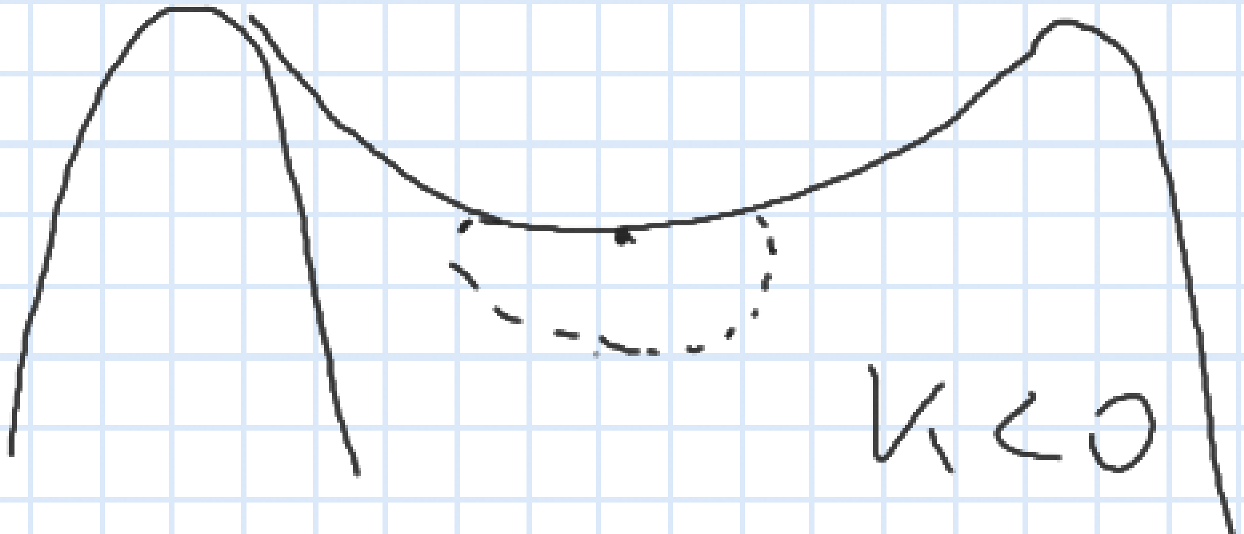
$1/R^2 = K$ ne $a \to 0$

$C = 2\pi a - \frac{\pi}{3} a^3 K + O(a^5)$ $\quad K = \left(\frac{-C + 2\pi a}{a^3}\right)\frac{3}{\pi} + O(a^5)$

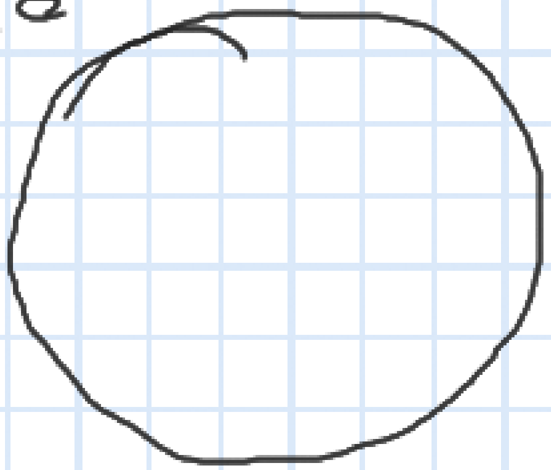
$$K = \lim_{a \to 0} \frac{3}{\pi}\left(\frac{-C + 2\pi a}{a^3}\right)$$



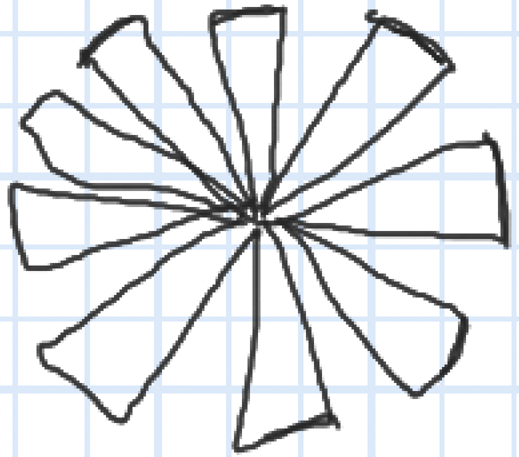




$C = 2\pi a$



$C < 2\pi a$



$C > 2\pi a$