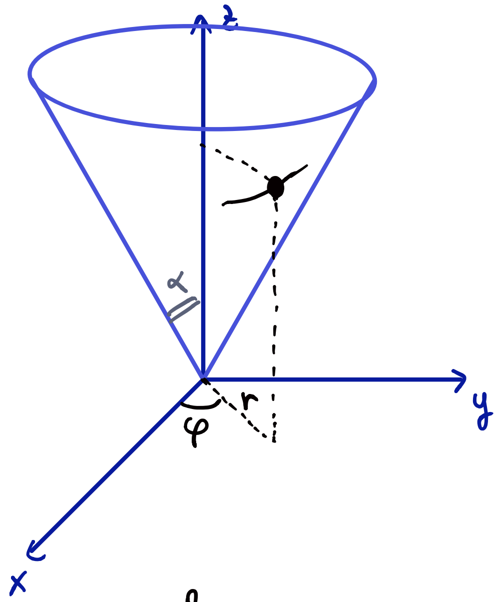
ES2



$r = \rho \sin\alpha$
$z = \rho\cos\alpha$

- Lagrangiana, eq. di Lagrange
- Costanti del moto: quali e perché
- Coord. ciclica e Lagrangiana ridotta
- Si tracci il grafico del pot. efficace e si derivi il diagramma di fase descrivendo a parole i vari pezzi
- Pti ep. e stab. nel problema ridotto
- Lin. L^* e freq. piccole oscillazioni.

1)
$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$z = r\frac{\cos\alpha}{\sin\alpha}$$

$$\dot x^2 + \dot y^2 + \dot z^2 = \dot r^2\left(1 + \frac{\cos^2\alpha}{\sin^2\alpha}\right) + r^2\dot\varphi^2 = \frac{\dot r^2}{\sin^2\alpha} + r^2\dot\varphi^2$$

$$T = \frac{1}{2}m\left(\frac{\dot r^2}{\sin^2\alpha} + r^2\dot\varphi^2\right)$$

$$L = T - V$$

$$V = mgr\frac{\cos\alpha}{\sin\alpha} = \frac{mgr}{\operatorname{tg}\alpha}$$

$$L = \frac{1}{2}m\left(\frac{\dot r^2}{\sin^2\alpha} + r^2\dot\varphi^2\right) - \frac{mgr}{\operatorname{tg}\alpha}$$

2)
$$\frac{d}{dt}\frac{\partial L}{\partial \dot r} = \frac{d}{dt}\frac{m\dot r}{\sin^2\alpha} = \frac{m}{\sin^2\alpha}\ddot r \qquad \frac{\partial L}{\partial r} = mr\dot\varphi^2 - \frac{mg}{\operatorname{tg}\alpha}$$

$$\frac{\sin^2\alpha}{\operatorname{tg}\alpha} = \frac{\sin^2\alpha}{\sin\alpha/\cos\alpha} = \sin\alpha\cos\alpha$$

$$\ddot r = r\sin^2\alpha\,\dot\varphi^2 - g\sin\alpha\cos\alpha$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\varphi} = \frac{d}{dt}\left(mr^2\dot\varphi\right) = 2mr\dot r\dot\varphi + mr^2\ddot\varphi \qquad \frac{\partial L}{\partial\varphi} = 0$$

$$2\dot r\dot\varphi + r\ddot\varphi = 0$$

3) Cost. del moto è $f : \mathbb{R}^{2n+1} \to \mathbb{R}$ t.c.
$(q,\dot q,t) \mapsto f(q,\dot q,t)$

preso $q(t)$ soluz. di eq. di Lag. allora $f(q(t),\dot q(t),t)$ è cost. in t.

1- Energia : $E = T+V$; infatti siccome $\frac{\partial L}{\partial t}=0$ allora E è cost. del moto.

2- Comp. lungo asse z del momento angolare ; esso è mom. coniug. a φ (angolo rotaz. attorno asse z) che è cost. del moto poiché φ è ciclica

$\Rightarrow \frac{d}{dt} p_\varphi = \frac{d}{dt} \frac{\partial L}{\partial \dot\varphi} = 0.$

4) φ coord. ciclica $\quad \ell = \frac{\partial L}{\partial \dot\varphi} = mr^2\dot\varphi \to \dot\varphi = \frac{\ell}{mr^2}$

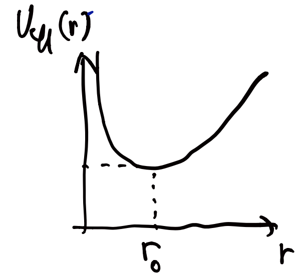
$$L = \frac{1}{2}m\left(\frac{\dot r^2}{\sin^2\alpha} + r^2\dot\varphi^2\right) - \frac{mgr}{\tan\alpha}$$

$\downarrow$

$$L^* = \frac{1}{2}m\frac{\dot r^2}{\sin^2\alpha} + \frac{\cancel{mr^2}}{2}\frac{\ell^2}{(mr^2)^{\cancel{2}}} - \frac{mgr}{\tan\alpha} - \frac{\ell^2}{mr^2}$$

$$= \frac{m}{2}\frac{\dot r^2}{\sin^2\alpha} - \frac{mgr}{\tan\alpha} - \frac{\ell^2}{2mr^2}$$

5) $V_{eff} = \frac{\ell^2}{2mr^2} + \frac{mgr}{tg\alpha}$

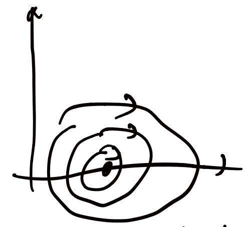


$V_{eff}' = -\frac{\ell^2}{mr^3} + \frac{mg}{tg\alpha}$ $\qquad r_0 = \left(\frac{\ell^2 tg\alpha}{m^2 g}\right)^{1/3}$

$V_{eff}'' = \frac{3\ell^2}{mr^4} > 0$

$\parallel$

$\frac{3\cancel{\ell^2}}{m r_0}\frac{m^2 g}{\cancel{\ell^2} tg\alpha} = \frac{3mg}{r_0 tg\alpha} = \frac{3mg}{tg\alpha}\left(\frac{m^2 g}{\ell^2 tg\alpha}\right)^{1/3}$



orbite limitate

6) $L^* = \frac{m}{2}\frac{\dot r^2}{\sin^2\alpha} - \frac{mgr}{tg\alpha} - \frac{\ell^2}{2mr^2}$

$r = \delta r + r_0$

$\frac{1}{2}\dot q A \dot q - \frac{1}{2} q B q \qquad B = V_{eff}''(r_0).$

$L^* = \frac{m}{2}\frac{\delta\dot r^2}{\sin^2\alpha} + \frac{3\ell^2}{2mr_0^4}\delta r^2$

$\omega^2 = \frac{B}{A} = \frac{3\ell^2}{mr_0^4}\frac{\sin^2\alpha}{m} = \frac{3\cancel{\ell^2}}{\cancel{m}r_0}\frac{\sin^2\alpha}{\cancel{m}}\frac{\cancel{m^2}g}{\cancel{\ell^2}tg\alpha}$

$= \frac{3g}{r_0}\sin\alpha\cos\alpha$

$\frac{1}{r_0} = \left(\frac{m^2 g}{\ell^2 tg\alpha}\right)^{1/3}$

Domande facoltative ES 1

$$F_2(q,\tilde{p}) = \frac{(q\tilde{p})^2}{2} \qquad p = \frac{\partial F_2}{\partial q} = (q\tilde{p})\,\tilde{p} = q\tilde{p}^2$$

$$\tilde{q} = \frac{\partial F_2}{\partial \tilde{p}} = (q\tilde{p})\,\tilde{q} = q^2\tilde{p}$$

$$q = \sqrt{\frac{\tilde{q}}{\tilde{p}}} \rightarrow p = \tilde{p}^2\sqrt{\frac{\tilde{q}}{\tilde{p}}}$$

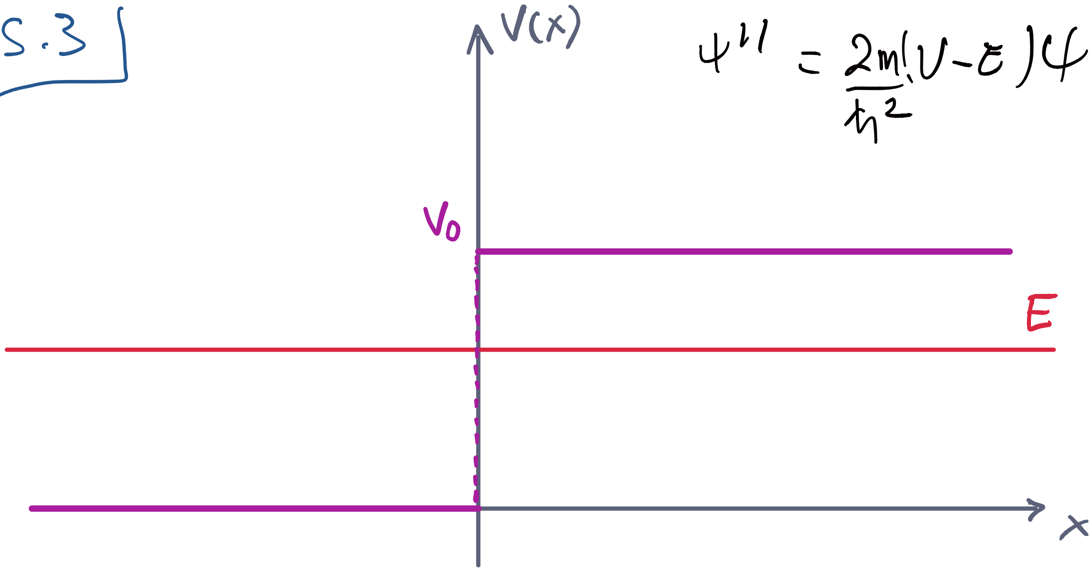
$$\begin{cases} q = \tilde{q}^{1/2}\,\tilde{p}^{-1/2} \\ p = \tilde{p}^{3/2}\,\tilde{q}^{1/2} \end{cases}$$

$$\{q,p\} = \left(\frac{1}{2}\tilde{q}^{-1/2}\tilde{p}^{-1/2}\,\frac{3}{2}\tilde{p}^{1/2}\tilde{q}^{1/2} + \right.$$

$$\left. + \frac{1}{2}\tilde{q}^{1/2}\tilde{p}^{-3/2}\,\frac{1}{2}\tilde{p}^{3/2}\tilde{q}^{-1/2}\right) = 1$$

$$\{q,q\} = 0 = \{p,p\}$$

ES.3



$$\psi'' = \frac{2m}{\hbar^2}(V-E)\psi$$

$p = \sqrt{2mE}$ $\quad q = \sqrt{2m(V_0-E)}$

$$c_1^+ e^{ipx/\hbar} + c_1^- e^{-ipx/\hbar} \qquad x<0$$

$$c_2^+ e^{-qx/\hbar}$$

$$-ip\,(c_1^+ + c_1^- = c_2^+)$$

$$ip(c_1^+ - c_1^-) = -q\,c_2^+ \qquad ip(c_1^+ - c_1^-) = -q(c_1^+ + c_1^-)$$

$$c_1^+ = \frac{ip-q}{2ip}\,c_2^+ = \frac{1}{2}\left(1+\frac{iq}{p}\right)$$

$$c_1^- = \frac{ip+q}{2ip}\,c_2^+ = \frac{1}{2}\left(1-\frac{iq}{p}\right)$$

$$c_1^- = \frac{q+ip}{-q+ip}\,c_1^+ = \frac{p-iq}{p+iq}\,c_1^+$$

$$c_2^+ = \frac{2p}{p+iq}$$

$c_2^+ \in \mathbb{R} \Rightarrow \psi^{(1)} \in \mathbb{R}$

$\psi^{(2)} \in \mathbb{R}$

$J \propto \operatorname{Im}(\psi'\psi^*)$

se $\psi \in \mathbb{R} \Rightarrow J = 0$

$J^{(1)} = J_I + J_R$

$J^{(2)} = J_T$

$\rightarrow J_T = 0 \quad J_R = -J_I$

$\Rightarrow T = 0 \quad R = 1$

$c_1^+ = 1$

$$\psi_E(x) = \begin{cases} e^{ipx/\hbar} + \dfrac{p-iq}{p+iq}\, e^{-ipx/\hbar} \\[2ex] \dfrac{2p}{p+iq}\, e^{-qx/\hbar} \end{cases}$$

$$\psi_E^* = \begin{cases} e^{-ipx/\hbar} + \dfrac{p+iq}{p-iq}\, e^{ipx/\hbar} \\[2ex] \dfrac{2p}{p-iq}\, e^{-qx/\hbar} \end{cases} = \begin{cases} \underline{\dfrac{p+iq}{p-iq}} \left(\dfrac{p-iq}{p+iq}\, e^{-ipx/\hbar} + e^{ipx/\hbar} \right) \\[2ex] \underline{\dfrac{p+iq}{p-iq}}\ \dfrac{2p}{p+iq}\, e^{-qx/\hbar} \end{cases}$$