ES 1)

$$F = \sum_s p_s \, v_s(\tilde{q}) \qquad q_h = v_h(\tilde{q})$$
$$\tilde{p}_h = \sum_s p_s \frac{\partial v_s}{\partial \tilde{q}_h}$$

6) Trasf. puntuali estese!

inversa di matrice

$$q_\ell = v_\ell(\tilde{q}, t) \qquad p_\ell = \sum_r \left(\frac{\partial v}{\partial \tilde{q}}\right)^{-1}_{r\ell} \tilde{p}_r$$

$$\{q_\ell, q_s\} = \sum_h \Big(\frac{\partial v_\ell}{\partial \tilde{q}_h} \underbrace{\frac{\partial v_s}{\partial \tilde{p}_h}}_{=0} - \underbrace{\frac{\partial v_\ell}{\partial \tilde{p}_h}}_{=0} \frac{\partial v_s}{\partial \tilde{q}_h} \Big) = 0 \qquad \checkmark$$

Per le p viene più comodo controllare $\{\tilde{p}, \tilde{p}\}$ assunto che p, q soddisfano:

$$\{\tilde{p}_\ell, \tilde{p}_s\} = \sum_{ij} \left\{ p_i \frac{\partial v_i}{\partial \tilde{q}_\ell}, p_j \frac{\partial v_j}{\partial \tilde{q}_s} \right\} =$$

$$= \sum_{ijh} p_i \sum_r \frac{\partial^2 v_i}{\partial \tilde{q}_\ell \partial \tilde{q}_r} \underbrace{\frac{\partial \tilde{q}_r}{\partial q_h} \cdot \frac{\partial v_h}{\partial \tilde{q}_s}}_{\delta_{rs}} - (\ell \leftrightarrow s)$$

$$= \sum_{ij} p_j \frac{\partial^2 v_i}{\partial \tilde{q}_\ell \partial \tilde{q}_s} - (\ell \leftrightarrow s) = 0$$

$$\{q_\ell, p_s\} = \sum_h \left(\frac{\partial v}{\partial \tilde{q}}\right)_{\ell h} \cdot \left(\frac{\partial v}{\partial \tilde{q}}\right)^{-1}_{hs} - 0 = \delta_{\ell s}$$

7) In coord. cilindriche $M_z = p_\varphi$
$(r, \varphi, z; p_r, p_\varphi, p_z)$

$$\{M_z, H\} = \{p_\varphi, H\} = 0$$

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\varphi^2}{r^2} + p_z^2\right) + V(r)$$

perché $\{p_\varphi, p_i\} = 0$ e

e $\{p_\varphi, f(r)\} = 0$.

ES 2

$$V(x,y) = k\left(\beta x^2 + \frac{y^2}{2}\right) - kL^2 \log\left(1 + \frac{x^2}{L^2} + \frac{y^2}{L^2}\right)$$

1) $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x,y)$

2) $m\ddot{x} = -\partial_x V = -\left[2k\beta x - kL^2 \frac{2x}{L^2+x^2+y^2}\right]$

$m\ddot{y} = -\partial_y V = -\left[ky - kL^2 \frac{2y}{L^2+x^2+y^2}\right]$

3) $\begin{cases} 2\beta x(L^2+x^2+y^2) - 2L^2 x = 0 \\ y(L^2+x^2+y^2) - 2L^2 y = 0 \end{cases}$

$2\beta x\left[x^2+y^2+L^2 - \frac{L^2}{\beta}\right] = 0$

$y\left[x^2+y^2+L^2 - 2L^2\right] = 0$

Su qte solutions

$x^2+y^2+L^2 = \left(\frac{1}{\beta}-1\right)L^2 + L^2 =$

$= L^2/\beta$

$(x,y) = (0,0),\ (0, \pm L),\ \left(\pm\sqrt{\left(\frac{1}{\beta}-1\right)}L, 0\right).$

esistono quando $0<\beta\leq 1$

quando $\beta=1$, coincidono con prima soluz.

$[1] = [2] = 0$ non ha soluzioni a meno che $\beta = 1/2$.
(in tal caso c'è un continuo di soluz. su circonferenza $x^2+y^2=L^2$

$$\partial^2 V = \begin{pmatrix} 2k\beta - \frac{2kL^2}{L^2+x^2+y^2} + \frac{4kL^2x^2}{(L^2+x^2+y^2)^2} & \frac{4kL^2xy}{(L^2+x^2+y^2)^2} \\ \frac{4kL^2xy}{(L^2+x^2+y^2)^2} & k - \frac{2kL^2}{L^2+x^2+y^2} + \frac{4kL^2y^2}{(L^2+x^2+y^2)^2} \end{pmatrix}$$

$$\partial^2 V(0,0) = \begin{pmatrix} 2k\beta - 2k & 0 \\ 0 & \underbrace{k - 2k}_{<0} \end{pmatrix}$$

instabile se $k>0$
(se $k<0$, stab. per $\beta>1$)

$$\partial^2 V(0,\pm L) = \begin{pmatrix} 2k\beta - k & 0 \\ 0 & k - \frac{2k\cancel{L^2}}{2\cancel{L^2}} + \frac{4k\cancel{L^4}}{4\cancel{L^4}} \end{pmatrix}$$

stab per $\beta > 1/2$ e $k>0$
(instab. se $k<0$)

$$\partial^2 V\left(\pm\sqrt{\tfrac{1}{\beta}-1}\,L, 0\right) = \begin{pmatrix} \cancel{2\beta k} - \cancel{2\beta k} + 4kL^2\left(\frac{1}{\beta}-1\right)L^2\frac{\beta^2}{L^4} & 0 \\ 0 & k - 2k\beta \end{pmatrix}$$

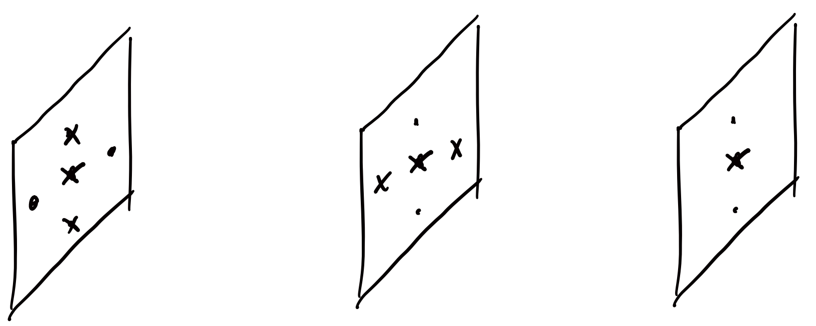
$$L^2+x^2 = L^2\left(1+\frac{1}{\beta}-1\right) = \frac{L^2}{\beta}$$

$$= \begin{pmatrix} \underbrace{4k\beta^2\left(\frac{1}{\beta}-1\right)}_{>0 \text{ per } 0<\beta<1} & \\ & \underbrace{k(1-2\beta)}_{>0 \text{ per } \beta<1/2} \end{pmatrix}$$

>0 per $\beta<1/2$ ($\Leftrightarrow \frac{1}{\beta}>2$)

stab. per $0<\beta<1/2$ e $k>0$
(complementare se $k<0$)

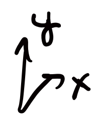
$k>0$:



$\beta < 1/2$ $\qquad$ $\frac{1}{2}<\beta<1$ $\qquad$ $\beta>1$

x → instab

• → stab



4) $B = \partial^2 V(0, L) = \begin{pmatrix} k(2\beta-1) & 0 \\ 0 & k \end{pmatrix}$ $\quad A = m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (°) $q_1 = x$, $q_2 = y - L$

stab. p. $k>0$ e $\beta > 1/2$ $\quad$ è matrice cinetica

$$\hat{L} = \frac{1}{2} \dot{\bar{q}} \cdot A \dot{\bar{q}} - \frac{1}{2} \bar{q} \cdot B \bar{q} \overset{(\circ)}{=} \frac{m}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2} k(2\beta-1) q_1^2 - \frac{1}{2} k q_2^2$$

$$(B - \lambda A) = 0 \to \quad \lambda_1 = \frac{k}{m}(2\beta-1) \quad \lambda_2 = \frac{k}{m}$$

(oppure si legge direttam. da $\hat{L}$ che è somma di due oscillatori armonici disaccoppiati)

5) C'è solo l'Energia (L indip. dal tempo)

6) $\beta = \frac{1}{2}$ $\to$ sist. invariante p. rotazioni attorno asse z $\Rightarrow$ M_z cost. del moto.

7) $x = r\cos\varphi$, $y = r\sin\varphi$

$$V = \frac{1}{2} k r^2 - kL^2 \log\left(1 + \frac{r^2}{L^2}\right) \to L(r, \varphi, \dot{r}, \dot{\varphi}) = T - V$$

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) \qquad \varphi \text{ ciclica}$$

$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi}$ è cost. del moto (p. eq. di Lag. $\frac{d}{dt} p_\varphi = \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} \overset{\uparrow}{=} \frac{\partial L}{\partial \varphi} = 0$) φ ciclica

p_φ cost. Invert. $\to \dot{\varphi} = \frac{p_\varphi}{m r^2}$

Lagrangiana ridotta:

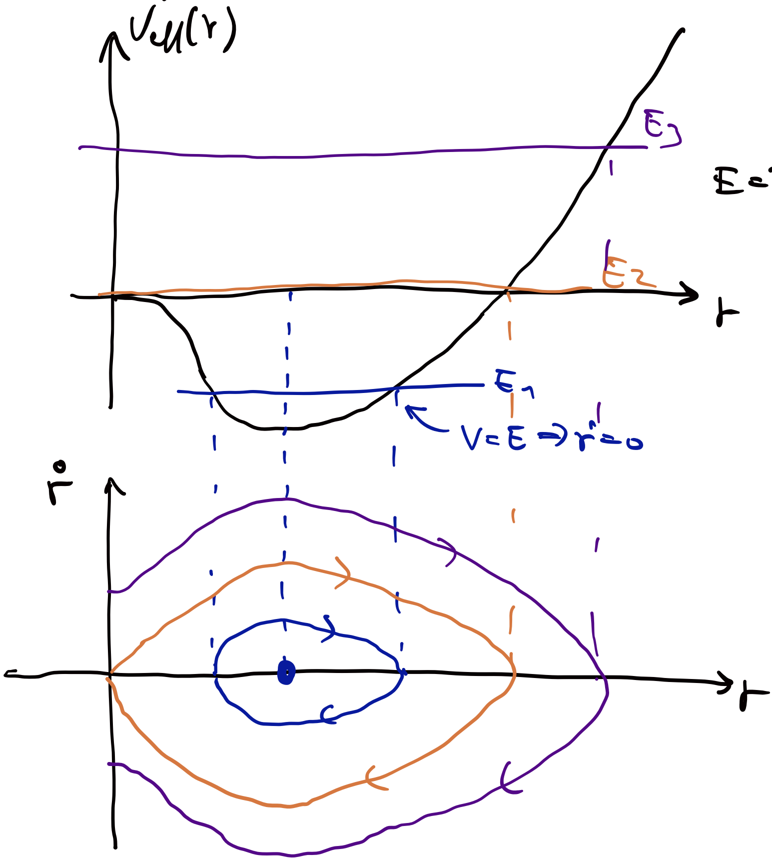
$$L^* = L\left(r, \varphi, \dot{r}, \frac{p_\varphi}{m r^2}\right) - p_\varphi \dot{\varphi} \Big|_{\dot{\varphi} = \frac{p_\varphi}{m r^2}} =$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \cancel{m r^2} \frac{p_\varphi^2}{(m r^2)^{\cancel{2}}} - \frac{1}{2} k r^2 + kL^2 \log\left(1 + \frac{r^2}{L^2}\right) - \frac{p_\varphi^2}{m r^2}$$

$$= \frac{m \dot{r}^2}{2} - \underbrace{\left(\frac{1}{2} k r^2 - kL^2 \log\left(1 + \frac{r^2}{L^2}\right) + \frac{\ell^2}{2 m r^2}\right)}_{\equiv V_{eff}(r)}$$

8) $V_{eff}(r)\Big|_{p_\varphi=0} = \frac{1}{2}kr^2 - kL^2 \log\left(1+\frac{r^2}{L^2}\right)$

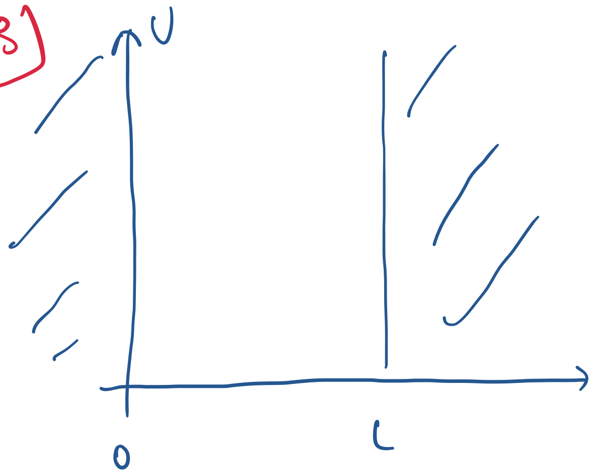
studio di funz. come alle scuole superiori



$E = T + V \Rightarrow \frac{m\dot r^2}{2} = E - V$

Semipiano sup. $\to \dot r > 0 \Rightarrow$
$\Rightarrow r(t)$ crescente $\Rightarrow$
$\Rightarrow$ frecce verso destra

ES 3)



$$p = \frac{\sqrt{2mE}}{\hbar}$$

1) $$\psi'' = -\frac{2mE}{\hbar^2}\psi$$

Solut in $0 \le x \le L$:

$$\psi(x) = C_+ e^{ipx} + C_- e^{-ipx}$$

(*) - continuità di ψ
- non occorre imporre continuità di ψ' perché V ha discontinuità ∞.

2) Condiz. raccordo (*)

$$\psi(0)=0 \rightarrow \begin{cases} C_+ + C_- = 0 \\ C_+ e^{ipL} + C_- e^{-ipL} = 0 \end{cases} \qquad \psi(L)=0$$

$$\begin{cases} C_- = -C_+ \\ e^{ipL} - e^{-ipL} = 0 \end{cases}$$

$\sin pL = 0 \rightarrow pL = n\pi$

$$\psi(x) = 2iC_+ \sin\left(n\pi \frac{x}{L}\right)$$ ← 1 sola solut. per questi valori di E → degen. = 1

$$p = n\frac{\pi}{L} \Rightarrow \underline{\underline{E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}}}$$

3) Normalizzaz.

$$\|\psi\|^2 = 4|C_+|^2 \int_0^L \sin^2\left(n\pi\frac{x}{L}\right)dx = 2|C_+|^2 \int_0^L \left(1 - \cos\left(2n\pi\frac{x}{L}\right)\right) =$$

$$= 2|C_+|^2 L + \cancel{2|C_+|^2 \frac{L}{2n\pi} \sin\left(2n\pi\frac{x}{L}\right)\Big|_0^L}$$

$$\Rightarrow C_+ = \frac{1}{\sqrt{2L}} \quad \text{a meno di una fase}$$

$d(\sin x)$

$$\langle x \rangle = \frac{1}{2L}\int_0^L x \sin^2\left(n\pi\frac{x}{L}\right)dx = \frac{1}{2L}\int_0^L x\,dx - \frac{1}{2L}\int_0^L \cos\left(2n\pi\frac{x}{L}\right) x\,dx$$

$$= \frac{L}{2} - \frac{1}{2L}\frac{L}{2n\pi}\cancel{x\sin\left(2n\pi\frac{x}{L}\right)\Big|_0^L} + \cancel{\frac{1}{2L}\int_0^L \cos\left(2n\pi\frac{x}{L}\right)dx} = \frac{L}{2}$$

$$\langle p \rangle = \frac{\hbar}{2Li}\int_0^L \sin\left(n\pi\frac{x}{L}\right)\frac{n\pi}{L}\cos\left(n\pi\frac{x}{L}\right) = \frac{\hbar n\pi}{4iL^2}\int_0^L \sin\left(\frac{2n\pi x}{L}\right)dx = 0$$

4) Sì: Funzione è L^2