

# Systems Dynamics

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Thomas Parisini  
Gianfranco Fenu

University of Trieste  
Department of Engineering and Architecture



**267MI –Fall 2021**

**Lecture 2**

**State and Output Movement of  
Linear Discrete-Time Systems**

## 2. State and Output Movement of Linear Discrete-Time Systems

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# General State-Space Solution

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# General State-Space Solution

Consider a linear discrete-time free (no inputs) dynamic system:

$$x(k+1) = A(k)x(k), \quad x(k_0) = x_0$$

Clearly,  $x(k)$ ,  $k > k_0$  can be determined by **iterating** the state equation:

$$x(k_0) = x_0$$

$$x(k_0 + 1) = A(k_0)x(k_0)$$

$$x(k_0 + 2) = A(k_0 + 1)x(k_0 + 1) = A(k_0 + 1)A(k_0)x(k_0)$$

$$\vdots$$

$$x(k) = A(k-1)A(k-2)A(k-3) \cdots A(k_0+1)A(k_0)x(k_0)$$

Hence:

$$x(k) = \varphi(k, k_0, x_0) = \Phi(k, k_0)x_0$$

where the **discrete-time state-transition matrix** is:

$$\Phi(k, k_0) = \prod_{j=k_0}^{k-1} A(j), \quad k > k_0; \quad \Phi(k_0, k_0) = I$$

## General State-Space Solution (cont.)

Now, consider a linear discrete-time dynamic system with inputs:

$$x(k+1) = A(k)x(k) + B(k)u(k), \quad x(k_0) = x_0$$

Clearly:

$$x(k_0) = x_0$$

$$x(k_0+1) = A(k_0)x(k_0) + B(k_0)u(k_0)$$

$$x(k_0+2) = A(k_0+1)x(k_0+1) + B(k_0+1)u(k_0+1)$$

$$= A(k_0+1)[A(k_0)x(k_0) + B(k_0)u(k_0)] + B(k_0+1)u(k_0+1)$$

$$= A(k_0+1)A(k_0)x(k_0) + A(k_0+1)B(k_0)u(k_0) + B(k_0+1)u(k_0+1)$$

$$x(k_0+3) = A(k_0+2)x(k_0+2) + B(k_0+2)u(k_0+2)$$

$$= A(k_0+2)A(k_0+1)A(k_0)x(k_0) + A(k_0+2)A(k_0+1)B(k_0)u(k_0)$$

$$+ A(k_0+2)B(k_0+1)u(k_0+1) + B(k_0+2)u(k_0+2)$$

⋮

## General State-Space Solution (cont.)

Therefore, using

$$\Phi(k, k_0) = \prod_{j=k_0}^{k-1} A(j), \quad k > k_0; \quad \Phi(k_0, k_0) = I$$

one gets

$$\begin{aligned} x(k) &= \varphi(k, k_0, x_0, \{u(k_0), \dots, u(k-1)\}) \\ &= \Phi(k, k_0)x_0 + \sum_{j=k_0}^{k-1} \Phi(k, j+1)B(j)u(j), \quad k > k_0 \end{aligned}$$

which expresses the **general solution** providing the state movement of a linear discrete-time dynamic system.

The determination of the state transition matrix  $\Phi(k, k_0)$  is clearly very important.

## General State-Space Solution (cont.)

- **Free state movement.** Setting  $u(k) = 0, \forall k \geq k_0$  gives:

$$x(k) = \varphi(k, k_0, x_0, 0) = \varphi_L(k) = \Phi(k, k_0)x_0, \quad k > k_0$$

- **Forced state movement.** Setting  $x_0 = 0$  gives:

$$\begin{aligned} x(k) &= \varphi(k, k_0, 0, \{u(k_0), \dots, u(k-1)\}) = \varphi_F(k) \\ &= \sum_{j=k_0}^{k-1} \Phi(k, j+1)B(j)u(j), \quad k > k_0 \end{aligned}$$

The **total state movement** is thus given by:

$$\varphi(k, k_0, x_0, \{u(k_0), \dots, u(k-1)\}) = \varphi_L(k) + \varphi_F(k)$$

which is a direct consequence of the **linearity** of the dynamic system.



## General State-Space Solution (cont.)

Now, let us add the output equation:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k), & x(k_0) = x_0 \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases}$$

one gets:

$$y(k) = C(k)\Phi(k, k_0)x_0 + \sum_{j=k_0}^{k-1} C(k)\Phi(k, j+1)B(j)u(j) + D(k)u(k), \quad k > k_0$$

- **Free output movement.** Setting  $u(k) = 0, \forall k \geq k_0$  gives:

$$y(k) = y_L(k) = C(k)\Phi(k, k_0)x_0, \quad k > k_0$$

- **Forced output movement.** Setting  $x_0 = 0$  gives:

$$y(k) = y_F(k) = \sum_{j=k_0}^{k-1} C(k)\Phi(k, j+1)B(j)u(j) + D(k)u(k), \quad k > k_0$$

The **total output movement** is thus given by:

$$y(k) = y_L(k) + y_F(k)$$

# **State-Space Solution: the Time-Invariant Case**

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## State-Space Solution: the Time-Invariant case

- In the time-invariant case, matrices  $A(k), B(k), C(k), D(k)$  do not depend on time-index  $k$ , that is they are **constant** matrices  $A, B, C, D$ .
- Hence, when considering a linear discrete-time free (no inputs) time-invariant dynamic system:

$$x(k+1) = Ax(k), \quad x(k_0) = x_0$$

one gets:

$$x(k) = \varphi(k, k_0, x_0) = \Phi(k, k_0)x_0$$

where the **discrete-time state-transition matrix** now takes on the form

$$\Phi(k, k_0) = \prod_{j=k_0}^{k-1} A = A^{(k-k_0)}, \quad k > k_0; \quad \Phi(k_0, k_0) = I$$

- With some abuse of notation, we denote  $\Phi(k - k_0)$  to highlight the dependence on  $(k - k_0)$  instead of  $k$  and  $k_0$  separately.

## State-Space Solution: the Time-Invariant case (cont.)

Now, consider a linear discrete-time time-invariant dynamic system with inputs:

$$x(k+1) = Ax(k) + Bu(k), \quad x(k_0) = x_0$$

Therefore, using

$$\Phi(k - k_0) = A^{(k-k_0)}, \quad k > k_0; \quad \Phi(k_0, k_0) = I$$

one gets

$$\begin{aligned} x(k) &= \varphi(k, k_0, x_0, \{u(k_0), \dots, u(k-1)\}) \\ &= A^{(k-k_0)}x_0 + \sum_{j=k_0}^{k-1} A^{k-(j+1)}Bu(j), \quad k > k_0 \end{aligned}$$

The explicit form  $\Phi(k - k_0) = A^{(k-k_0)}$  will be used later on to determine the state and output evolution over time in **closed-form**.

## State-Space Solution: the Time-Invariant case (cont.)

- **Free state movement.** Setting  $u(k) = 0, \forall k \geq k_0$  gives:

$$x(k) = \varphi(k, k_0, x_0, 0) = \varphi_L(k) = A^{(k-k_0)}x_0, \quad k > k_0$$

- **Forced state movement.** Setting  $x_0 = 0$  gives:

$$\begin{aligned} x(k) &= \varphi(k, k_0, 0, \{u(k_0), \dots, u(k-1)\}) = \varphi_F(k) \\ &= \sum_{j=k_0}^{k-1} A^{k-(j+1)}Bu(j), \quad k > k_0 \end{aligned}$$

The **total state movement** is thus given by:

$$\varphi(k, k_0, x_0, \{u(k_0), \dots, u(k-1)\}) = \varphi_L(k) + \varphi_F(k)$$

which is a direct consequence of the **linearity** of the dynamic system.

## State-Space Solution: the Time-Invariant case (cont.)

Now, by adding the output equation:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k), & x(k_0) = x_0 \\ y(k) = Cx(k) + Du(k) \end{cases}$$

one gets:

$$y(k) = CA^{(k-k_0)}x_0 + \sum_{j=k_0}^{k-1} CA^{k-(j+1)}Bu(j) + Du(k), \quad k > k_0$$

- **Free output movement.** Setting  $u(k) = 0, \forall k \geq k_0$  gives:

$$y(k) = y_L(k) = CA^{(k-k_0)}x_0, \quad k > k_0$$

- **Forced output movement.** Setting  $x_0 = 0$  gives:

$$y(k) = y_F(k) = \sum_{j=k_0}^{k-1} CA^{k-(j+1)}Bu(j) + Du(k), \quad k > k_0$$

The **total output movement** is thus given by:

$$y(k) = y_L(k) + y_F(k)$$

# **Input-Output Dynamic Description for Linear Systems**

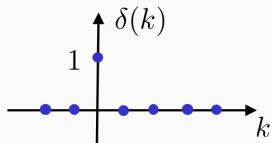
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# Input-Output Dynamic Description of Linear Systems

## Preliminaries

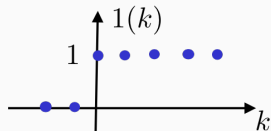
Discrete-time unit **impulse** sequence

$$\delta(k) = \begin{cases} 0, & k \neq 0, k \in \mathbb{Z} \\ 1, & k = 0 \end{cases}$$



Discrete-time unit **step** sequence

$$1(k) = \begin{cases} 0, & k < 0, k \in \mathbb{Z} \\ 1, & k \geq 0, k \in \mathbb{Z} \end{cases}$$



$$\implies \delta(k) = 1(k) - 1(k-1); \quad 1(k) = \begin{cases} \sum_{j=0}^{\infty} \delta(k-j), & k \geq 0 \\ 0, & k < 0 \end{cases}$$

Moreover, an arbitrary sequence  $\{x(k)\}$  can be expressed as

$$x(k) = \sum_{j=-\infty}^{\infty} x(j)\delta(k-j)$$



# Input-Output Dynamic Description of Linear Systems (cont.)

- Consider a linear discrete-time system with scalar input and output



- Moreover, consider the "external" input/output relationship

$$y(k) = \sum_{j=-\infty}^{\infty} h(k, j)u(j) \quad (\star)$$

**Assumption.** The sequences  $\{h(k, j)\}$  for any given  $k$  and  $\{u(j)\}$  are such that the relationship  $(\star)$  is well-defined. For example,  $\{h(k, j)\} \in l_2$  and  $\{u(j)\} \in l_2$ .

- Under the above assumption, relationship  $(\star)$  is **linear**.

# Input-Output Dynamic Description of Linear Systems (cont.)

- Denote by  $h(k, j)$  the output response at time  $k$  produced by a unit impulse  $\delta(j)$  applied at time  $j$
- By linearity, the output response at time  $k$  produced by a impulse of amplitude  $u(j)$  applied at time  $j$  is  $h(k, j)u(j)$
- By linearity, the output response at time  $k$  produced by two impulses of amplitude  $u(j_1)$  and  $u(j_2)$  applied at times  $j_1$  and  $j_2$ , respectively, is  $h(k, j_1)u(j_1) + h(k, j_2)u(j_2)$

## Input-Output Model

At time  $k$ , the system output  $y(k)$  produced by the input sequence  $\{u(j)\}$  is given by

$$y(k) = \sum_{j=-\infty}^{\infty} h(k, j)u(j)$$

where  $h(k, j)$  denotes the output response at time  $k$  produced by a unit impulse  $\delta(k - j)$  applied at time  $j$

## Properties

- Due to **causality**, the response to an input sequence has to be **identically zero before the input sequence is applied**. Hence:

$$h(k, j) = 0, \quad \forall j, \forall k < j$$

Hence:

$$\begin{aligned} y(k) &= \sum_{j=-\infty}^k h(k, j)u(j) \\ \implies y(k) &= \sum_{j=-\infty}^{k_0-1} h(k, j)u(j) + \sum_{j=k_0}^k h(k, j)u(j) \\ &= Y(k; k_0 - 1) + \sum_{j=k_0}^k h(k, j)u(j) \end{aligned}$$

# Input-Output Dynamic Description of Linear Systems (cont.)

- The system is **at rest** at time  $k_0$  if

$$u(k) = 0, \forall k \geq k_0 \implies y(k) = 0, \forall k \geq k_0$$

and this implies  $Y(k; k_0 - 1) = 0$ .

- Hence, if the system is **at rest** at time  $k_0$ , it follows that

$$y(k) = \sum_{j=k_0}^{\infty} h(k, j)u(j)$$

and due to causality, one gets

$$y(k) = \sum_{j=k_0}^k h(k, j)u(j)$$

# Input-Output Dynamic Description of Linear Systems (cont.)

- If the system is **time-invariant**, denoting by  $\{h(k, 0)\}$  the response to  $\{\delta(k)\}$ , it follows that  $\{h(k - j, 0)\}$  is the response to  $\{\delta(k - j)\}$
- Letting (with some abuse of notation)

$$h(k - j) := h(k - j, 0)$$

one gets the well-known **convolution formula**:

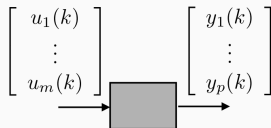
$$y(k) = u(k) * h(k) = \sum_{j=-\infty}^{\infty} h(k - j)u(j)$$

or equivalently (via a change of variables)

$$y(k) = h(k) * u(k) = \sum_{i=-\infty}^{\infty} h(i)u(k - i)$$

# Input-Output Dynamic Description of Linear Systems (cont.)

- Consider a linear discrete-time system with **vector input and output**



- The scalar case (with all properties) can be generalised as:

$$y(k) = \sum_{j=-\infty}^{\infty} H(k, j)u(j)$$
$$H(k, j) = \begin{bmatrix} h_{11}(k, j) & h_{12}(k, j) & \cdots & h_{1m}(k, j) \\ h_{21}(k, j) & h_{22}(k, j) & \cdots & h_{2m}(k, j) \\ \cdots & \cdots & \cdots & \cdots \\ h_{p1}(k, j) & h_{p2}(k, j) & \cdots & h_{pm}(k, j) \end{bmatrix}$$

where  $h_{rs}(k, j)$  denotes the  $r$ -th component of the response at time  $k$  produced by a unit impulse applied at time  $j$  on the  $s$ -th component of the input, while all other input components are set to zero.

# Relationship between State-Space and Input-Output Dynamic Descriptions

Consider a state-space description with **initial state set to zero**:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k), & x(k_0) = 0 \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases}$$

Recalling that

$$y(k) = \sum_{j=k_0}^{k-1} C(k)\Phi(k, j+1)B(j)u(j) + D(k)u(k), \quad k > k_0$$

one gets immediately

$$H(k, j) = \begin{cases} C(k)\Phi(k, j+1)B(j), & k > j \\ D(k) & k = j \\ 0 & k < j \end{cases}$$

which, in the **time-invariant** case, becomes

$$H(k-j) = \begin{cases} CA^{k-(j+1)}B, & k > j \\ D & k = j \\ 0 & k < j \end{cases}$$

# **Determination of the State/Output Movement**

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# **Determination of the State/Output Movement**

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**Response Modes**

# Determination of the State/Output Movement

Recall that in the general **time-varying** case one has:

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k), & x(k_0) = x_0 \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases}$$

one gets:

$$y(k) = C(k)\Phi(k, k_0)x_0 + \sum_{j=k_0}^{k-1} C(k)\Phi(k, j+1)B(j)u(j) + D(k)u(k), \quad k > k_0$$

where

$$\Phi(k, k_0) = \prod_{j=k_0}^{k-1} A(j), \quad k > k_0; \quad \Phi(k_0, k_0) = I$$

is the **state-transition matrix**.

## Determination of the State/Output Movement (cont.)

In the **time-invariant** case, recall that the solution specialises as follows:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k), & x(k_0) = x_0 \\ y(k) = Cx(k) + Du(k) \end{cases}$$

one gets:

$$y(k) = CA^{(k-k_0)}x_0 + \sum_{j=k_0}^{k-1} CA^{k-(j+1)}Bu(j) + Du(k), \quad k > k_0$$

where the **state-transition matrix** now is given by:

$$\Phi(k - k_0) = A^{(k-k_0)}, \quad k > k_0; \quad \Phi(k_0, k_0) = I$$

- Without loss of generality we let  $k_0 = 0$  and we "expand" matrix  $A^{k-k_0} = A^k$  in "matrix partial fractions".
- Clearly

$$\det(zI - A) = \prod_{i=1}^{\sigma} (z - \lambda_i)^{n_i}$$

where  $\lambda_1, \dots, \lambda_{\sigma}$  are the **distinct** eigenvalues of  $A$  and  $n_i$  is the **algebraic multiplicity** of such eigenvalues.

- Of course  $\sum_{i=1}^{\sigma} n_i = n$ .

## Response Modes (cont.)

- Let's assume  $\lambda_i \neq 0, i = 1, \dots, \sigma$ .
- It can be shown that:

$$A^k = \sum_{i=1}^{\sigma} \left[ A_{i0} \lambda_i^k \mathbf{1}(k) + \sum_{l=1}^{n_i-1} A_{il} k(k-1) \cdots (k-l+1) \lambda_i^{k-l} \mathbf{1}(k-l) \right]$$

where

$$A_{il} = \frac{1}{l!} \frac{1}{(n_i - 1 - l)!} \lim_{z \rightarrow \lambda_i} \left\{ \frac{d^{n_i-1-l}}{dz^{n_i-1-l}} [(z - \lambda_i)^{n_i} (zI - A)^{-1}] \right\}$$

## Response Modes (cont.)

Hence:

- $A^k$  can be expressed as a sum of terms  $A_{il}l! \binom{k}{l} \lambda_i^{k-l}$  which are called **Response Modes**
- If an eigenvalue  $\lambda_i$  has algebraic multiplicity  $n_i$ , then, in general,  $n_i$  response modes

$$A_{il}l! \binom{k}{l} \lambda_i^{k-l}, l = 0, 1, \dots, n_i - 1$$

can be associated to  $\lambda_i$ .

- When all eigenvalues of  $A$  are distinct, one has  $\sigma = n$ ;  $n_i = 1, i = 1, \dots, n$  and

$$A^k = \sum_{i=1}^n A_i \lambda_i^k$$

with

$$A_i = \lim_{z \rightarrow \lambda_i} [(z - \lambda_i)(zI - A)^{-1}]$$

## Response Modes: A particular Case

- If an eigenvalue  $\lambda_j$  of  $A$  is zero, then the terms in the expression of  $A^k$ , corresponding to the zero eigenvalue must be modified.
- The terms corresponding to  $\lambda_j = 0$  are

$$A_{j0} \cdot \delta(k) + \sum_{l=1}^{n_j-1} A_{jl} l! \delta(k-l)$$

where

$$A_{jl} = \frac{1}{l!} \frac{1}{(n_j - 1 - l)!} \lim_{z \rightarrow 0} \left\{ \frac{d^{n_j-1-l}}{dz^{n_j-1-l}} [z^{n_j} (zI - A)^{-1}] \right\}$$

## Response Modes: A different Characterisation

In the special case of **distinct eigenvalues** of  $A$ :

- In such a case:  $\det(zI - A) = \prod_{i=1}^n (z - \lambda_i)$  and  $A^k = \sum_{i=1}^n A_i \lambda_i^k$
- It can be shown that  $A_i = v_i \tilde{v}_i^\top$  where:
  - $(\lambda_i I - A)v_i = 0$ :  $v_i$  right eigenvector associated with  $\lambda_i$
  - $\tilde{v}_i^\top (\lambda_i I - A) = 0$ :  $\tilde{v}_i^\top$  left eigenvector associated with  $\lambda_i$

In fact:

$$Q := [v_1 | v_2 | \dots | v_n] \implies P = Q^{-1} = \begin{bmatrix} \tilde{v}_1^\top \\ \vdots \\ \tilde{v}_n^\top \end{bmatrix}; \tilde{v}_i^\top v_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

and then

$$\begin{aligned} (zI - A)^{-1} &= [zI - Q \text{diag} [\lambda_1, \dots, \lambda_n] Q^{-1}]^{-1} \\ &= Q [zI - \text{diag} [\lambda_1, \dots, \lambda_n]]^{-1} Q^{-1} \\ &= Q \text{diag} [(z - \lambda_1)^{-1}, \dots, (z - \lambda_n)^{-1}] Q^{-1} = \sum_{i=1}^n v_i \tilde{v}_i^\top (z - \lambda_i)^{-1} \end{aligned}$$



## Response Modes: A different Characterisation (cont.)

- If the initial state vector  $x_0$  is "parallel" to eigenvector  $v_j$  of  $A$ , then the only response mode showing up in the state movement is  $\lambda_j^k$ :

$$x_0 = \alpha v_j \implies x(k) = A^k x_0 = v_1 \tilde{v}_1^\top x_0 \lambda_1^k + \dots + v_n \tilde{v}_n^\top x_0 \lambda_n^k = \alpha v_j \lambda_j^k$$

**Example:** consider  $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ ;  $\lambda_1 = -1$ ,  $\lambda_2 = 1$

$$\implies Q = [v_1 \mid v_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} \tilde{v}_1^\top \\ \tilde{v}_2^\top \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$A^k = v_1 \tilde{v}_1^\top \lambda_1^k + v_2 \tilde{v}_2^\top \lambda_2^k = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} (-1)^k + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} 1^k$$

and thus, if  $x_0 = \alpha v_1 = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  then the response mode  $1^k$  **does not show up** in the free state response starting from such an initial state  $x_0$

# Calculation of $A^k$ by Similarity Transformation

Consider:

- $x(k+1) = Ax(k)$ ,  $x(0) = x_0 \implies x(k) = A^k x_0$
- $T \in \mathbb{R}^{n \times n}$ ,  $\det(T) \neq 0 \implies x = T\hat{x}$ ,  $\hat{x} = T^{-1}x$

Hence  $\hat{x}(k+1) = T^{-1}Ax(k) = T^{-1}AT\hat{x}(k)$ ,  $\hat{x}_0 = T^{-1}x_0$  which yields

$$\hat{x}(k) = (T^{-1}AT)^k T^{-1}x_0$$

Letting  $J := T^{-1}AT$ , one gets the closed-form expression for the free-state response expressed in the original state coordinates

$$x(k) = TJ^k T^{-1}x_0$$

Suppose now that the similarity transformation is such that

$$J = T^{-1}AT$$

takes on the **Jordan Canonical Form**.

## Calculation of $A^k$ by Similarity Transformation (cont.)

**Case 1.** Suppose that matrix  $A$  admits the construction of a basis of  $n$  linearly-independent eigenvectors  $v_i$  associated with the eigenvalues  $\lambda_i$ ,  $i = 1, \dots, n$  (**not necessarily distinct**).

Thus:

$$T = [v_1 | v_2 | \dots | v_n] \implies J = T^{-1}AT = \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix}$$

Hence:

$$J^k = \begin{bmatrix} \lambda_1^k & \dots & 0 \\ & \ddots & \\ 0 & \dots & \lambda_n^k \end{bmatrix}$$
$$\implies x(k) = TJ^kT^{-1}x_0 = T \begin{bmatrix} \lambda_1^k & \dots & 0 \\ & \ddots & \\ 0 & \dots & \lambda_n^k \end{bmatrix} T^{-1}x_0$$

## Calculation of $A^k$ by Similarity Transformation (cont.)

**Case 2.** Consider the general case in which matrix  $A$  has multiple eigenvalues. It is always possible to construct a basis of  $n$  linearly-independent vectors  $v_i$  such that:

$$T = [v_1|v_2|\cdots|v_n] \implies J = T^{-1}AT = \begin{bmatrix} J_0 & \cdots & \cdots & 0 \\ \vdots & J_1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & J_s \end{bmatrix}$$

where

$$J_0 = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_k \end{bmatrix}$$

## Calculation of $A^k$ by Similarity Transformation (cont.)

and  $J_i, i \geq 1$  is a  $n_i \times n_i$  matrix taking on the special form

$$J_i = \begin{bmatrix} \lambda_{k+i} & 1 & 0 & \cdots & 0 \\ 0 & \lambda_{k+i} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & \lambda_{k+i} \end{bmatrix}$$

where **not necessarily**  $\lambda_{k+i} \neq \lambda_{k+j}, i \neq j$  and

$$k + n_1 + \cdots + n_s = n$$

Matrix  $J$  is block-diagonal and its special structure makes it possible to compute  $A^k$  in **closed-form**.

## Calculation of $A^k$ by Similarity Transformation (cont.)

In fact:

$$J^k = \begin{bmatrix} J_0^k & \cdots & \cdots & 0 \\ & J_1^k & & \\ & & \ddots & \\ 0 & \cdots & \cdots & J_s^k \end{bmatrix}$$

where

$$J_0^k = \begin{bmatrix} \lambda_1^k & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \lambda_r^k \end{bmatrix}$$

Then:

$$x(k) = T J^k T^{-1} x_0 = T \begin{bmatrix} J_0^k & \cdots & \cdots & 0 \\ & J_1^k & & \\ & & \ddots & \\ 0 & \cdots & \cdots & J_s^k \end{bmatrix} T^{-1} x_0$$

## Calculation of $A^k$ by Similarity Transformation (cont.)

Concerning the computation of  $J_i^k$ ,  $i = 1, \dots, s$  we can write:

$$J_i = \lambda_{r+i} I_i + N_i$$

where  $I_i$  is the identity matrix with dimension  $n_i \times n_i$  and  $N_i$  is a matrix of dimension  $n_i \times n_i$  having the form:

$$N_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

Matrix  $N_i$  is a nilpotent matrix, that is, it holds:

$$N_i^k = 0, \forall k \geq n_i$$

## Calculation of $A^k$ by Similarity Transformation (cont.)

On the other hand, one immediately gets:

$$\begin{aligned} J_i^k &= (\lambda_{r+i} I_i + N_i)^k \\ &= \lambda_{r+i}^k I + k \lambda_{r+i}^{k-1} N_i + \frac{k(k-1)}{2!} \lambda_{r+i}^{k-2} N_i^2 + \dots + k \lambda_{r+i} N_i^{k-1} + N_i^k \end{aligned}$$

thus getting to **discrete-time response modes** of the form

$$\lambda^k, \binom{k}{n_i} \lambda_i^{k-n_i}$$



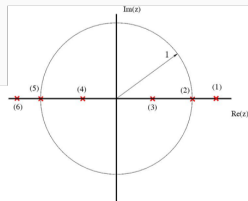
# **Determination of the State/Output Movement**

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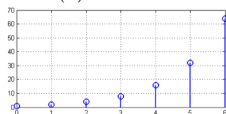
**Qualitative Behaviour of Response  
Modes**

# Qualitative Behaviour of Response Modes

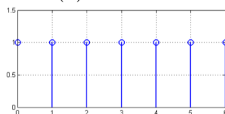
- $\binom{k}{n_i} \lambda_i^{k-n_i}$  with  $\lambda \in \mathbb{R}$ , multiplicity = 1



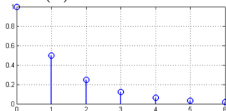
(1)  $\lambda > 1$



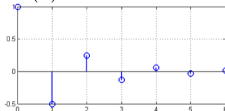
(2)  $\lambda = 1$



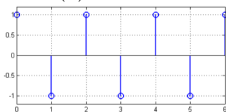
(3)  $0 < \lambda < 1$



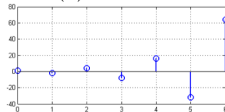
(4)  $-1 < \lambda < 0$



(5)  $\lambda = -1$

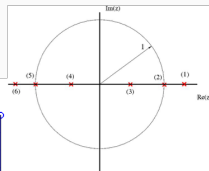


(6)  $\lambda < -1$

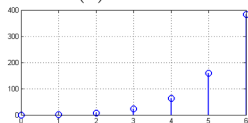


# Qualitative Behaviour of Response Modes

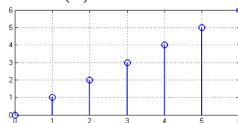
- $\binom{k}{n_i} \lambda_i^{k-n_i}$  with  $\lambda \in \mathbb{R}$ , multiplicity  $> 1$



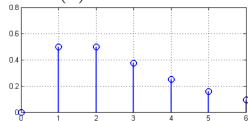
(1)  $\lambda > 1$



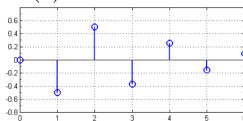
(2)  $\lambda = 1$



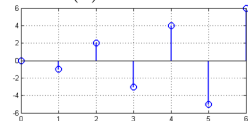
(3)  $0 < \lambda < 1$



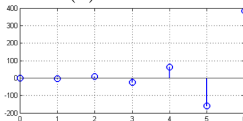
(4)  $-1 < \lambda < 0$



(5)  $\lambda = -1$

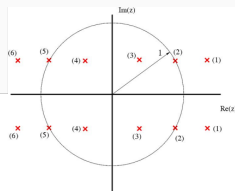


(6)  $\lambda < -1$

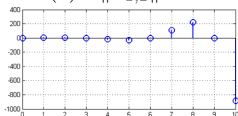


# Qualitative Behaviour of Response Modes

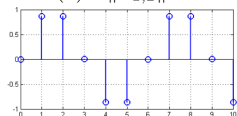
- $\binom{k}{n_i} \lambda_i^{k-n_i}$  with  $\lambda \in \mathbb{C}$ , multiplicity = 1



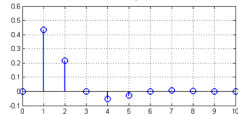
(1)  $\|\lambda_{1,2}\| > 1$



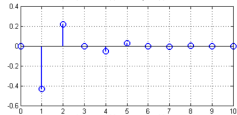
(2)  $\|\lambda_{1,2}\| = 1$



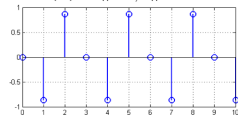
(3)  $\|\lambda_{1,2}\| < 1$



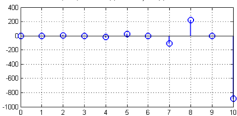
(4)  $\|\lambda_{1,2}\| < 1$



(5)  $\|\lambda_{1,2}\| = 1$

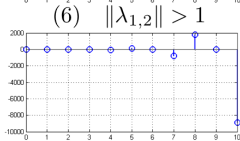
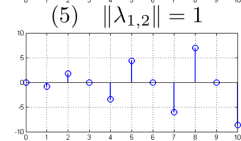
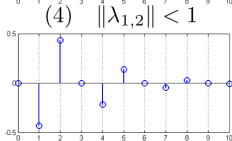
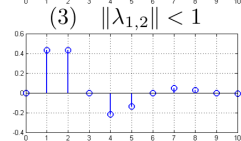
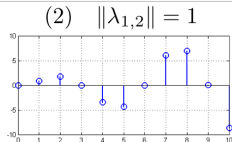
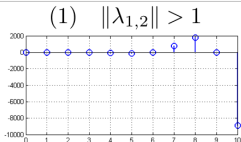
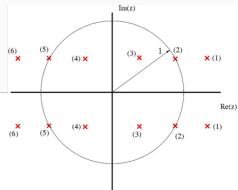


(6)  $\|\lambda_{1,2}\| > 1$



# Qualitative Behaviour of Response Modes

- $\binom{k}{n_i} \lambda_i^{k-n_i}$  with  $\lambda \in \mathbb{C}$ , multiplicity  $> 1$



# **External Description of LTI Dynamic Systems: Transfer Function**

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# External Description of LTI Dynamic Systems: Transfer Function

Recall the relationship between the state space description and the impulse response (**an external description**):

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k), & x(k_0) = 0 \\ y(k) = C(k)x(k) + D(k)u(k) \end{cases}$$

Recalling that

$$y(k) = \sum_{j=k_0}^{k-1} C(k)\Phi(k, j+1)B(j)u(j) + D(k)u(k), \quad k > k_0$$

one gets immediately

$$H(k, j) = \begin{cases} C(k)\Phi(k, j+1)B(j), & k > j \\ D(k) & k = j \\ 0 & k < j \end{cases}$$

which, in the **time-invariant** case, becomes

$$H(k-j) = \begin{cases} CA^{k-(j+1)}B, & k > j \\ D & k = j \\ 0 & k < j \end{cases}$$

# Transfer Function

Consider the time-invariant dynamic system:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k), & x(k_0) = 0 \\ y(k) = Cx(k) + Du(k) \end{cases}$$

Applying the  $\mathcal{Z}$  Transform to both sides one gets:

$$\begin{aligned} z[X(z) - x_0] &= AX(z) + BU(z) \\ \implies (zI - A)X(z) &= z x_0 + BU(z) \\ \implies \begin{cases} X(z) = (zI - A)^{-1} z x_0 + (zI - A)^{-1} BU(z) \\ Y(z) = CX(z) + DU(z) \end{cases} \\ \implies Y(z) &= C(zI - A)^{-1} z x_0 + [C(zI - A)^{-1} B + D] U(z) \end{aligned}$$

Letting  $x_0 = 0$ , it follows that:

$$Y(z) = [C(zI - A)^{-1} B + D] U(z) = H(z) U(z)$$

and  $H(z)$  is called **transfer function**.



## Transfer Function (cont.)

Let's analyse the structure of the transfer function:

$$H(z) = \begin{bmatrix} H_{11}(z) & \cdots & H_{1m}(z) \\ \vdots & & \vdots \\ H_{i1}(z) & \cdots & H_{im}(z) \\ \vdots & & \vdots \\ H_{p1}(z) & \cdots & H_{pm}(z) \end{bmatrix}$$

$H(z)$  is a  $p \times m$  matrix where the  $i$ -th component of the output vector is given by:

$$Y_i(z) = \sum_{j=1}^m H_{ij}(z)U_j(z) = H_{i1}(z)U_1(z) + H_{i2}(z)U_2(z) + \cdots$$

Hence:

$$\begin{aligned} x(0) &= x_0 \\ u_r(k) &= 0, \quad r \neq j \end{aligned} \quad \Longrightarrow \quad H_{ij}(z) = \frac{Y_i(z)}{U_j(z)}$$

# Transfer Function of equivalent dynamic systems

Recall:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases}$$

Let  $\hat{x} := T^{-1}x$ , where  $T \in \mathbb{R}^{n \times n}$  is a generic non-singular  $n \times n$  matrix ( $\det(T) \neq 0$ ). Then, the equivalent state-space description is given by:

$$\begin{cases} \hat{x}(k+1) = T^{-1}x(k+1) = T^{-1}AT\hat{x}(k) + T^{-1}Bu(k) = \hat{A}\hat{x}(k) + \hat{B}u(k) \\ y(k) = CT\hat{x}(k) + Du(k) = \hat{C}\hat{x}(k) + Du(k) \end{cases}$$

Hence:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases} \iff \begin{cases} \hat{x}(k+1) = \hat{A}\hat{x}(k) + \hat{B}u(k) \\ y(k) = \hat{C}\hat{x}(k) + Du(k) \end{cases}$$

## Transfer Function of equivalent dynamic systems (cont.)

$$\begin{aligned}\hat{H}(z) &= \hat{C}(zI - \hat{A})^{-1}\hat{B} + \hat{D} \\ &= C \left[ T (zI - T^{-1}AT)^{-1} T^{-1} \right] B + D \\ &= C \left[ T (zT^{-1}T - T^{-1}AT)^{-1} T^{-1} \right] B + D \\ &= C \left[ T (T^{-1}(zI - A)T)^{-1} T^{-1} \right] B + D \\ &= C \left[ TT^{-1} (zI - A)^{-1} TT^{-1} \right] B + D \\ &= C \left[ (zI - A)^{-1} \right] B + D \\ &= H(z)\end{aligned}$$

Hence: the transfer function does not depend on the specific choice of the state variables

# Transfer Function: Properties

Consider the scalar case, that is,  $u(k) \in \mathbb{R}$ ,  $y(k) \in \mathbb{R}$ :

$$H(z) = C \left[ (zI - A)^{-1} \right] B + D$$

and

$$(zI - A)^{-1} = \begin{bmatrix} z - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & z - a_{22} & & \vdots \\ \vdots & & \ddots & \\ -a_{n1} & \cdots & & z - a_{nn} \end{bmatrix}^{-1}$$

## Transfer Function: Properties (cont.)

The inverse can be expressed as:

$$(zI - A)^{-1} = \frac{1}{\det(zI - A)} K(z)$$

where  $K(z)$  is the matrix of the algebraic complements.

Clearly:

- $\det(zI - A)$  is a polynomial with degree  $n$
- $K(z) = [k_{ij}(z); i, j = 1, \dots, n]$  where  $k_{ij}(z)$  is a polynomial of degree  $< n, \forall i, j$
- $C(zI - A)^{-1}B = \frac{1}{\det(zI - A)} CK(z)B = \frac{M(z)}{\varphi(z)}$  where  $M(z)$  is a polynomial of degree  $< n,$

## Transfer Function: Properties (cont.)

Therefore:

$$\begin{aligned} H(z) &= C(zI - A)^{-1}B + D = \frac{M(z)}{\varphi(z)} + D \\ &= \frac{M(z) + D\varphi(z)}{\varphi(z)} = \frac{N(z)}{\varphi(z)} \end{aligned}$$

where:

- $N(z)$  in general is a polynomial of degree  $n$
- In case of a **strictly proper** system, that is  $D = 0$ ,  $N(z)$  in general is a polynomial of degree  $< n$
- All the above holds if **no common factors** between  $N(z)$  and  $\varphi(z)$  are present

## Transfer Function: Properties (cont.)

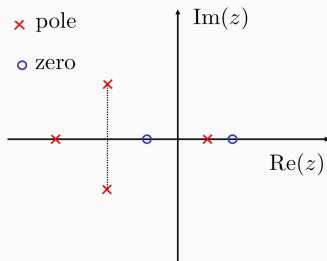
In the presence of common factors between  $N(z)$  and  $\varphi(z)$ :

$$H(z) = \frac{\overline{N}(z)}{\overline{\varphi}(z)}$$

- $\overline{\varphi}(z)$  is a factor of  $\varphi(z)$  of degree  $\nu < n$
- $\overline{N}(z)$  has degree  $m < \nu$  and has degree  $\nu$  only if  $D \neq 0$  (non strictly proper systems)

## Transfer Function: Poles and Zeros (scalar case)

- **Poles:** roots of polynomial  $\varphi(z)$
- **Zeros:** roots of polynomial  $N(z)$



- The poles are eigenvalues of  $A$
- An eigenvalue of  $A$  might not belong to the set of poles when common factors are present
- In case of more than one input and/or more than one output extra-care has to be exercised



## Transfer Function: Example

$$\begin{cases} x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k) \\ y(k) = [0 \ 1] x(k) \end{cases} \quad n = 2$$

Hence:

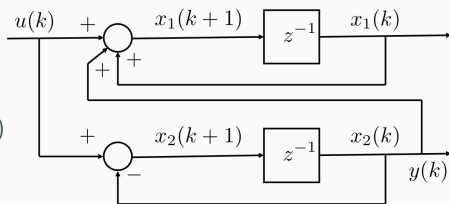
$$\begin{aligned} G(z) &= [0 \ 1] \begin{bmatrix} z-1 & -1 \\ 0 & z+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= [0 \ 1] \frac{1}{(z-1)(z+1)} \begin{bmatrix} z+1 & 1 \\ 0 & z-1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{(z-1)}{(z-1)(z+1)} = \frac{1}{z+1} \end{aligned}$$

Thus:  $\overline{\varphi}(z) = z + 1$  is a factor of  $\varphi(z) = (z-1)(z+1)$

## Transfer Function: Example (cont.)

The state equations have the form:

$$\begin{cases} x_1(k+1) = x_1(k) + x_2(k) + u(k) \\ x_2(k+1) = -x_2(k) + u(k) \\ y(k) = x_2(k) \end{cases}$$



**Only** the dynamics  $\begin{cases} x_2(k+1) = -x_2(k) + u(k) \\ y(k) = x_2(k) \end{cases}$  shows up in the transfer function  $G(z) = \frac{1}{z+1}$  and the time-evolution of  $x_1(k)$  is not influencing the output  $y(k)$ .

## Transfer Function: Example in the Non-Scalar Case

$$\begin{cases} x(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(k) + \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} u(k) \\ y(k) = [-3 \ 3] x(k) \end{cases}$$

Hence, one gets:

$$\begin{aligned} H(z) &= [-3 \ 3] \begin{bmatrix} z & -1 \\ 1 & z+2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} \\ &= [-3 \ 3] \frac{1}{(z+1)^2} \begin{bmatrix} z+2 & 1 \\ -1 & z \end{bmatrix} \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{z+1} & \frac{3(z-1)}{(z+1)^2} \end{bmatrix} \begin{bmatrix} 0 & -1/2 \\ 1 & 1/2 \end{bmatrix} = \begin{bmatrix} \frac{3(z-1)}{(z+1)^2} & \frac{3}{z+1} \end{bmatrix} \end{aligned}$$

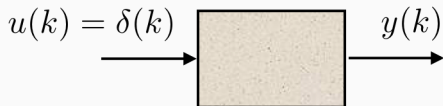
The notion of zeros and poles of a transfer function in the non-scalar case is more complicated (and less useful though)

## Transfer Function: Alternative Definition in the Scalar Case

$$x(0) = 0$$

$$u(k) = \delta(k)$$

$$\implies U(z) = \mathcal{Z}[\delta(k)] = 1$$



Therefore:

$$H(z) = \frac{Y(z)}{U(z)} = \frac{Y(z)}{1} = Y(z)$$

that is:

$$H(z) = \mathcal{Z}[\text{Impulse Response}]$$

# **Determination of Response Modes: Examples**

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## Determination of Response Modes: Example 1

Consider:

$$\begin{cases} x(k+1) &= \begin{bmatrix} -0.5 & 2 \\ 0 & 0.1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 2 & -1.5 \end{bmatrix} x(k) \end{cases}$$

Determine the free-state movement  $x_l(k) = A^k x(0)$  starting from the initial state  $x(0) = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$

The free-state movement is given by

$$x(k) = A^k x(0) + \sum_{i=0}^{k-1} A^{k-i-1} B u(i)$$

We are going to determine the free-state movement in two ways:

- by the  $\mathcal{Z}$  transform
- by calculating the matrix  $A^k$ .

## Determination of Response Modes: Example 1 (cont.)

### Calculation by the $\mathcal{Z}$ transform

$$x_l(k) = A^k x(0) \implies X_l(z) = z (zI - A)^{-1} x(0)$$

$$(zI - A) = \begin{bmatrix} z + 0.5 & -2 \\ 0 & z - 0.1 \end{bmatrix}$$
$$\implies (zI - A)^{-1} = \begin{bmatrix} \frac{2}{2z + 1} & \frac{40}{(2z + 1)(10z - 1)} \\ 0 & \frac{10}{10z - 1} \end{bmatrix}$$

Hence:

$$X_l(z) = \begin{bmatrix} \frac{20z(10z - 21)}{(10z - 1)(2z + 1)} \\ -\frac{100z}{10z - 1} \end{bmatrix}$$

## Determination of Response Modes: Example 1 (cont.)

First, we proceed with the inverse  $\mathcal{Z}$  transform:

$$X_l(z) = \begin{bmatrix} X_{l1}(z) \\ X_{l2}(z) \end{bmatrix} = \begin{bmatrix} \frac{20z(10z-21)}{(10z-1)(2z+1)} \\ -\frac{100z}{10z-1} \end{bmatrix}$$

Hence:

$$X_{l1}(z) = \frac{20z(10z-21)}{(10z-1)(2z+1)}$$
$$\implies \frac{X_{l1}(z)}{z} = \frac{20(10z-21)}{(10z-1)(2z+1)} = \frac{C_1}{z - \frac{1}{10}} + \frac{C_2}{z + \frac{1}{2}}$$

$$C_1 = \lim_{z \rightarrow \frac{1}{10}} \frac{20(10z-21)}{10(2z+1)} = -\frac{100}{3}; \quad C_2 = \lim_{z \rightarrow -\frac{1}{2}} \frac{20(10z-21)}{2(10z-1)} = \frac{130}{3}$$

thus getting: 
$$X_{l1}(z) = -\frac{100}{3} \frac{z}{(z - \frac{1}{10})} + \frac{130}{3} \frac{z}{(z + \frac{1}{2})}$$



## Determination of Response Modes: Example 1 (cont.)

Then, it follows that:

$$X_l(z) = \begin{bmatrix} -\frac{100}{3} \frac{z}{(z - \frac{1}{10})} + \frac{130}{3} \frac{z}{(z + \frac{1}{2})} \\ -10 \frac{z}{(z - \frac{1}{10})} \end{bmatrix}$$

and thus:

$$x_l(k) = \begin{bmatrix} \left\{ -\frac{100}{3} \left(\frac{1}{10}\right)^k + \frac{130}{3} \left(-\frac{1}{2}\right)^k \right\} \cdot 1(k) \\ -10 \left(\frac{1}{10}\right)^k \cdot 1(k) \end{bmatrix}$$

## Determination of Response Modes: Example 1 (cont.)

Now, as alternative technique, we proceed with calculating the matrix  $A^k$ .

- $A = \begin{bmatrix} -0.5 & 2 \\ 0 & 0.1 \end{bmatrix}$
- Eigenvalues:  $\lambda_1 = -0.5$ ,  $\lambda_2 = 0.1$ . Hence, matrix  $A$  admits a diagonal similar matrix because the eigenvalues are distinct
- The characteristic polynomial is given by:

$$p_A(\lambda) = \det(\lambda I - A) = (\lambda + 0.5)(\lambda - 0.1)$$

- A basis of linearly independent eigenvectors is now determined.

## Determination of Response Modes: Example 1 (cont.)

- $Az = \lambda_1 z$  with  $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$$\begin{bmatrix} -0.5 & 2 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = -0.5 \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \implies \begin{cases} -0.5z_1 + 2z_2 = -0.5z_1 \\ 0.1z_2 = -0.5z_2 \end{cases}$$

For example:  $z_2 = 0 \implies z^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- $Az = \lambda_2 z$

$$\begin{bmatrix} -0.5 & 2 \\ 0 & 0.1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0.1 \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \implies \begin{cases} -0.5z_1 + 2z_2 = 0.1z_1 \\ 0.1z_2 = 0.1z_2 \end{cases}$$

For example:  $z_2 = \frac{3}{10}z_1 \implies z^{(2)} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$

## Determination of Response Modes: Example 1 (cont.)

One now proceeds with calculating the equivalent state-space representation of matrix  $A$ :

$$T = \left[ z^{(1)} \mid z^{(2)} \right] = \begin{bmatrix} 1 & 10 \\ 0 & 3 \end{bmatrix} \implies T^{-1} = \frac{1}{3} \begin{bmatrix} 3 & -10 \\ 0 & 1 \end{bmatrix}$$

thus obtaining:

$$\tilde{A} = T^{-1}AT = \frac{1}{3} \begin{bmatrix} 3 & -10 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 2 \\ 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} 1 & 10 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{10} \end{bmatrix}$$

## Determination of Response Modes: Example 1 (cont.)

The calculation of  $A^k$  is now straightforward:

$$\begin{aligned} A^k &= M \tilde{A}^k M^{-1} = M \begin{bmatrix} \left(-\frac{1}{2}\right)^k & 0 \\ 0 & \left(\frac{1}{10}\right)^k \end{bmatrix} M^{-1} \\ &= \begin{bmatrix} 1 & 10 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \left(-\frac{1}{2}\right)^k & 0 \\ 0 & \left(\frac{1}{10}\right)^k \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} 3 & -10 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \left(-\frac{1}{2}\right)^k & \left(-\frac{10}{3}\left(-\frac{1}{2}\right)^k + \frac{10}{3}\left(\frac{1}{10}\right)^k\right) \\ 0 & \left(\frac{1}{10}\right)^k \end{bmatrix} \end{aligned}$$

## Determination of Response Modes: Example 1 (cont.)

Finally, from

$$A^k = \begin{bmatrix} \left(-\frac{1}{2}\right)^k & \left(-\frac{10}{3}\left(-\frac{1}{2}\right)^k + \frac{10}{3}\left(\frac{1}{10}\right)^k\right) \\ 0 & \left(\frac{1}{10}\right)^k \end{bmatrix}$$

and  $x(0) = \begin{bmatrix} 10 \\ -10 \end{bmatrix}$ , one gets:

$$x_l(k) = \begin{bmatrix} \left\{ -\frac{100}{3}\left(\frac{1}{10}\right)^k + \frac{130}{3}\left(-\frac{1}{2}\right)^k \right\} \cdot 1(k) \\ -10\left(\frac{1}{10}\right)^k \cdot 1(k) \end{bmatrix}$$

## Determination of Response Modes: Example 2

Consider:

$$\begin{cases} x_1(k+1) = x_1(k) + 4x_2(k) \\ x_2(k+1) = x_1(k) + x_2(k) \end{cases}$$

Setting  $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , show in two different ways that

$$\lim_{k \rightarrow \infty} \frac{x_1(k)}{x_2(k)} = 2$$

We are going to determine the free-state movement yielding  $x_1(k), x_2(k), \forall k \geq 0$  in two ways:

- by the  $\mathcal{Z}$  transform
- by calculating the matrix  $A^k$ .

## Determination of Response Modes: Example 2 (cont.)

Using the  $\mathcal{Z}$  transform:

$$\begin{cases} zX_1(z) - z &= X_1(z) + 4X_2(z) \\ zX_2(z) - z &= X_1(z) + X_2(z) \end{cases} \implies \begin{cases} X_1(z) &= \frac{z(z+3)}{(z+1)(z-3)} \\ X_2(z) &= \frac{z^2}{(z+1)(z-3)} \end{cases}$$

Hence:

$$\begin{cases} x_1(k) = \left[ \left(-\frac{1}{2}\right) (-1)^k + \frac{3}{2} 3^k \right] 1(k) \\ x_2(k) = \left[ \frac{1}{4} (-1)^k + \frac{3}{4} 3^k \right] 1(k) \end{cases}$$
$$\implies \lim_{k \rightarrow \infty} \frac{x_1(k)}{x_2(k)} = \lim_{k \rightarrow \infty} \frac{\left(\frac{3}{2}\right) 3^k}{\left(\frac{3}{4}\right) 3^k} = 2$$



## Determination of Response Modes: Example 2 (cont.)

Using the calculation of  $A^k$ :

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \implies \det(\lambda I - A) = \lambda^2 - 2\lambda - 3 = 0 \implies \begin{array}{l} \text{distinct} \\ \text{eigenvalues} \\ \lambda_1 = 3, \\ \lambda_2 = -1 \end{array}$$

$$\begin{aligned} \ker(A - 3I) &= \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} & T &= \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix} \\ & \implies & & \\ \ker(A + I) &= \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\} & T^{-1} &= -\frac{1}{4} \begin{bmatrix} 1 & -2 \\ -1 & -2 \end{bmatrix} \end{aligned}$$

Thus

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

## Determination of Response Modes: Example 2 (cont.)

By some algebra:

$$A^k = T \tilde{A}^k T^{-1} = \begin{bmatrix} \frac{1}{2} 3^k + \frac{1}{2} (-1)^k & 3^k - (-1)^k \\ \frac{1}{4} (3^k - (-1)^k) & \frac{1}{2} 3^k + \frac{1}{2} (-1)^k \end{bmatrix}$$

and then:

$$x(k) = A^k x(0) = \begin{cases} x_1(k) & = \left[ \left(-\frac{1}{2}\right) (-1)^k + \frac{3}{2} 3^k \right] 1(k) \\ x_2(k) & = \left[ \frac{1}{4} (-1)^k + \frac{3}{4} 3^k \right] 1(k) \end{cases}$$

$$\implies \lim_{k \rightarrow \infty} \frac{x_1(k)}{x_2(k)} = \lim_{k \rightarrow \infty} \frac{\left(\frac{3}{2}\right) 3^k}{\left(\frac{3}{4}\right) 3^k} = 2$$

**267MI –Fall 2021**

**Lecture 2**

**State and Output Movement of  
Linear Discrete-Time Systems**

**END**