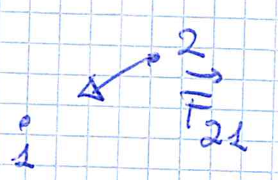
THEORY OF POTENTIAL
PLEASE, REFER TO BINNEY AND TREMAINE BOOK

B12 NUM

You know that



$$\bar F_{21} = -\frac{G m_1 m_2}{r_{12}^2}\,\hat r$$

$\vec F$ is a conservative force as all "central" forces

$$\text{Work} = \int_A^B \bar F\, d\bar s$$

$$\ldots\ \hat r \cdot d\vec s$$

$$\ldots\ \hat r\, d\hat r \longrightarrow \int_{r_A}^{r_B}$$

W = potential energy

$$\Delta W = -\,\text{Work} = -\int \bar F\, d\bar s$$

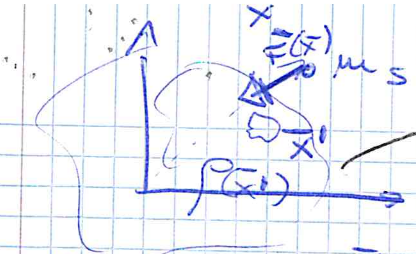
$$W = -\frac{G m_1 m_2}{r_{12}} + \text{const} \quad \to 0 \ \text{as} \ r \to \infty$$

· 1 · 2 · 3 ~ particles

binding energy

$$W = \frac{-G m_1 m_2}{r_{12}} - \frac{G m_2 m_3}{r_{23}} - \frac{G m_1 m_3}{r_{13}} \ldots$$

New Binney-Tremaine (2)



ad distribuzione di massa

calcolare la forza generata da $\bar{x}'$ su m_s

$$\delta \bar{F}(\bar{x}) = G m_s \frac{\bar{x}' - \bar{x}}{|\bar{x}' - \bar{x}|^3} \delta m(\bar{x}') = G m_s \frac{\bar{x}' - \bar{x}}{|\bar{x}' - \bar{x}|^3} \rho(\bar{x}') d^3(x')$$

the overall force ⟵ integrating in the whole space

$$\bar{F}(\bar{x}) = \int \delta \bar{F}(\bar{x})$$

$$\bar{F}(\bar{x}) = m_s \bar{g}(\bar{x})$$

gravitational field

$$\boxed{\bar{g}(\bar{x}) \equiv G \int d^3\bar{x}' \frac{(\bar{x}' - \bar{x})}{|\bar{x}' - \bar{x}|^3} \rho(x')} \qquad (2.2)$$

We define the gravitational potential

$$\phi(\bar{x}) \equiv -G \int d^3\bar{x}' \frac{\rho(\bar{x}')}{|\bar{x}' - \bar{x}|} \qquad (2.3)$$

$$\bar{\nabla}\phi(\bar{x}) = -G \bar{\nabla}_x \int d^3\bar{x}' \frac{\rho(\bar{x}')}{|\bar{x}' - \bar{x}|} =$$

the $\int$ does not depend on $\bar{x}$ ...

$$= -G \int d^3\bar{x}' \rho(\bar{x}') \bar{\nabla}_x \frac{1}{|\bar{x}' - \bar{x}|}$$

but $\bar{\nabla}_x \left(\frac{1}{|\bar{x}' - \bar{x}|} \right) = \frac{\bar{x}' - \bar{x}}{|\bar{x}' - \bar{x}|^3}$

See other papers * su "credit to Flavino Menotti"

$$= -G \underbrace{\int d^3\bar{x}' \rho(\bar{x}') \frac{\bar{x}' - \bar{x}}{|\bar{x}' - \bar{x}|^3}}_{\bar{g}(\bar{x})} \Rightarrow$$

$$\boxed{g(\bar{x}) = -\bar{\nabla}\phi} \qquad (2.5)$$

the potential is very useful ...
a scalar field is easier to visualize than a vector field

Now

laplacian

$$\bar{\nabla}(\bar{g}(\bar{x})) = -\bar{\nabla}\bar{\nabla}\phi = -\nabla^2\phi$$

divergence

$$\bar{\nabla}(\bar{g}(\bar{x})) = G\int d^3\bar{x}'\,\bar{\nabla}_x\left(\frac{\bar{x}'-\bar{x}}{|\bar{x}'-\bar{x}|^3}\right)\rho(\bar{x}') \qquad (2.6)$$

We know that

$$\bar{\nabla}_x\left(\frac{\bar{x}'-\bar{x}}{|\bar{x}'-\bar{x}|^3}\right) = -\frac{3}{|\bar{x}'-\bar{x}|^3} + \underbrace{\frac{3(\bar{x}'-\bar{x})(\bar{x}'-\bar{x})}{|\bar{x}'-\bar{x}|^5}}_{} \qquad (2.7)$$

see the papers

when $\bar{x}'-\bar{x} \neq 0$ $\qquad \dfrac{3\cancel{(\bar{x}'-\bar{x})}\cancel{(\bar{x}'-\bar{x})}}{|\bar{x}'-\bar{x}|^{\cancel{5}\,3}}$

i.e

$$\bar{\nabla}_x\left(\frac{\bar{x}'-\bar{x}}{|\bar{x}'-\bar{x}|^3}\right) = 0 \qquad (\bar{x}'\neq\bar{x}) \qquad (2.8)$$

So the contribution to the integral (2.6) comes from the point $\bar{x}'=\bar{x}$

around this point $\rho(\bar{x}') \sim$ const.

$$(2.6) \rightarrow \bar{\nabla}\bar{g}(\bar{x}) = G\rho(\bar{x})\int_{|\bar{x}'-\bar{x}|\leq h} d^3\bar{x}'\,\bar{\nabla}_x\left(\frac{\bar{x}'-\bar{x}}{|\bar{x}'-\bar{x}|^3}\right)$$

↑ small

~~symmetric~~ changing variable

$$= -G\rho(\bar{x})\int_{|\bar{x}'-\bar{x}|\leq h} d^3\bar{x}'\,\bar{\nabla}_{x'}\left(\frac{\bar{x}'-\bar{x}}{|\bar{x}'-\bar{x}|^3}\right)$$

divergence theorem

$$\int_V \bar{\nabla}\bar{F}\,d^3\bar{x} = \int_S \bar{F}\,d^2\bar{S}$$

$$= -G\rho(\bar{x})\int_{|\bar{x}'-\bar{x}|=h} d^2\bar{S}\,\frac{(\bar{x}'-\bar{x})}{|\bar{x}'-\bar{x}|^3}$$

but on the sphere

$\rightarrow r^2 d^2\Omega$

$h = |\bar{x}'-\bar{x}|$

$$d^2\bar{S} = (\bar{x}'-\bar{x})\,h\,d^2\Omega$$

oriented !

$$\bar{\nabla}\bar{g}(\bar{x}) = -G\rho(\bar{x})\int d^2\Omega = -4\pi G\rho(\bar{x})$$

eq. 2.5

$$\boxed{\nabla^2\phi = 4\pi G\rho}$$

POISSON EQ. (2.10) diff. ~~equation~~ equation.

$\rho \rightarrow \phi \rightarrow \bar{g}$

For isolated system $\phi \to 0$ as $|\bar{x}| \to \infty$

In the special case $\boxed{\rho=0}$
Poisson eq $\rightarrow$

$$\boxed{\nabla^2\phi=0} \qquad \text{Laplace eq.} \qquad (2.11)$$

BT 2 NVN

We integrate eq. (2.10)

$$\int \nabla^2\phi \, d^3\bar{x} = \int 4\pi G \rho(\bar{x}) \, d^3\bar{x}$$

$M = \int \rho(\bar{x}) d^3\bar{x}$
$\hookrightarrow$ mass

$$\int \bar{\nabla}(\bar{\nabla}\phi) \, d^3\bar{x} = 4\pi G M$$

$$\bar{\nabla}\phi = -\vec{g}$$

e
ole

divergence theor.

$$\int d^2\bar{S}\, \bar{\nabla}\phi = 4\pi G M$$

the integral of the normal component on $\bar{S} \rightarrow 4\pi G M$ contained in $\bar{S}$

or

$$\int \bar{g} \, d^2\bar{S} = -4\pi G M$$

(remember? $\int_S \bar{E}\, d^2\bar{s} = \frac{Q}{\varepsilon_0}$ for $\bar{E}$ field)

$$\boxed{\int d^2\bar{S}\, \bar{\nabla}\phi = 4\pi G M} \qquad (2.12)$$

integrale delle componente normale di $\nabla\phi$ su una qq superficie chiusa equivale a $4\pi G \times$ massa contenuta in quella superficie

The Gauss theorem

$\vec{g} = -\vec{\nabla}\phi$

$\bar{g}$ comes from $\bar{\nabla}\phi$ so $\vec{g}$ is conservative!
(un gradiente del potenziale)

lavoro fatto contro le forze gravitazionali per assemblare una certa configurazione di massa è indipendente dai dettagli dell'assemblaggio

Potential energy

$\rho(\bar{x})$ — δm — si crea un aumento di potenziale $\delta\phi$

a mass is already in place $\rho(\bar{x})$, $\phi(\bar{x})$

We bring an additional small mass δm from ∞ to $\bar{x}$, the work done is

$\delta m\,\phi(\bar{x})$

So a small increment of density $\delta\rho(\bar{x})$ gives a change in the potential energy (a small change)

in. potenziale

$$\delta W = \int d^3\bar{x}\;\delta\rho(\bar{x})\,\phi(\bar{x}) \qquad (2.13)$$

Poisson eq. $\nabla^2(\delta\phi) = 4\pi G(\delta\rho) \longrightarrow$

$$\delta W = \int d^3\bar{x}\;\phi(\bar{x})\,\frac{1}{4\pi G}\,\nabla^2(\delta\phi) =$$

$$= \frac{1}{4\pi G}\int d^3\bar{x}\;\phi(\bar{x})\,\nabla^2(\delta\phi) =$$

$$= \frac{1}{4\pi G}\int d^3\bar{x}\;\underbrace{\phi(x)}_{f}\,\bar{\nabla}\,\underbrace{(\bar{\nabla}(\delta\phi))}_{\bar{F}}$$

da integrale x parti $\int f'g\,dx = fg - \int fg'\,dx$

$\bar{\nabla}\bar{F} \to f$

"extended" theorem of divergence $\left(\int_V f(\bar{\nabla}\bar{F})\,d^3\bar{x} = \int_S f\bar{F}\,d^2\bar{S} - \int_V (\bar{F}\cdot\bar{\nabla})f\,d^3\bar{x}\right)$

see Appendix of Binney and Tremaine

$$= \frac{1}{4\pi G}\int \phi\,\bar{\nabla}(\delta\phi)\,d^2\bar{S} - \frac{1}{4\pi G}\int d^3\bar{x}\;\bar{\nabla}(\delta\phi)\,\bar{\nabla}\phi(\bar{x}) \qquad (2.15)$$

$=0$ why $\phi \propto r^{-1}$ $|\bar{\nabla}(\delta\phi)| \propto r^{-2}$ as $r\to\infty$

Σ surface where $r \to \infty$

$\underbrace{\phi\bar{\nabla}(\delta\phi)}_{\text{integrand}} \propto r^{-3}$ while the area $\propto r^2$

Moreover, for the second term

$$\bar{\nabla}\phi \cdot \bar{\nabla}(\delta\phi) = \bar{\nabla}\phi \cdot \delta(\bar{\nabla}\phi) =$$

(In fact $\delta(\bar{\nabla}\phi) = \bar{\nabla}(\phi + \delta\phi) - \bar{\nabla}\phi =$
$= \bar{\nabla}\phi + \bar{\nabla}(\delta\phi) - \bar{\nabla}\phi = \bar{\nabla}(\delta\phi)$)

$$= \frac{1}{2}\delta(\bar{\nabla}\phi \cdot \bar{\nabla}\phi)$$

{ remember? In fact
For a vector $\bar{v}$ $\underbrace{\frac{1}{2} d|\bar{v}|^2}$ $= \frac{1}{2} d(\bar{v}\cdot\bar{v}) = \cancel{\frac{1}{2}}\cancel{2}\,\underbrace{\bar{v}\, d\bar{v}}$

$$= \frac{1}{2}\delta|(\bar{\nabla}\phi)|^2$$

2.15 $\rightarrow$

$$\delta W = -\frac{1}{4\pi G}\int d^3\bar{x}\, \frac{1}{2}\,\delta|(\bar{\nabla}\phi)|^2 =$$

(sum of variations = variation of the sum)

$$= -\frac{1}{8\pi G}\,\delta\left(\int d^3\bar{x}\, |\bar{\nabla}\phi|^2\right) \quad 2.16$$

Now we sum up all of the contributions δW

$$W = \sum \delta W$$

$$W = -\frac{1}{8\pi G}\int d^3\bar{x}\, |\bar{\nabla}\phi|^2 \quad 2.17$$

energia potenziale per un corpo assemblato

To obtain an alternative expression for W

2.17 $\longrightarrow$ $$W = -\frac{1}{8\pi G}\int d^3\bar{x}\; \bar{\nabla}\phi\cdot\bar{\nabla}\phi$$

but we know that $\rightarrow$ $(\bar{\nabla}(\lambda\vec{u}) = \lambda\bar{\nabla}\vec{u} + \vec{u}\,\bar{\nabla}\lambda)$

$$\bar{\nabla}(\phi\bar{\nabla}\phi) = \phi\bar{\nabla}(\bar{\nabla}\phi) + \bar{\nabla}\phi\cdot\bar{\nabla}\phi$$

$$\rightarrow \bar{\nabla}\phi\cdot\bar{\nabla}\phi = \bar{\nabla}(\phi\bar{\nabla}\phi) - \phi\nabla^2\phi$$

$$= -\frac{1}{8\pi G}\left[\cancel{\int d^3\bar{x}\,\bar{\nabla}(\phi\bar{\nabla}\phi)}_{=0} - \int d^3x\,\phi\nabla^2\phi\right] =$$

div. Theor.

$$\int d^2\bar{s}\;\phi\,\bar{\nabla}\phi$$

$$\underset{\sim r^2}{d^2\bar{s}} \quad \underbrace{\underset{r^{-1}}{\phi} \quad \underset{r^{-2}}{\bar{\nabla}\phi}}_{r^{-3}}$$

I use eq. of Poisson $\quad \nabla^2\phi = 4\pi G\rho$

$$= \frac{1}{8\pi G}\int d^3\bar{x}\; 4\pi G\rho\phi$$

$$\boxed{W = \frac{1}{2}\int d^3\bar{x}\;\rho(\bar{x})\,\phi(\bar{x})} \qquad (2.18)$$

Potential - energy for a body

Chandrasekhar

The potential-energy tensor

tensor $\overline{\overline{W}}$ $\begin{pmatrix} 11 & 12 & 13 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ 3 x 3

ponent
OMPONENT

$$\left[W_{jk} \equiv -\int d^3\bar{x}\, \rho(\bar{x})\, x_j \frac{\partial \phi}{\partial x_k} \right]$$

over all space

2.19

SS eq. 0.10 B.

2.3

$$\phi_{(\bar{x})} = -G\int d^3\bar{x}' \frac{\rho(\bar{x}')}{|\bar{x}'-\bar{x}|}$$

definito prima

la traccia di $\overline{\overline{W}}$ sarà W

trace$(\overline{\overline{W}}) = W$

$$\cancel{W_{jk} = G\int d^3\bar{x} \int d^3\bar{x}' \{\rho(\bar{x})\, x_j \frac{\partial}{\partial x_k} \}}$$

$$W_{jk} = G\int d^3\bar{x}\, \rho(\bar{x})\, x_j \frac{\partial}{\partial x_k} \int d^3\bar{x}' \frac{\rho(\bar{x}')}{|\bar{x}'-\bar{x}|} = \quad 2.20$$

solo componente k does not depend on $\bar{x}$

$$W_{jk} = G\int d^3\bar{x} \int d^3\bar{x}'\, \rho(\bar{x})\, \rho(\bar{x}') \frac{x_j(x'_k - x_k)}{|\bar{x}'-\bar{x}|^3} \quad 2.21a$$

x and x' are dummy variables of integr.

$$W_{jk} = G\int d^3\bar{x}' \int d^3\bar{x}\, \rho(\bar{x}')\, \rho(\bar{x})\, x'_j \frac{(x_k - x'_k)}{|\bar{x}-\bar{x}'|^3} = \quad 2.21b$$

change of the order of integration

$$= G\int d^3\bar{x} \int d^3\bar{x}'\, \rho(\bar{x})\, \rho(\bar{x}') \frac{(-x'_j)(x'_k - x_k)}{|\bar{x}'-\bar{x}|^3}$$

2.21a+b

$$2W_{jk} = G\int d^3\bar{x} \int d^3\bar{x}'\, \rho(\bar{x})\rho(\bar{x}') \cdot (-1) \frac{(x'_j - x_j)(x'_k - x_k)}{|\bar{x}'-\bar{x}|^3}$$

$$W_{jk} = -\frac{1}{2} G\int d^3\bar{x} \int d^3\bar{x}'\, \rho(\bar{x})\, \rho(\bar{x}') \frac{(x'_j - x_j)(x'_k - x_k)}{|\bar{x}'-\bar{x}|^3} \quad 2.22$$

8

2.22

$\to \overline{\overline{W}}$ is symmetric

$W_{jk} = W_{kj}$

trace $\begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}$

Solo 3 terms $\boxed{j=k}$

$$\mathrm{Trace}(\overline{\overline{W}}) = \sum_{j=1}^{3} W_{jj} =$$

$$= -\frac{1}{2} G \int d^3\bar{x}\, \rho(\bar{x}) \int d^3\bar{x}' \frac{\rho(\bar{x}')}{|\bar{x}' - \bar{x}|^3} \cancel{|\bar{x}' - \bar{x}|^2}$$

def. of ϕ

$$= \frac{1}{2} \int d^3\bar{x}\, \rho(\bar{x})\, \phi(\bar{x})$$ is eq. 2.18 for W (2.24)

$$\boxed{\mathrm{trace}(\overline{\overline{W}}) = W}$$

If we make directly the trace using ~~of~~ 2.18

$$\sum_{j=1}^{3} W_{jj} = -\sum_{j=1}^{3} \int d^3\bar{x}\, \rho(\bar{x})\, x_j \frac{\partial \phi}{\partial x_j} =$$

$$= -\int d^3\bar{x}\, \rho(\bar{x})\, \bar{x} \cdot \bar{\nabla}\phi$$

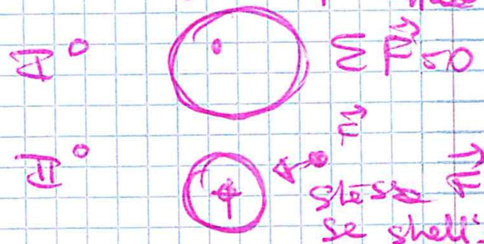
So

$$W = -\int d^3\bar{x}\, \rho(\bar{x})\, \bar{x}\, \bar{\nabla}\phi$$ (2.24)

alternative expression for the potential energy of a body

Spherical systems

Newton's theorems → (page 60)



COROLLARY del I° teo

gravitational potential inside an empty shell is constant since

$$\overline{\nabla}\phi = -\overline{F} = 0 \implies \phi = \text{const.}$$

$$\phi(r) = ? \longleftarrow \text{from eq. 2.3} \quad \text{def. of gravit. potent.}$$

$$\phi(\vec{x}) = -G \int d^3x' \frac{\rho(x')}{|\vec{x}' - \vec{x}|}$$

$$\hookrightarrow \phi = -\frac{GM}{R} \quad (2.25)$$

al centro! e quindi ovunque $\vec{x} = \phi$ $x' = R$

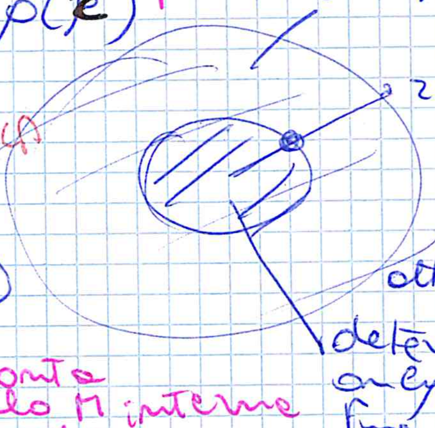
FOR 1 shell !

→ also $\phi = -\frac{G}{R}\int dV \rho(r')$

shell esterne conta x potential

ATTRAZIONE GRAVITAZIONALE DI UNA DISTRIBUZIONE DI DENSITÀ SFERICA

from I and II Theorems



$$\vec{F}(\vec{r}) = -\frac{GM(r)}{r^2}\hat{e}_r \quad 2{,}27$$

where

$$M(r) = 4\pi \int_0^r dr' \, r'^2 \rho(r')$$

conta solo M interna ad r

Total gravitational potential is given by the sum of the potential of the shells of mass

$$dM(r) = 4\pi \rho(r) r^2 dr$$

$$\phi(r) = \underbrace{-\frac{G}{r}\int_0^r dM(r')}_{\text{internal shells}} \underbrace{- G\int_r^\infty \frac{dM(r')}{r}}_{\text{external}}$$

↳ cioè potential esso quello ad un ...