5 STABILITY OF COLLISIONLESS SYSTEMS

MATS

1 PRELIMINARIES

dynamics of self-gravitating collisionless stellar systems

collisionless Boltzmann eq. + Poisson equation

$$\frac{\partial f}{\partial t} + \bar{v}\frac{\partial f}{\partial \bar{x}} - \bar{\nabla}\phi\frac{\partial f}{\partial \bar{v}} = 0 \qquad 5.1$$

$$\nabla^2\phi(\bar{x},t) = 4\pi G\int f(\bar{x},\bar{v},t)\,d^3\bar{v} \qquad 5.2$$

equilibrium stellar system $f_0(\bar{x},\bar{v})$, $\phi_0(\bar{x})$

$$\bar{v}\frac{\partial f_0}{\partial \bar{x}} - \nabla\phi_0\frac{\partial f_0}{\partial \bar{v}} = 0$$

$$\nabla^2\phi_0 = 4\pi G\int f_0\,d^3\bar{v} \qquad 5.3$$

Now small perturbation $f(\bar{x},\bar{v},t) = f_0(\bar{x},\bar{v}) + \epsilon f_1(\bar{x},\bar{v},t)$

$\phi(\bar{x},t) = \phi_0(\bar{x}) + \epsilon\phi_1(\bar{x},t)$

$\epsilon \ll 1 \longrightarrow$ 5.1 e 5.2 using 5.3

$$\cancel{\frac{\partial f_0}{\partial t}} + \epsilon\frac{\partial f_1}{\partial t} + \cancel{\bar{v}\frac{\partial f_0}{\partial \bar{x}}} + \epsilon\bar{v}\frac{\partial f_1}{\partial \bar{x}} - \cancel{\bar{\nabla}\phi_0\frac{\partial f_0}{\partial \bar{v}}} - \epsilon\bar{\nabla}\phi_0\frac{\partial f_1}{\partial \bar{v}} +$$

$$-\epsilon\bar{\nabla}\phi_1\frac{\partial f_0}{\partial \bar{v}} - \cancel{\epsilon^2\bar{\nabla}\phi_1\frac{\partial f_1}{\partial \bar{v}}} = 0$$

$/\epsilon$

$$\frac{\partial f_1}{\partial t} + \bar{v}\frac{\partial f_1}{\partial \bar{x}} - \bar{\nabla}\phi_0\frac{\partial f_1}{\partial \bar{v}} - \bar{\nabla}\phi_1\cdot\frac{\partial f_0}{\partial \bar{v}} = 0 \qquad \boxed{5.5}$$

eq. of B. collisionless and linearized

$$\cancel{\nabla^2\phi_0} - \epsilon\nabla^2\phi_1 = 4\pi G\cancel{\int f_0\,d^3\bar{v}} + \epsilon\,4\pi G\int f_1\,d^3\bar{v}$$

$/\epsilon$

$$\nabla^2\phi_1 = 4\pi G\int f_1\,d^3\bar{v} \qquad \boxed{5.6}$$

SUMMARY FOR FLUIDS

ct. eq. $\frac{\partial \rho}{\partial t} + \bar\nabla(\rho\bar v) = 0$ 5.7

Euler's eq. $\frac{\partial \bar v}{\partial t} + (v\bar\nabla)\bar v = -\frac{1}{\rho}\bar\nabla p - \bar\nabla\phi$ 5.8

Poisson eq. $\nabla^2\phi = 4\pi G\rho$ 5.9

$p(\bar x,t) = p[\rho(x),t]$ 5.10 **barotropic** $p = p(\rho)$ ~~5.10~~

→ linearized eqs for the fluid

equilibrium fluid system

$\rho_0(\bar x),\ p_0(\bar x),\ \bar v_0(\bar x),\ \phi_0(\bar x)$

small perturbation $\rho(\bar x,t) = \rho_0(x) + \varepsilon\rho_1(\bar x,t)$

$\varepsilon \ll 1$

$p(\bar x,t) =$

$\bar v(\bar x,t) =$

$\phi(\bar x,t) =$

5.7 →

$$\cancel{\frac{\partial\rho_0}{\partial t}} + \varepsilon\frac{\partial\rho_1}{\partial t} + \cancel{\bar\nabla(\rho_0\bar v_0)} + \varepsilon\bar\nabla(\rho_0\bar v_1) + \varepsilon\bar\nabla(\rho_1\bar v_0) + \cancel{\varepsilon^2(\rho_1\bar v_1)} = 0$$

$/\varepsilon$

$$\frac{\partial\rho_1}{\partial t} + \bar\nabla(\rho_0\bar v_1) + \bar\nabla(\rho_1\bar v_0) = 0 \quad \boxed{5.12a}$$

5.8 →

$$\frac{\partial\bar v_1}{\partial t} + (\bar v_0\bar\nabla)v_1 + (\bar v_1\bar\nabla)\bar v_0 = \underbrace{\frac{\rho_1}{\rho_0^2}\bar\nabla p_0 - \frac{1}{\rho_0}\bar\nabla p_1}_{} - \bar\nabla\phi_1 \quad \boxed{5.12b}$$

VEDI FOGLIO

$$= -\bar\nabla h_1 - \bar\nabla\phi_1$$

$\boxed{5.15}$

$$h_1 = \left(\frac{dp}{d\rho}\right)_0\frac{\rho_1}{\rho_0} = v_s^2\frac{\rho_1}{\rho_0} = \frac{p_1}{\rho_0}$$

5.9 →

$$\cancel{\nabla^2\phi_0} + \varepsilon\nabla^2\phi_1 = \cancel{4\pi G\rho_0} + 4\pi G\varepsilon\rho_1$$

$$\nabla^2\phi_1 = 4\pi G\rho_1 \quad \boxed{5.12c}$$

2

5.10 → $p_0 + \varepsilon p_1 = p(\rho_0 + \varepsilon\rho_1)$ $\quad \cancel{p_0} + \varepsilon p_1 = \cancel{p(\rho_0)} + \left.\frac{dp}{d\rho}\right|_{\rho=0}\cdot\varepsilon\rho_1$

$\boxed{5.12d}$

$$p_1 = \left(\frac{dp}{d\rho}\right)_0\rho_1 = v_s^2\rho_1$$

From 5.8 $\rightarrow$ 5.12 b

$$\cancel{\frac{\partial \bar{v}_0}{\partial t}} + \boxed{\varepsilon \frac{\partial \bar{v}_1}{\partial t}} + \cancel{(\bar{v}_0 \bar{\nabla}) \bar{v}_0} + \boxed{\varepsilon (\bar{v}_1 \bar{\nabla}) \bar{v}_0} + \boxed{\varepsilon (\bar{v}_0 \bar{\nabla}) \bar{v}_1} + \cancel{\varepsilon^2 (\bar{v}_1 \bar{\nabla}) \bar{v}_1} =$$

$$= -\frac{\bar{\nabla} p_0}{\rho_0 + \varepsilon \rho_1} \boxed{- \frac{\varepsilon \bar{\nabla} p_1}{\rho_0 + \varepsilon \rho_1}} - \cancel{\bar{\nabla} \phi_0} \boxed{- \varepsilon \bar{\nabla} \phi_1}$$

$$-\frac{\bar{\nabla} p_0}{(\rho_0^2 - \cancel{\varepsilon^2 \rho_1^2})} (\rho_0 - \varepsilon \rho_1) = \cancel{\frac{-\bar{\nabla} p_0}{\rho_0}} + \boxed{\frac{\varepsilon \rho_1 \bar{\nabla} p_0}{\rho_0^2}} *$$

$$\boxed{} = -\frac{\varepsilon \bar{\nabla} p_1}{(\rho_0^2 - \cancel{\varepsilon^2 \rho_1^2})} (\rho_0 - \varepsilon \rho_1) = \boxed{\frac{-\varepsilon \bar{\nabla} p_1}{\rho_0}} * + \cancel{\frac{\varepsilon^2 \rho_1 \bar{\nabla} p_1}{\rho_0^2 - \varepsilon^2 \rho_1^2}}$$

From eq. of the equilib.

II order

remnant parts $/\varepsilon$ $\Longrightarrow$ eq. $\boxed{5\text{-}12b}$ * $\longrightarrow$ 5-12c

(A) From definition of $h(\bar{x},t)$ 5-(A) $= -\bar{\nabla} h_1 - \bar{\nabla} \phi_1$ *

$$h(\bar{x},t) \equiv \int_0^{\rho(\bar{x},t)} \frac{dp(\rho')}{d\rho'}$$

$$h = h_0 + \varepsilon h_1$$

$$\cancel{h_0} + \varepsilon h_1 = \cancel{\int_0^{\rho_0} \frac{dp(\rho')}{\rho'}} + \int_{\rho_0}^{\rho_0 + \varepsilon \rho_1} \frac{dp(\rho')}{\rho'}$$

$$\varepsilon h_1 = \left(\frac{dp}{d\rho}\right)_0 \frac{1}{\rho_0} \varepsilon \rho_1$$

$\frac{dp}{d\rho}$ $\frac{dp}{\rho}$

$$h_1 = \left(\frac{dp}{d\rho}\right)_0 \frac{\rho_1}{\rho_0} = v_s^2 \frac{\rho_1}{\rho_0} = \frac{p_1}{\rho_0}$$

5-12d

5.15

5-12d $\quad p_1 = \left(\frac{dp}{d\rho}\right)_0 \rho_1 \equiv v_s^2 \rho_1$

(B)

$$\bar{\nabla} h_1 = \bar{\nabla}\left(\frac{p_1}{\rho_0}\right) =$$

$$= \frac{\rho_0 \bar{\nabla} p_1 - \rho_1 \bar{\nabla} \rho_0}{\rho_0^2} =$$

$$= \frac{\bar{\nabla} p_1}{\rho_0} - \frac{\left(\frac{dp}{d\rho}\right)_0 \rho_1 \bar{\nabla} \rho_0}{\rho_0^2} =$$

$$= \frac{\bar{\nabla} p_1}{\rho_0} - \frac{\frac{\bar{\nabla} p_0}{\cancel{\bar{\nabla} \rho_0}} \rho_1 \cancel{\bar{\nabla} \rho_0}}{\rho_0^2}$$

$$\bar{\nabla} h_1 = -\frac{\rho_1}{\rho_0^2} \bar{\nabla} p_0 + \frac{1}{\rho_0} \bar{\nabla} p_1 \longrightarrow \boxed{5\text{-}12\,b}$$

THE JEANS INSTABILITY

An infinite homogeneous gravitating system cannot be in static equilibrium
(no pressure to balance gravity!)

we costruct a fictitious infinite system (homogeneous!) (in equilibrium!) using the JEANS SWINDLE

if $\vec{v}_0 \equiv 0$ and p_0 and ρ_0 are const.

From Euler's eq. $\rightarrow \vec{\nabla}\phi_0 = 0$
Poisson eq. $\nabla^2\phi_0 = 4\pi G\rho_0$ $\Big\} \Rightarrow \rho_0 = 0$

AD HOC ASSUMPTION

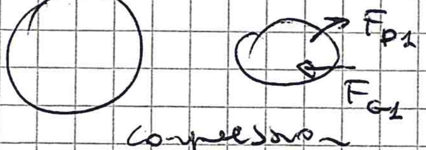
Poisson eq. only for the perturbed density and ϕ while the unperturbed ϕ is $\not\equiv$ zero! $\boxed{\phi_0 \equiv 0}$

This should be valid for $\lambda \ll \lambda_0$
small scale

PHYSICAL BASIS OF THE JEANS INSTABILITY

perturbation related to a scale r = radius of a sphere

homog. fluid $\rho_0, p_0, \vec{v}_0 = 0$



perturbation
$\rho_1 \sim \alpha\rho_0$
$p_1 \sim \left(\frac{dp}{d\rho}\right)_0 \alpha\rho_0$ from 4.12d
$\sim \alpha v_s^2 \rho_0$

$V \longrightarrow (1-\alpha)V$ $\alpha \ll 1$

pressure force per unit mass

$$\vec{F}_p = -\frac{\vec{\nabla}p}{\rho}$$

$\frac{F_p}{m} \approx \frac{p\, S}{m} = \frac{p}{x}\cdot\frac{Sx}{m} = \frac{\vec{\nabla}p}{\rho}$

extra pressure force $|\vec{F}_{p1}| = \left|\frac{\vec{\nabla}p_1}{\rho_0}\right| \sim \frac{p_1}{\rho_0 r} \sim$

$\sim \frac{\alpha v_s^2}{\ldots}$

3

extra gravitational force

$\overline{F}_{G1} = -\overline{\nabla}\phi_1 \qquad |\overline{F}_{G1}| \sim \frac{GM\alpha}{r^2} \qquad M = \frac{4}{3}\pi r^3 \rho_0$

$|\overline{F}_{G1}| \sim G\rho_0 r \alpha$

If $\overline{F}_{p1} + \overline{F}_{G1}$ is outward $\rightarrow$ perturbation is stable

is inward $\rightarrow$ unstable

instability if

$|F_{G1}| > |F_{p1}|$

$G\rho_0 r \alpha \gtrsim \alpha v_s^2 / r$

$r^2 \gtrsim \frac{v_s^2}{G\rho_0}$

if scale of perturbation $\lambda > \frac{v_s}{\sqrt{G\rho_0}}$ perturb. is UNSTABLE!

$\equiv$

$T_{dyn} < t_{sound}$

$\frac{1}{\sqrt{G\rho_0}} \qquad t_s \approx \frac{r}{v_s}$

4

JEANS INSTABILITY FOR A FLUID

linearized fluid eqs. 5.12a → c + 5.15

equilibrium $\rho_0 = 0$ $\bar{v}_0 = 0$ Jeans swindle $\phi_0 = 0$ but

5.12a + $\bar{v}_0 = 0$ $\quad \dfrac{\partial \rho_1}{\partial t} + \rho_0 \bar{\nabla} \bar{v}_1 = 0 \qquad$ 5.17 a

5.12b $\bar{v}_0 = 0$ $\quad \dfrac{\partial \bar{v}_1}{\partial t} = -\bar{\nabla} h_1 - \bar{\nabla}\phi_1 \qquad$ b

5.12c $\quad \nabla^2 \phi_1 = 4\pi G \rho_1 \qquad$ c

5.15 $\quad h_1 = \dfrac{v_s^2 \rho_1}{\rho_0} \qquad$ d

$\dfrac{d}{dt}$ (5.17a) and $\underset{\text{div}}{\bar{\nabla}}$ (5.17b) and eliminating $\bar{v}_1$, ϕ_1, h_1 with other

$$\frac{\partial^2 \rho_1}{\partial t^2} + \frac{\partial}{\partial t}\left(\rho_0 \bar{\nabla} \bar{v}_1\right) = 0 \qquad \frac{\partial^2 \rho_1}{\partial t^2} + \rho_0 \frac{\partial (\bar{\nabla} \bar{v}_1)}{\partial t} = 0$$

$$\bar{\nabla} \frac{\partial \bar{v}_1}{\partial t} = -\nabla^2 h_1 - \nabla^2 \phi_1$$

5.17d 5.17c

$$\frac{\partial^2 \rho_1}{\partial t^2} + \rho_0\left(-\nabla^2 h_1 - \nabla^2 \phi_1\right) = 0$$

$$\frac{\partial^2 \rho_1}{\partial t^2} - \frac{v_s^2}{\rho_0} \nabla^2 \rho_1 - 4\pi G \rho_0 \rho_1 = 0 \qquad \boxed{5.18}$$

wave equation for perturbed density

solution

$$\rho_1(\bar{x}, t) = C e^{i(\bar{k}\bar{x} - \omega t)} \qquad 5.19$$

ω → frequency

k → numero d'onda wave number

TRY : $\dfrac{\partial \rho_1}{\partial t} = C \cdot e^{\square} (-\omega i)$, $\dfrac{\partial^2 \rho_1}{\partial t^2} = C e^{\square} (-\omega^2)$

$$\frac{\partial^2 \rho_1}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left[C e^{\square} (i k_1)\right] = C e^{\square} (-1 \cdot k_1^2)$$

5

$$C e^{\circlearrowleft}(-\omega^2) - v_s^2 \, C e^{\circlearrowleft}\left[-1(k_1^2+k_2^2+k_3^2)\right] - 4\pi G \rho_0 C e^{\circlearrowleft} = 0$$

$$\boxed{-\omega^2 + k^2 v_s^2 - 4\pi G \rho_0 = 0} \qquad \boxed{5-20}$$

dispersion relation $\omega = \omega(k)$ $\qquad k = |\bar{k}|$

general solution of 5.19 is a superimposition of 5.19

$$\rho_1(\bar{x}, t) = \int C(\bar{k}) \, e^{i[\bar{k}\bar{x} - \omega(k)t]} \, d^3\bar{k}$$

if ρ_0 small or $\lambda = \frac{2\pi}{k}$ is small

or k is big

5-20 $\rightarrow$ that of sound wave $\omega^2 = v_s^2 k^2$ (1E-21)

when $\lambda \uparrow$ $\rightarrow$ $\omega^2 \downarrow$
or $k \downarrow$

AT some point
$\omega^2 < 0$ assume
$\omega^2 = -\gamma^2$

5.19 $\quad e^{\pm \gamma t}$

$\omega = +i\gamma \qquad \omega = -i\gamma$

exponential growing and decaying solutions

$\downarrow$

system unstable!

$\downarrow$

IF $\omega^2 < 0$ $\quad \boxed{k^2 < k_J^2 \equiv \frac{4\pi G \rho_0}{v_s^2} \quad \boxed{5.22}}$

$\downarrow$

JEANS WAVE NUMBER

$\lambda_J = \frac{2\pi}{k_J}$ $\qquad$ or

$$\lambda^2 > \lambda_J^2 \equiv \frac{\pi v_s^2}{G \rho_0}$$

JEANS

JEANS MASS $M_J = \frac{4\pi}{3}\rho_0\left(\frac{1}{2}\lambda_J\right)^3 = \frac{1}{6}\pi\rho_0\left(\frac{\pi v_s^2}{G\rho_0}\right)^{3/2}$

THE JEANS INSTABILITY FOR STELLAR SYSTEMS

we use linearized eqs. 5.5 - 5.6

$f_0(\bar{x},\bar{v},t) = \underline{f_0(\bar{v})}$ homogeneous and time indep.

and $\phi_0 = 0$ JEANS swindle

5.5 $\to \nabla\phi_0 = 0$: $\frac{\partial f_1}{\partial t} + \bar{v}\frac{\partial f_1}{\partial \bar{x}} - \bar{\nabla}\phi_1 \cdot \frac{\partial f_0}{\partial \bar{v}} = 0$

5.6: $\nabla^2\phi_1 = 4\pi G\int f_1\, d^3\bar{v}$ (5.25)

solution is

$f_1(\bar{x},\bar{v},t) = f_a(\bar{v})\, e^{i(\bar{k}\bar{x}-\omega t)}$

$\phi_1(\bar{x},t) = \phi_a\, e^{i(\bar{k}\bar{x}-\omega t)}$

5.25a

$f_a e(-i\omega) + f_a \bar{v} e\,[i(k_1,k_2,k_3)] - \phi_a e\,[i(k_1,k_2,k_3)]\cdot\frac{\partial f_0}{\partial \bar{v}} = 0$ $\vec{k}$

5.25b

$\phi_a e\,[-1(k^2)] = 4\pi G e\int f_a\, d^3\bar{v}$

$f_a = \frac{\phi_a \bar{k}\frac{\partial f_0}{\partial \bar{v}}}{\bar{k}\bar{v} - \omega}$

$-k^2\phi_a = 4\pi G\int \frac{\phi_a \bar{k}\frac{\partial f_0}{\partial \bar{v}}}{\bar{k}\bar{v}-\omega}\, d^3 v$

$k^2\phi_a + 4\pi G\int \frac{\phi_a \bar{k}\frac{\partial f_0}{\partial \bar{v}}}{\bar{k}\bar{v}-\omega}\, d^3\bar{v} = 0 \quad / k^2\phi_a$

$$\boxed{1 + \frac{4\pi G}{k^2}\int \frac{\bar{k}\frac{\partial f_0}{\partial \bar{v}}}{\bar{k}\bar{v}-\omega}\, d^3\bar{v} = 0} \quad \boxed{5.27}$$

dispersion relation

Assume that f_0 is Maxwellian

$$f_0(\vec{v}) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}} \, e^{-\frac{1}{2}\frac{v^2}{\sigma^2}} \qquad 5-28$$

$|\vec{v}|^2$

$$\frac{\partial f_0}{\partial \vec{v}} = \frac{-\rho_0/\sigma^2 \, e^{-\frac{v^2}{2\sigma^2}}}{(2\pi\sigma^2)^{3/2}} \cdot \vec{v}$$

v_x-axis along $\vec{k}$ $\quad \vec{v}_x \parallel \vec{k}$

$\vec{k}\cdot\vec{v} = k_x v_x$

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}v_y^2/\sigma^2} dv_y = \sqrt{2\pi\sigma^2}$$

$dv_z = \ldots$

5-27 $\rightarrow$

$$1 + \frac{4\pi G}{k^2}\left[\frac{-\rho_0/\sigma^2}{(2\pi\sigma^2)^{3/2}} \int \frac{e^{-\frac{1}{2}\frac{(v_x^2+v_y^2+v_z^2)}{\sigma^2}} k v_x}{(k v_x - \omega)} d^3\vec{v}\right] = 0$$

$$1 - \frac{4\pi G \rho_0}{k^2 \sigma^5 (2\pi)^{3/2}} \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{v_z^2}{2\sigma^2}} dv_z}_{\sqrt{2\pi\sigma^2}} \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{v_y^2}{2\sigma^2}} dv_y}_{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} \frac{e^{-\frac{v_x^2}{2\sigma^2}} \cdot v_x}{(v_x - \frac{\omega}{k})} dv_x = 0$$

$$1 - \frac{4\pi G \rho_0 \, 2\pi\sigma^2}{k^2 \sigma^5 (2\pi)^{3/2}} \int \frac{e^{-\frac{v_x^2}{2\sigma^2}} \cdot v_x}{(v_x - \frac{\omega}{k})} dv_x = 0$$

$$1 - \frac{2(2\pi)^{\cancel{1}} \rho_0 G}{k^2 \sigma^3 \cancel{(2\pi)} \sqrt{2\pi}} \int = 0$$

$$1 - \frac{2\sqrt{2\pi}\rho_0 G}{k \sigma^3} \underbrace{\int \frac{v_x e^{-\frac{v_x^2}{2\sigma^2}}}{(k v_x - \omega)} dv_x}_{} = 0 \qquad 5.29$$

By analogy with the fluid, boundary for stability at $\omega = 0$

$$\int \frac{e^{\ldots}}{k} dv_x = \frac{\sqrt{2\pi\sigma^2}}{k}$$

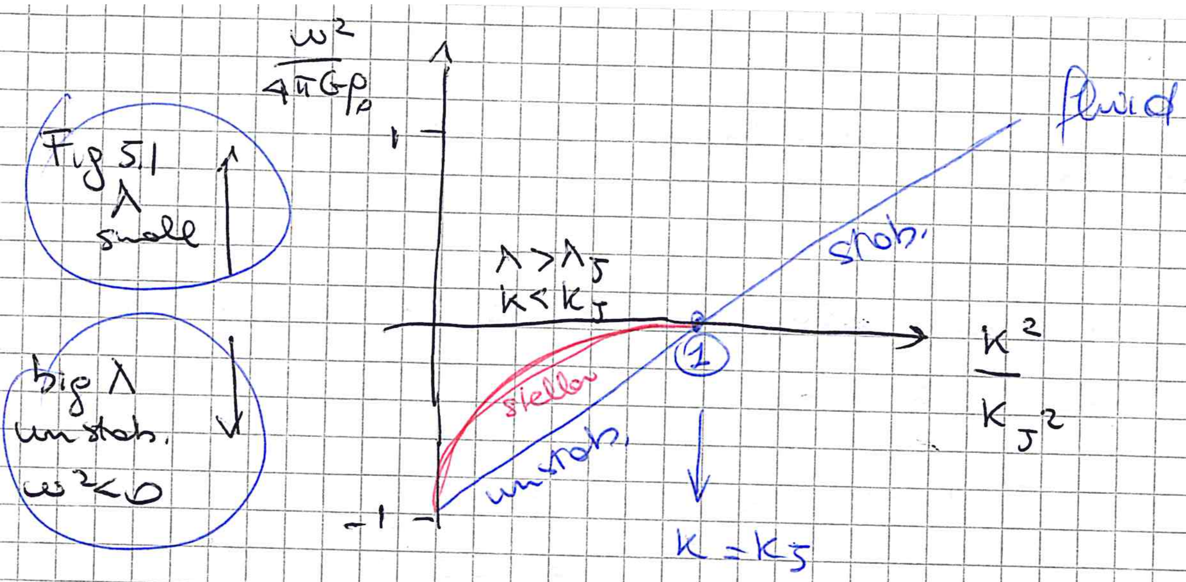
$$1 - \frac{4\pi G\rho_0}{k^2\sigma^2} = 0$$

$$\boxed{k^2(\omega=0) \equiv k_J^2 = \frac{4\pi G\rho_0}{\sigma^2}} \qquad 5.30$$

JEANS WAVENUMBER FOR THE STELLAR SYSTEM

8) cf. with 5-22

$$\boxed{\sigma \longleftrightarrow v_s}$$



The dispersion relation

$$5\text{-}20 \quad k \ll k_J^2 = \frac{4\pi G\rho_0}{v_s^2} \quad \text{unstable}$$

$$5\text{-}31 \quad k^2 \ll k_J^2 = \frac{4\pi G\rho_0}{\sigma^2}$$

For $k > k_J$, $\lambda < \lambda_J$ — The resemblance breaks down

Stable perturbations are damped (Landau) damping

$\Rightarrow$ There is a dissipation even if there are no dissipative term in the B. eq.

ASSUMP. system is infinite and homogeneous + Jeans swindle

stellar system of mass M and v dispersion σ

$M \sim \frac{\sigma^2 \lambda_0}{G} \rightarrow$ charact. size of the system

$$\lambda \propto \frac{GM}{\sigma^2} \qquad \rho \sim \frac{M}{\lambda_0^3} \qquad \lambda_0^3 \sim \frac{\sigma^2 \lambda_0}{G\rho}$$

$$\lambda_0^2 \sim \frac{\sigma^2}{G\rho} \qquad \lambda_J^2 \approx \frac{\pi\sigma^2}{G\rho} \sim \frac{\sigma^2}{G\rho} \sim \lambda_0^2$$

comparable!

$\lambda_J \sim \lambda_0 \Rightarrow$ homogeneity is not valid

homogenity assumption ~~valid~~ assumed to be valid for $\lambda \ll \lambda_0$

$\Rightarrow$ stationary stellar systems are stables on small scales

10