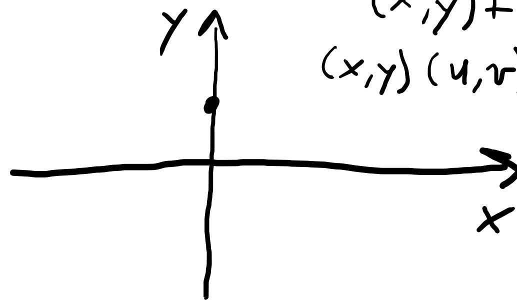
24 Settembre

$$(x,y)+(u,v)=(x+u,\,y+v)$$
$$(x,y)(u,v)=(xu-yv,\,xv+yu)$$



$$c\,(x,y)=(cx,\,cy)$$

$$(c,0)\,(x,y)=(cx,\,cy)$$

$$(0,1)^2=(0,1)(0,1)=(-1,0)$$

Il generico numero complesso viene denotato con z

Abbiamo appena scoperto che l'equazione

$$z^2 + (1,0) = 0 \quad \text{ammette due soluzioni}$$

$$z = \pm(0,1) \qquad\qquad z^2 = (-1,0)$$

Scriveremo $\quad i = (0,1)$

$$z = (x,y) = (x,0) + (0,y) = \overset{x}{x(1,0)} + y(0,1) =$$

D'ora innanzi identificheremo $(x,0)$ con x

$1 = (1,0)$

$$= x + y\,i = x + i\,y$$

Ricordando che $i^2 = -1$, $z = x + iy$, $w = u + iv$

$$z \cdot w = (x+iy)(u+iv) = xu + i^2 yv + xiv + iyu$$

$$= xu - yv + i\,(xv + yu) = (xu - yv,\ xv + yu)$$

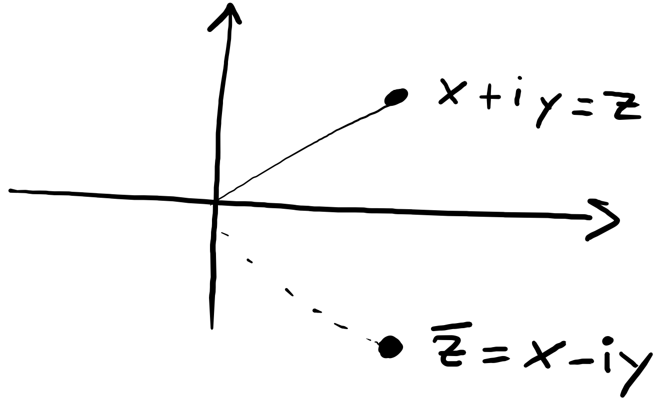
$$z = x + iy = (x, y)$$

$z = x + iy$

$x = \operatorname{Re} z$ parte reale di z

$y = \operatorname{Im} z$ " immaginaria

$z = x+iy$



$|z| = \sqrt{x^2+y^2}$

$\bar{z} = x-iy$ complesso coniugato.

$z = \bar{z} \iff y=0 \iff z \in \mathbb{R}$ (= asse x)

$$(a+b)(a-b)=a^2-b^2$$

$$z\bar{z} = (x+iy)(x-iy)$$

$$= x^2-(iy)^2 = x^2 - iy\, iy = x^2 - i^2y^2 = x^2+y^2$$

$$= |z|^2$$

$$z\bar{z} = |z|^2 = \bar{z}z$$

$$\overline{z+w} = \bar{z}+\bar{w}$$

$$\overline{zw} = \bar{z}\,\bar{w}$$

$$\bar{z} = -z \Longleftrightarrow \mathrm{Re}\, z = 0$$

$\bar{z} = -z \iff \operatorname{Re} z = 0$

$z = x + iy \qquad \bar{z} = x - iy$

$x - iy = -(x+iy)$

$\bar{z} = -z \iff x + iy = -(x - iy) = -x + iy$

$\iff x + iy = -\cancel{x} + iy \iff (x, y) = (-x, y)$

$\iff x = -x$

$y = y$ vere $\forall y \in \mathbb{R}$

$x = -x \iff 2x = 0 \iff x = 0$

Conclusione:

$\bar{z} = -z \iff z = x + iy$ con $x = 0$

$\Updownarrow$

$\operatorname{Re} z = 0$

$0\,z = 0 \quad \forall\, z \in \mathbb{C}$

$1 \cdot z = z \quad \forall\, z \in \mathbb{C}$

<u>Lemma</u> Sia $z \neq 0$. Allora esiste ed è unico, un $w \in \mathbb{C}$ che è l'inverso di z, cioè t.c.

$$wz = zw = 1$$

Scriviamo $w = \dfrac{1}{z}$

w è dato $\quad w = \dfrac{\bar{z}}{|z|^2}$

Dim Dimostriamo solo che se pongo $w = \frac{\bar{z}}{|z|^2}$

ho $z w = 1$

$$z \frac{\bar{z}}{|z|^2} = z \bar{z} \frac{1}{|z|^2} = |z|^2 \frac{1}{|z|^2} = 1$$

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} \qquad z \neq 0$$

$$z = \frac{3+4i}{2+3i} = (3+i4) \quad \frac{1}{2+3i} = (3+i4) \quad \frac{2-3i}{2^2+3^2} =$$

$$= (3+4i) \quad (2-3i) \quad \frac{1}{13}$$

$$= \left(18+i\,(8-9)\right) \frac{1}{13} = \frac{18}{13} - \frac{i}{13}$$

$$|z\,w| = |z|\;|w|$$

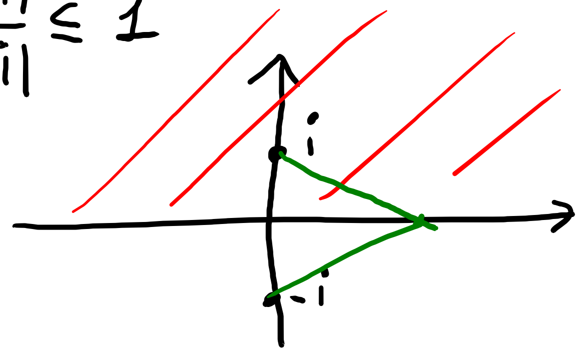
$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

dati z e w in $\mathbb{C}$ la loro distanza $= |z-w| =$

$$= \sqrt{(x-u)^2+(y-v)^2}$$

$$\left|\frac{z-i}{z+i}\right| \leq 1 \iff \frac{|z-i|}{|z+i|} \leq 1$$

$$z+i = z-(-i)$$



$$\frac{|z-i|}{|z+i|} \leq 1 \iff |z-i| \leq |z+i|$$

$$z = x+iy \qquad |x+iy-i| \leq |x+iy+i|$$

$$|x+i(y-1)| \leq |x+i(y+1)|$$

$$\Updownarrow$$

$$|x+i(y-1)|^2 \leq |x+i(y+1)|^2$$

$$\cancel{x^2}+(y-1)^2 \leq \cancel{x^2}+(y+1)^2$$

$$\cancel{y^2}-2y+\cancel{1} \leq \cancel{y^2}+2y+\cancel{1} \iff -2y \leq 2y \iff 4y \geq 0$$

$$\iff \boxed{y \geq 0}$$

$$\left|\frac{2z+i}{z-i}\right| \le 1 \iff |2z+i| \le |z-i| \qquad z = x+iy$$

$$|2x+(2y+1)i|^2 \le |x+(y-1)i|^2$$

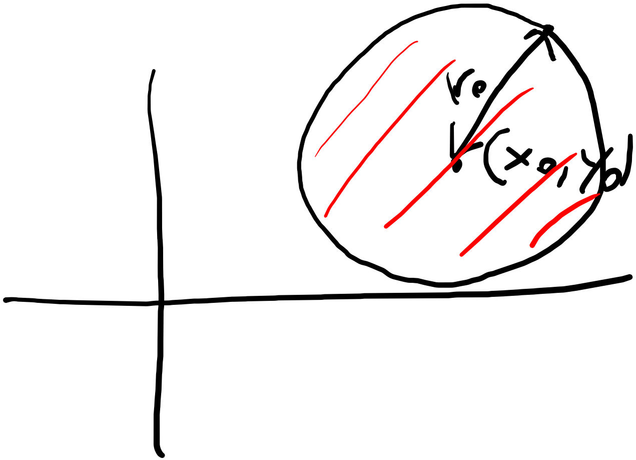
$$|(2x, 2y+1)| = \sqrt{4x^2+(2y+1)^2}$$

$$4x^2+(2y+1)^2 \le x^2+(y-1)^2$$

$$4x^2+4y^2+4y+\cancel{1} \le x^2+y^2-2y+\cancel{1}$$

$$3x^2+3y^2+6y \le 0 \iff x^2+y^2+2y \le 0$$

Devo trovare (x_0, y_0) $r_0 > 0$ t.c.



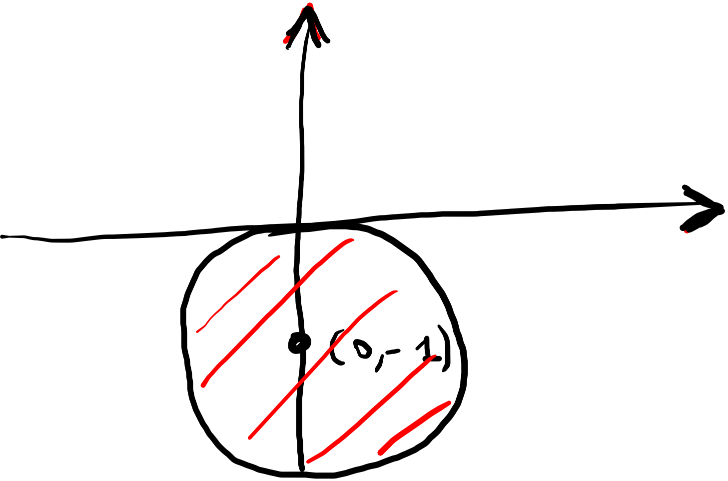
$$(x-x_0)^2+(y-y_0)^2 \leq r_0^2 \quad *$$

$$x^2+y^2+2y \leq 0$$

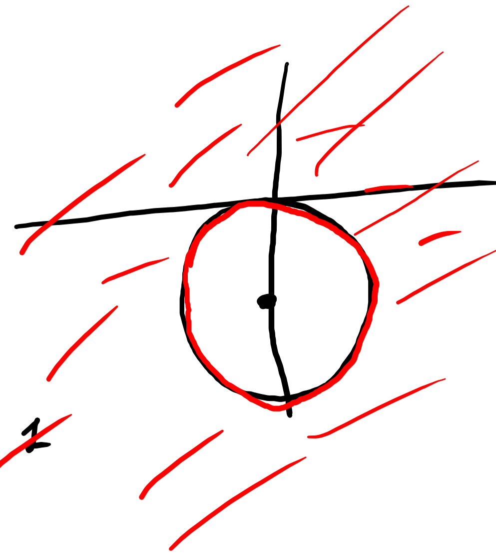
$$x^2+y^2+2y = x^2+y^2+2y\ 1 = x^2+(y+1)^2-1 \leq 0$$

$$x^2+(y+1)^2 \leq 1$$

centro $(0,-1)$ raggio $= 1$


(0,-1)

$$\left|\frac{z-i}{2z+i}\right| \le 1 \qquad |z-i| \le |2z+i| \qquad z = x+iy$$



$$|x+(y-1)i|^2 \le |2x+(2y+1)i|^2$$

$$x^2+(y-1)^2 \le 4x^2+(2y+1)^2$$

$$x^2+y^2-2y+\cancel{1} \le 4x^2+4y^2+4y+\cancel{1}$$

$$3x^2+3y^2+6y \ge 0 \qquad x^2+y^2+2y \ge 0$$

$$x^2+(y+1)^2-1 \ge 0 \qquad \boxed{x^2+(y+1)^2 \ge 1} \qquad (0,-1)$$

$z = x + iy$

$$\mathrm{Im}\left(\frac{1}{z^2+z}\right) =$$

$$= \mathrm{Im}\left(\frac{\overline{z^2+z}}{|z^2+z|^2}\right) = \frac{1}{|z^2+z|^2}\,\mathrm{Im}\left(\bar{z}^2+\bar{z}\right)$$

$$\mathrm{Im}\left(\bar{z}^2+\bar{z}\right) = \mathrm{Im}\left((x-iy)^2 + x - iy\right) = \mathrm{Im}\left((x-iy)^2 - iy\right)$$

$$= \mathrm{Im}\left(\cancel{x^2-y^2} - 2ixy - iy\right) = \mathrm{Im}\left(-(2x+1)y\,i\right) = -(2x+1)y$$

$$\mathrm{Im}\left(\frac{1}{z^2+z}\right) = \frac{-(2x+1)y}{|z^2+z|^2}$$

$$\mathrm{Im}\,\frac{1}{z^2+z} \geq 0$$

$$\boxed{-(2x+1)y \geq 0}$$

$$-(2x+1)y \geq 0 \iff (2x+1)y \leq 0$$

$$(2x+1)y = 0$$