Progettazione di Materiali e Processi

Università degli Studi di Trieste
Facoltà di Ingegneria
Corso di Laurea in Ingegneria di Processo e dei Materiali
A.A. 2021-22

PRODUCT (MATERIALS) AND PROCESS DESIGN

Intro

- Design, Product, Process
- Product Design; Process Design; Product and Process Design
- Material, process, shape, properties, function

Design process

- Example
- Fundamentals
- Identification of needs (market; coevolution; true need)
- Types of design
- **Design tools**
  - **Databases**
  - **Analytical tools**
  - Simulation tools

- **Selection and design of materials and processes (Lughi)**
  - **Tools for optimal systematic selection**
  - Design of materials: case studies (nano, meso, microstructures; hybrid materials; composites)
- **Design and optimization of chemical processes (Fermeglia)**
- **Advanced tools and methods** (ad-hoc lectures and seminars: FEM, product/process economics, Life Cycle Assessment, …)
- **Special topic seminars** (Intellectual Property, product evaluation, materials in industrial design, theory of scenarios, rapid plant assessment, material selection in engines, design for recycle, refurbish, reuse)

Progettazione di Materiali e Processi

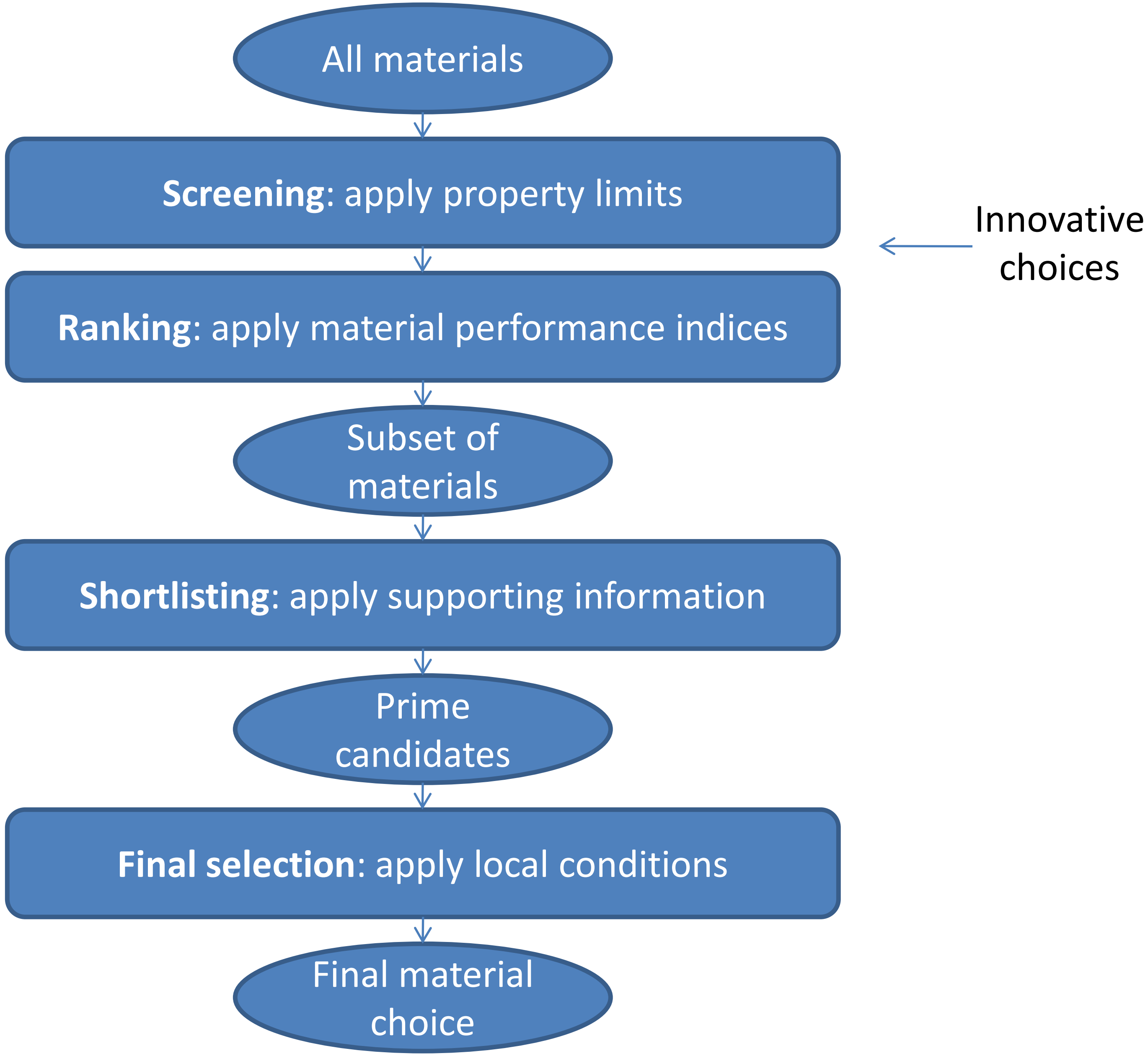
Modulo 1
Selezione sistematica di materiali e processi

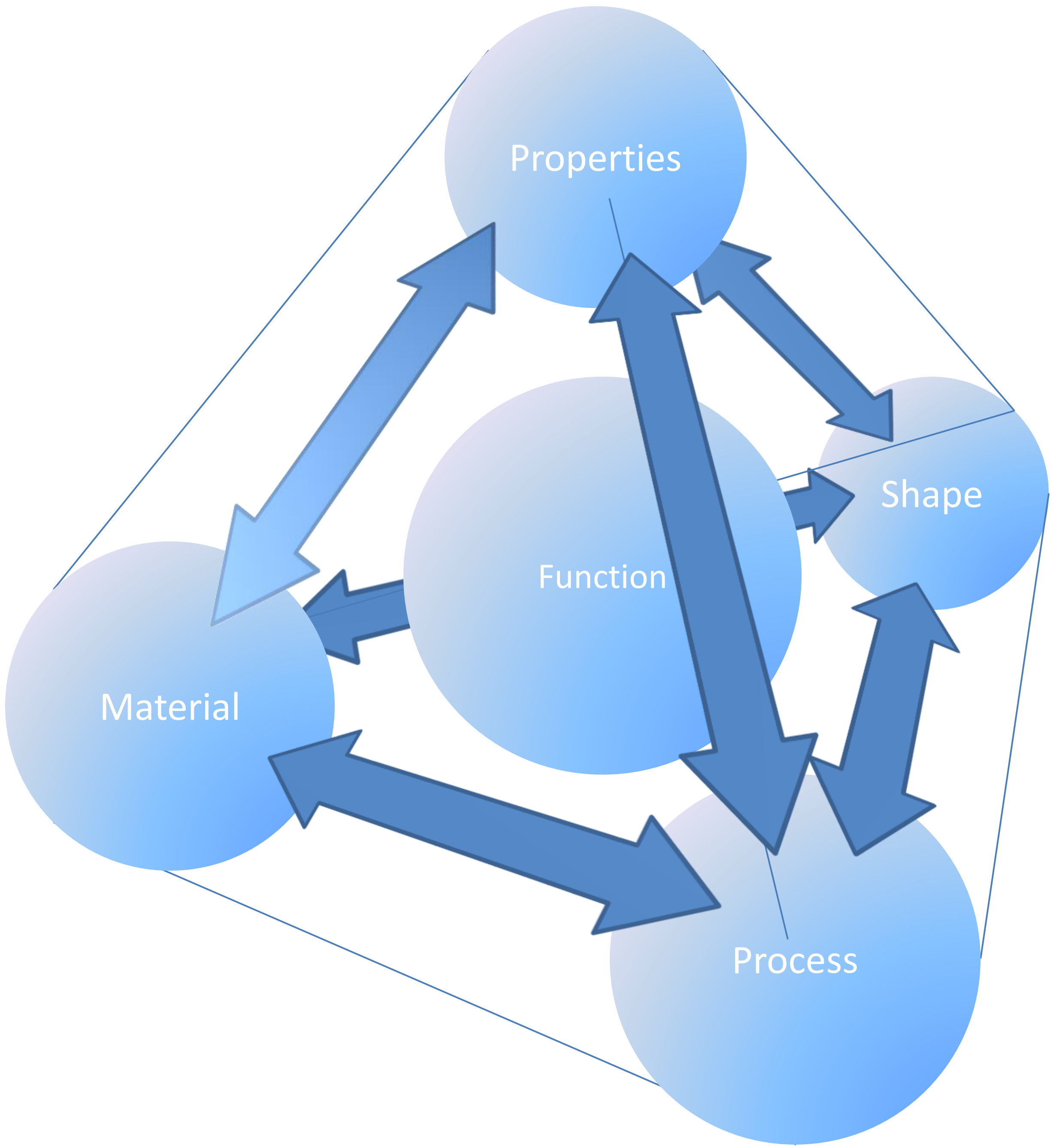
Lecture 2

- Selection process
- Materials indexes

The selection process

The selection process





FUNCTIONS:

- Carry load
- Transmit load
- Transmit heat
- Transmit current
- Store energy
- …

OBJECTIVES:

- Minimize mass
- Minimize cost
- Minimize impact
- …

Function – Objectives - Constraints

Function	**What does the component do?** e.g.: support load, seal, transmit heat, bycicle fork, etc.
Objective	**What do we want to maximize (minimize)?** e.g.: minimize cost, maximize energy storage, minimize weight, etc.
Constraints	**What conditions must be met?** (non-negotiable or negotiable) e.g. geometry, resist a certain load, resist a certain environment, etc.

- Implicit functions (e.g. tie, beam, shaft, column)
- Constraints often translate to property limits (temperature, conductivity, cost, …)
- Some constraints are more complex (e.g. stiffness, strength, etc.) as they are coupled with geometry -> need of a specific objective
- Material indices help unravel such complexity

The material index

Material index

$$\text{Performance} = f\,(F, G, M)$$

Functional requirements

Geometry

Material properties

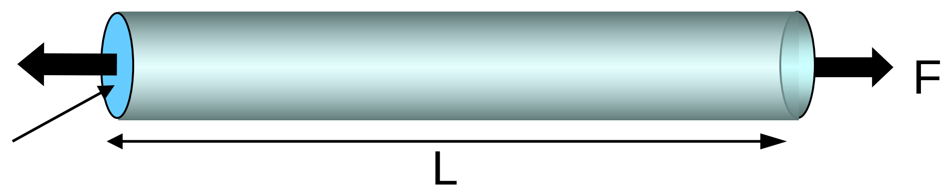
If separable:

$$\text{Performance} = f_1(F)\, f_2(G)\, f_3(M)$$

Material index

Index for a stiff, light tie-rod

Stiff tie of length L and minimum mass



Function — *Tie-rod*

Constraints

- *Length L is specified*
- *Must have axial stiffness > S**

m = mass
A = area
L = length
ρ = density
σ_y = yield strength

Equation for constraint on A:

$$S = \frac{F}{\Delta L} = \frac{E\,A}{L}$$

Objective

Minimize mass m:

$$m = A\,L\,\rho$$

Performance metric m

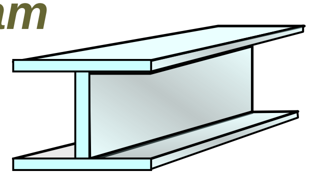
$$m = S^{*} L^{2} \left(\frac{\rho}{E}\right)$$

Chose materials with largest $\left(\frac{E}{\rho}\right)$

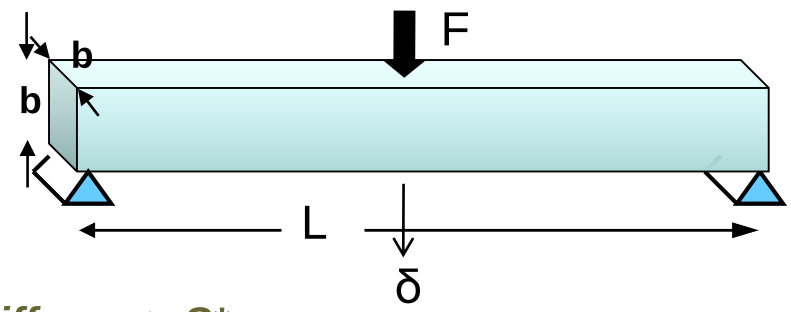
Index for a stiff, light beam

Function

Beam



Stiff beam of length L and minimum mass



Square section, area A = b^2

Constraints

- ***Length L is specified***
- ***Must have bending stiffness > S****

Equation for constraint on A:

$$S = \frac{F}{\delta} = \frac{C\,E\,I}{L^3} = \frac{C\,E\,A^2}{12\,L^3}$$

Objective

Minimize mass m:

$$m = A\,L\,\rho$$

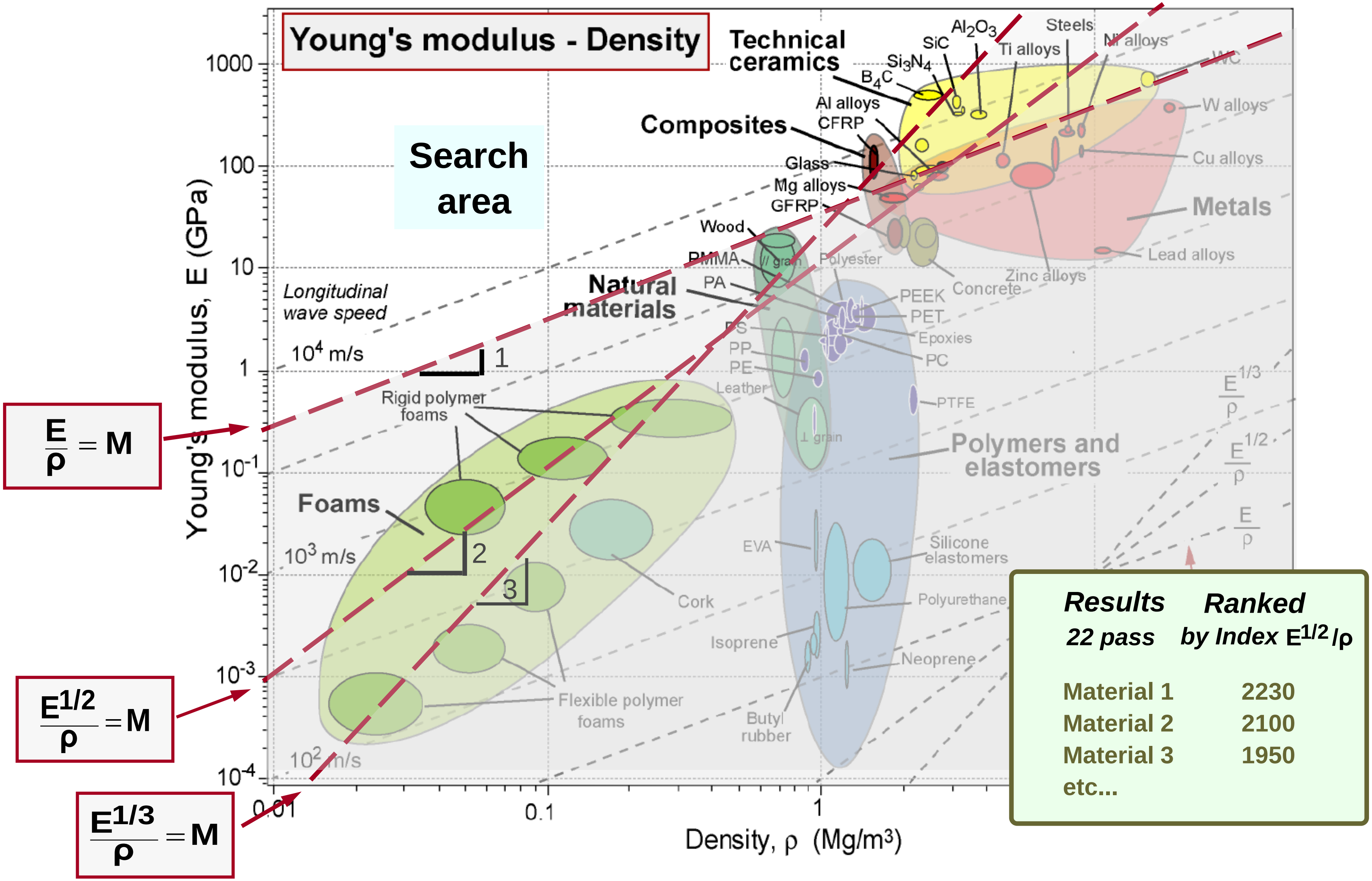
m = mass
A = area
L = length
ρ = density
E = Young's modulus
I = second moment of area
($I = b^4/12 = A^2/12$)
C = constant (here, 48)
S = stiffness (F/δ)

Performance metric m

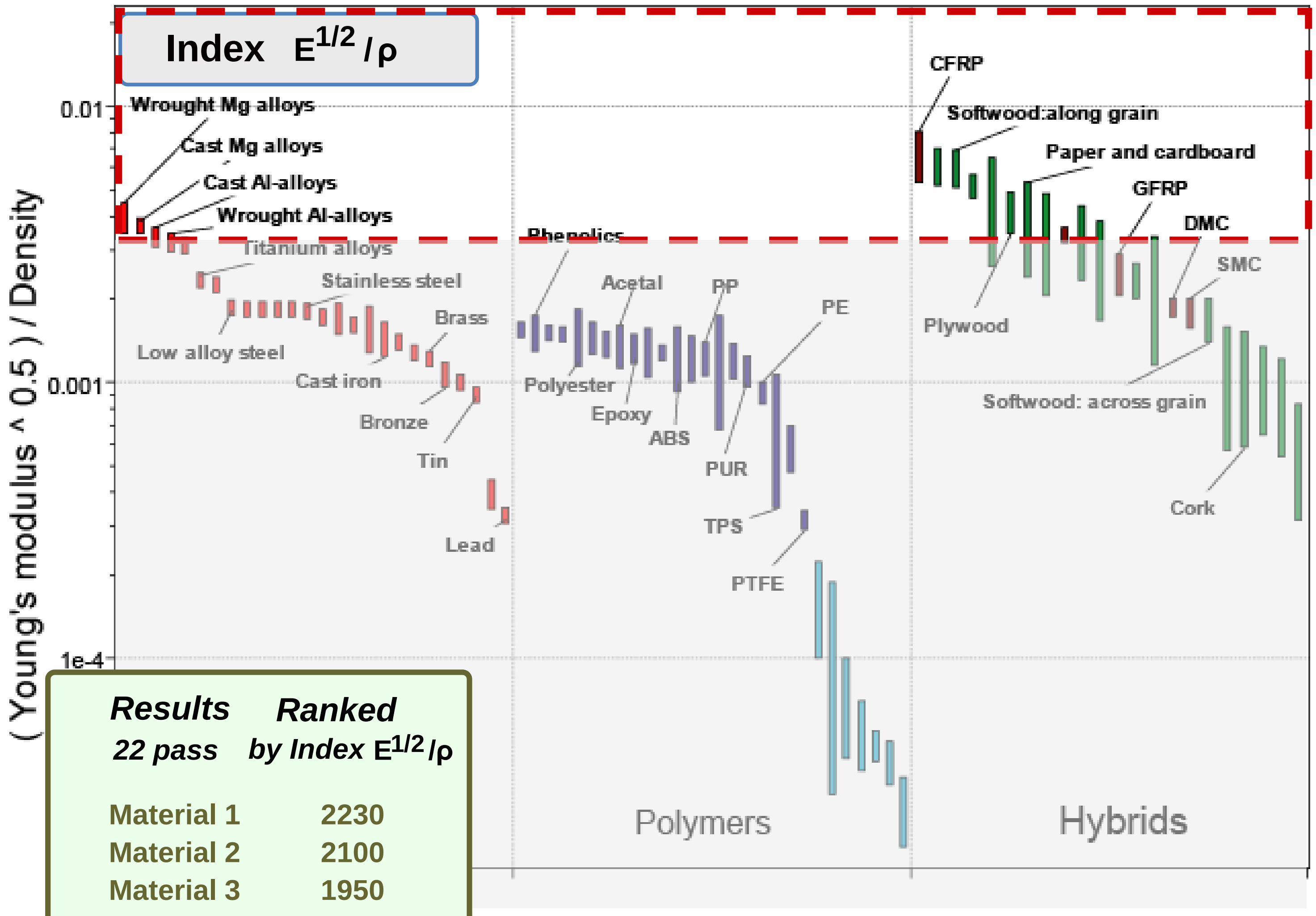
$$m = \left(\frac{12\,L^5\,S^*}{C}\right)^{1/2} \left(\frac{\rho}{E^{1/2}}\right)$$

Chose materials with largest $\left(\frac{E^{1/2}}{\rho}\right)$

Optimized selection using charts



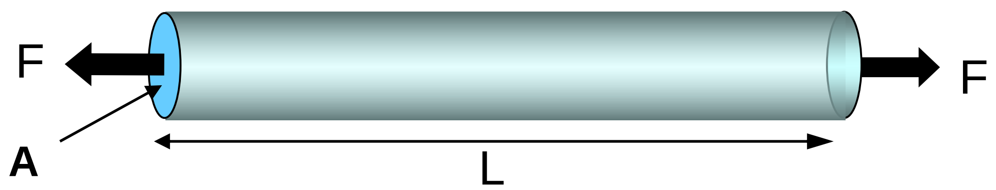
Plotting indices as functions



Results	Ranked
22 pass	by Index $E^{1/2}/\rho$
Material 1	2230
Material 2	2100
Material 3	1950
etc...	

Index for a strong, light tie-rod

Strong tie of length L and minimum mass



Function — *Tie-rod*

Constraints
- *Length L is specified*
- *Must not fail under load F*

Equation for constraint on A:

$$F/A < \sigma_y$$

m = mass
A = area
L = length
ρ = density
σ_y = yield strength

Objective — *Minimize mass m:*

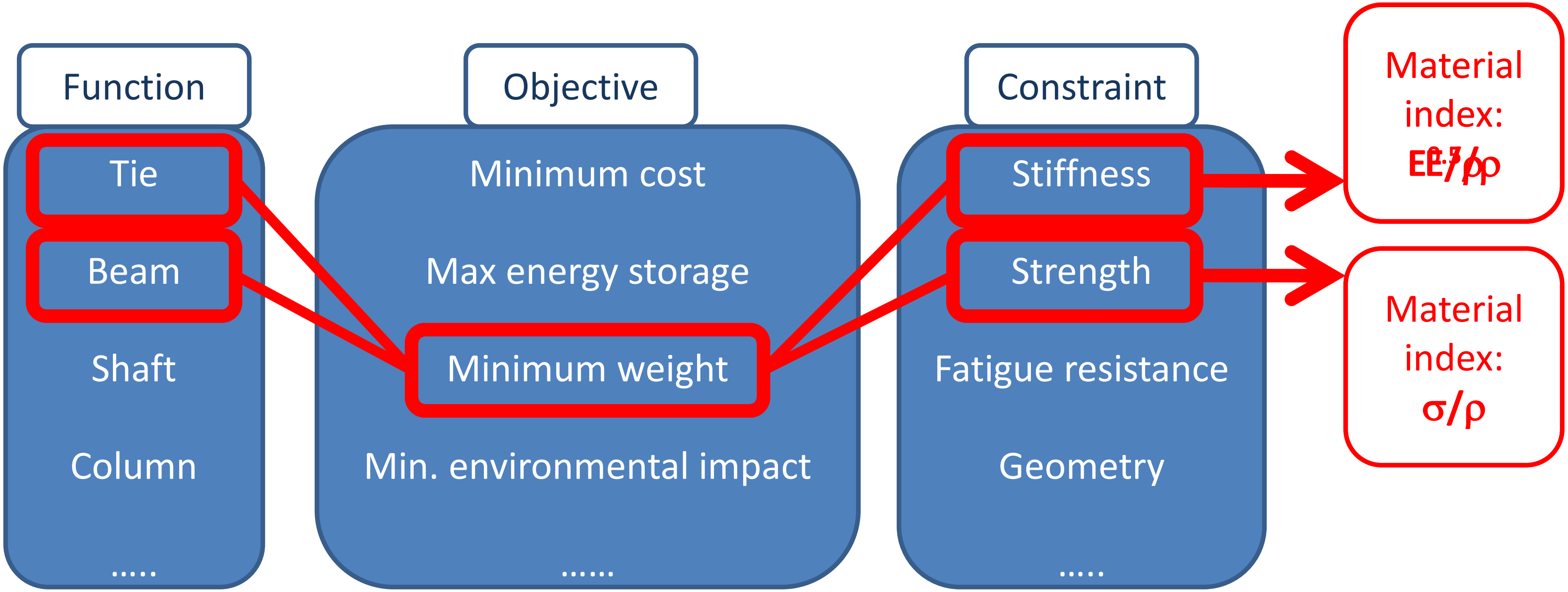
$$m = A L \rho$$

Performance metric m

$$\mathbf{m} = \mathbf{F\,L}\left(\frac{\boldsymbol{\rho}}{\boldsymbol{\sigma_y}}\right)$$

Chose materials with largest $\left(\frac{\boldsymbol{\sigma_y}}{\boldsymbol{\rho}}\right)$

Material indices

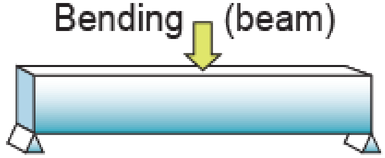
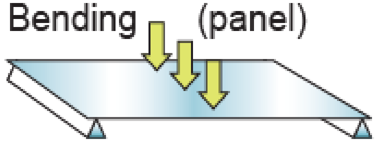


Criteria of excellence: material indices

- **Material index** = combination of material properties that limit performance

- Sometimes a single property
- Sometimes a combination

} Either is a **material index**

Objective minimise **mass**	**Constraints** Stiffness	 Strength
Tension (tie) 	E/ρ	σ_y/ρ
Bending (beam) 	$E^{1/2}/\rho$	$\sigma_y^{2/3}/\rho$
Bending (panel) 	$E^{1/3}/\rho$	$\sigma_y^{1/2}/\rho$

Explore these! (Stiffness column)

Remember this one too! ($\sigma_y^{2/3}/\rho$)

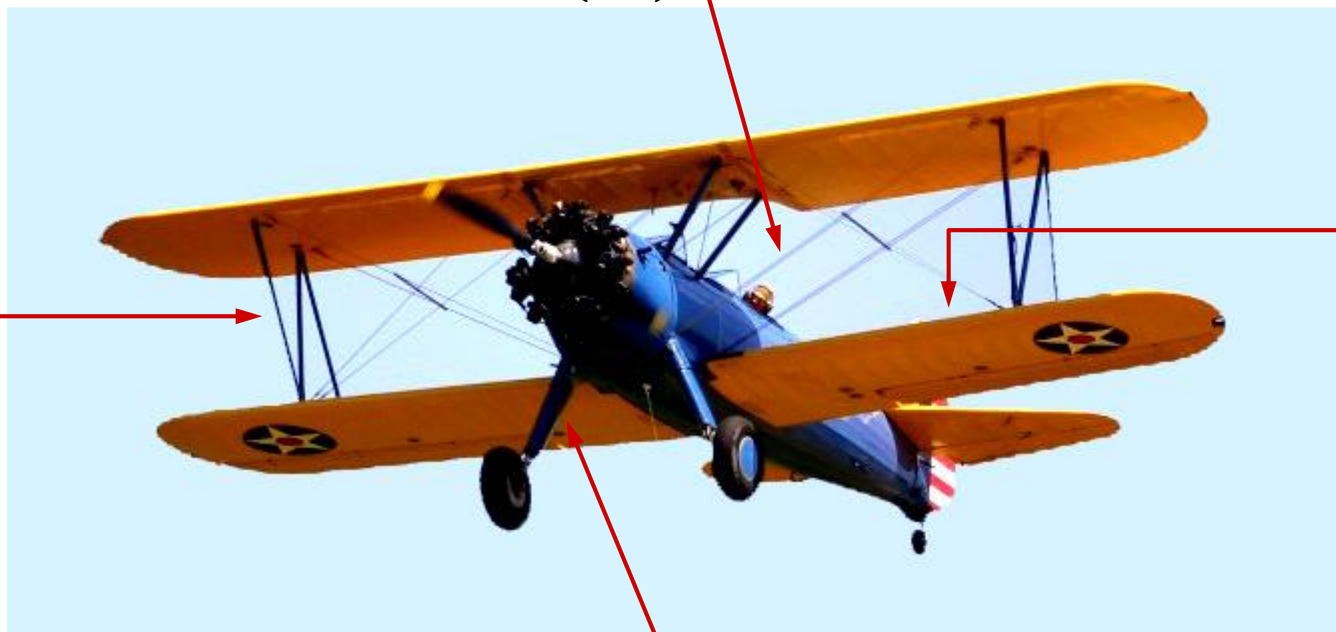
Minimum weight design

Tensile ties

$$\left(\frac{\sigma_y}{\rho}\right)$$

Compression strut

$$\left(\frac{E^{1/2}}{\rho}\right)$$



Main spar - beam

$$\left(\frac{E^{1/2}}{\rho}\right)$$

Undercarriage- compression

$$\left(\frac{\sigma_y^{2/3}}{\rho}\right)$$

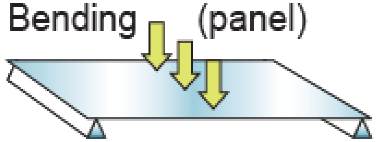
E = Young's modulus
ρ = Density
σ_y = Yield strength

Criteria of excellence: material indices

- **Material index** = combination of material properties that limit performance

- Sometimes a single property
- Sometimes a combination

Either is a **material index**

Objective minimise **cost**	**Constraints** Stiffness	Strength
Tension (tie) 	$E/C_m\rho$	$\sigma_y/C_m\rho$
Bending (beam) 	$E^{1/2}/C_m\rho$	$\sigma_y^{2/3}/C_m\rho$
Bending (panel) 	$E^{1/3}/C_m\rho$	$\sigma_y^{1/2}/C_m\rho$

Material cost/kg (C_m)

Minimum cost design

Structural panels $\left(\dfrac{\sigma_y^{1/2}}{C_m\rho}\right)$

Structural beam $\left(\dfrac{\sigma_y^{2/3}}{C_m\rho}\right)$

Tensile ties $\left(\dfrac{\sigma_y}{C_m\rho}\right)$

Compression strut (column) $\left(\dfrac{\sigma_y}{C_m\rho}\right)$

C_m = Material cost / kg

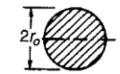
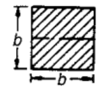
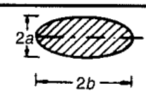
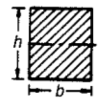
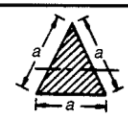
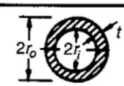
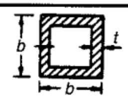
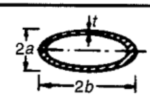
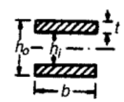
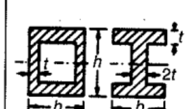
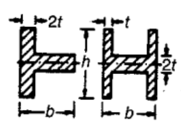
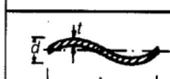
ρ = Density

σ_y = Yield strength

What if we change the free variable?

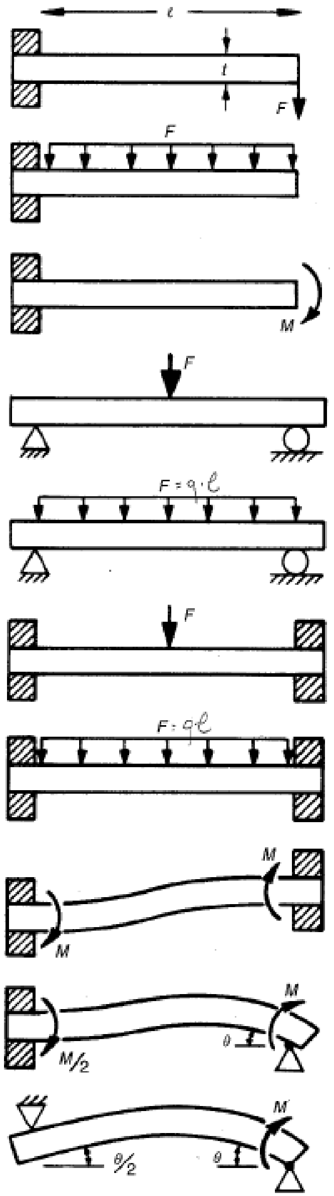
Tables

Table 7.1 Moments of areas of sections for common shapes

Section Shape	$A(m^2)$	$I_{xx}(m^4)$	$K(m^4)$	$Z(m^3)$	$Q(m^3)$
	πr^2	$\frac{\pi}{4}r^4$	$\frac{\pi}{2}r^4$	$\frac{\pi}{4}r^3$	$\frac{\pi}{2}r^3$
	b^2	$\frac{b^4}{12}$	$0.14b^4$	$\frac{b^3}{6}$	$0.21b^3$
	πab	$\frac{\pi}{4}a^3b$	$\frac{\pi a^3b^3}{(a^2+b^2)}$	$\frac{\pi}{4}a^2b$	$\frac{\pi a^2b}{2}$ $(a<b)$
	bh	$\frac{bh^3}{12}$	$\frac{b^3h}{3}\left(1-0.58\frac{b}{h}\right)$ $(h>b)$	$\frac{bh^2}{6}$	$\frac{b^2h^2}{3h+1.8b}$ $(h>b)$
	$\frac{\sqrt{3}}{4}a^2$	$\frac{a^4}{32\sqrt{3}}$	$\frac{a^4\sqrt{3}}{80}$	$\frac{a^3}{32}$	$\frac{a^3}{20}$
	$\pi(r_o^2-r_i^2)$ $\approx 2\pi rt$	$\frac{\pi}{4}(r_o^4-r_i^4)$ $\approx \pi r^3t$	$\frac{\pi}{2}(r_o^4-r_i^4)$ $\approx 2\pi r^3t$	$\frac{\pi}{4r_o}(r_o^4-r_i^4)$ $\approx \pi r^2t$	$\frac{\pi}{2r_o}(r_o^4-r_i^4)$ $\approx 2\pi r^2t$
	$4bt$	$\frac{2}{3}b^3t$	$b^3t\left(1-\frac{t}{b}\right)^4$	$\frac{4}{3}b^2t$	$2b^2t\left(1-\frac{t}{b}\right)^2$
	$\pi(a+b)t$	$\frac{\pi}{4}a^3t\left(1+\frac{3b}{a}\right)$	$\frac{4\pi(ab)^{5/2}t}{(a^2+b^2)}$	$\frac{\pi a^2t}{4}\left(1+\frac{3b}{a}\right)$	$2\pi t(a^3b)^{1/2}$ $(b>a)$
	$b(h_o-h_i)$ $\approx 2bt$	$\frac{b}{12}(h_o^3-h_i^3)$ $\approx \frac{1}{2}bth_o^2$	—	$\frac{b}{6h_o}(h_o^3-h_i^3)$ $\approx bth_o$	—
	$2t(h+b)$	$\frac{1}{6}h^3t\left(1+\frac{3b}{h}\right)$	Ɪ: $\approx \frac{2tb^2h^2}{h+b}$; □: $\frac{2}{3}bt^3\left(1+\frac{4h}{b}\right)$	$\frac{h^2t}{3}\left(1+\frac{3b}{h}\right)$	Ɪ: $2tbh$; □: $\frac{2}{3}bt^2\left(1+\frac{4h}{b}\right)$
	$2t(h+b)$	$\frac{t}{6}(h^3+4bt^2)$	H: $\frac{t^3}{3}(8b+h)$; ⊢: $\frac{2}{3}ht^3\left(1+\frac{4b}{h}\right)$	$\frac{t}{3h}(h^3+4bt^2)$	H: $\frac{t^2}{3}(8b+h)$; ⊢: $\frac{2}{3}ht^2\left(1+\frac{4b}{h}\right)$
	$t\lambda\left(1+\frac{\pi^2d^2}{4\lambda^2}\right)$	$\frac{t\lambda d^2}{8}$	—	$\frac{t\lambda d}{4}$	—

Tables

Elastic bending of beams



	C_1	C_2
1	3	2
2	8	6
3	2	1
4	48	16
5	$\frac{384}{5}$	24
6	192	–
7	384	–
8	6	–
9	–	4
10	–	3

$$\delta = \frac{F\ell^3}{C_1 EI} = \frac{M\ell^2}{C_1 EI}$$

$$\theta = \frac{F\ell^2}{C_2 EI} = \frac{M\ell}{C_2 EI}$$

E = Young's modulus (N/m^2)

δ = deflection (m)

F = force (N)

M = moment (Nm)

ℓ = length (m)

b = width (m)

t = depth (m)

θ = end slope (–)

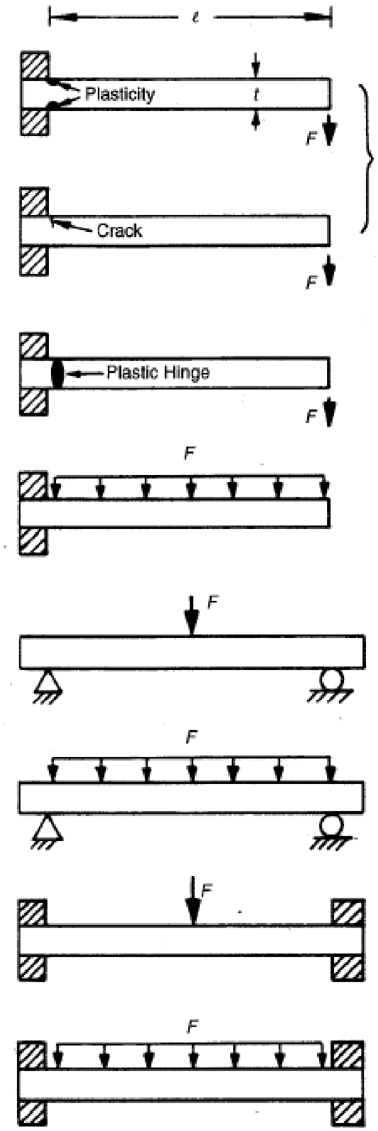
I = see Table 2 (m^4)

y = distance from N.A. (m)

R = radius of curvature (m)

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

Failure of beams and panels



	C
1–2 (Plasticity, Crack)	1
3 (Plastic Hinge)	1
4	2
5	4
6	8
7	8
8	16

$$M_f = \left(\frac{I}{y_m}\right)\sigma^* \text{ (Onset)}$$

$$M_f = H\sigma_y \text{ (Full plasticity)}$$

$$F_f = C\left(\frac{I}{y_m}\right)\frac{\sigma^*}{\ell} \text{ (Onset)}$$

$$F_f = \frac{CH\sigma_y}{\ell} \text{ (Full plasticity)}$$

M_f = failure moment (Nm)

F_f = force at failure (N)

ℓ = length (m)

t = depth (m)

b = width (m)

I = see Table 2 (m^4)

$\frac{I}{y_m}$ = see Table 2 (m^3)

H = see Table 2 (m^3)

σ_y = yield strength (N/m^2)

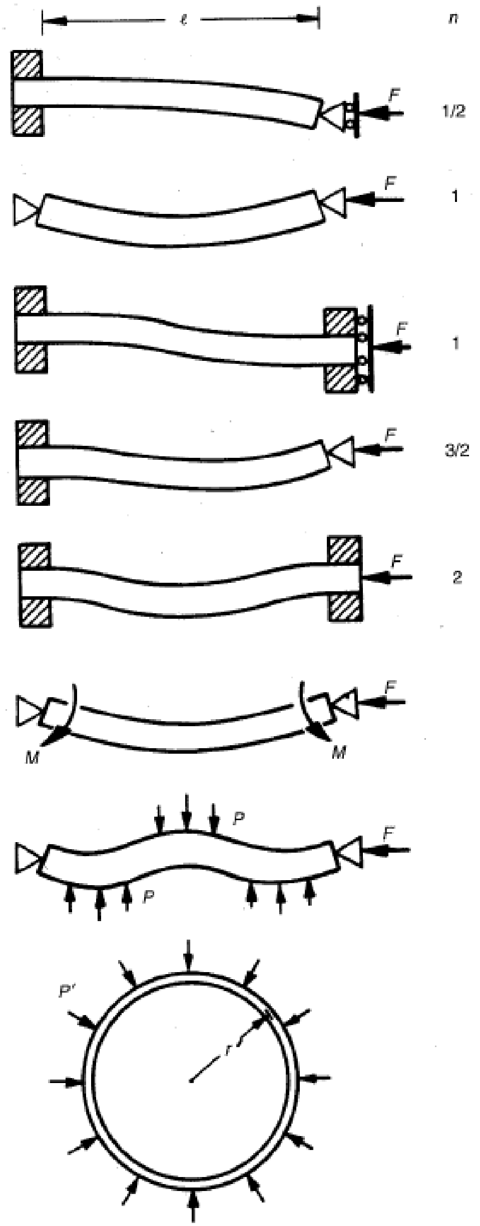
σ_f = modulus of rupture (N/m^2)

σ^* = σ_y (plastic material)

= σ_f (brittle material)

Tables

Buckling of columns and plates



$$F_{\mathrm{CRIT}} = \frac{n^2\pi^2 EI}{\ell^2}$$

or

$$\frac{F_{\mathrm{CRIT}}}{A} = \frac{n^2\pi^2 E}{(\ell/r)^2}$$

F = force (N)

M = moment (Nm)

E = Young's modulus (N/m^2)

ℓ = length (m)

A = section area (m^2)

I = see Table 2 (m^4)

r = gyration rad. $\left(\frac{I}{A}\right)^{1/2}$ (m)

k = foundation stiffness (N/m^2)

n = half-wavelengths in buckled shape

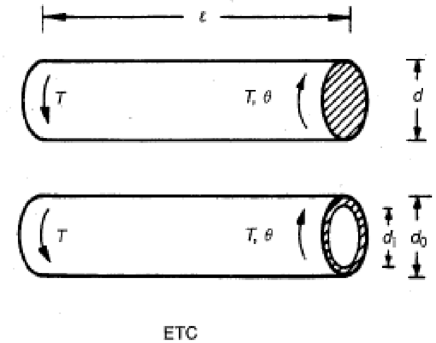
p' = pressure (N/m^2)

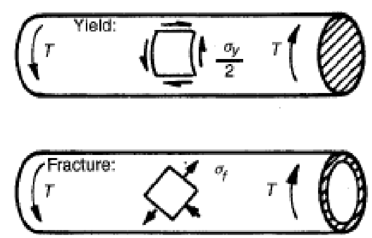
$$F_{\mathrm{CRIT}} = \frac{\pi^2 EI}{\ell^2} - \frac{M^2}{4EI}$$

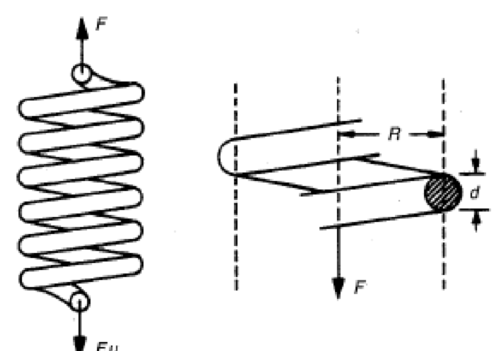
$$F_{\mathrm{CRIT}} = \frac{n^2\pi^2 EI}{\ell^2} + \frac{k\ell^2}{n^2}$$

$$p'_{\mathrm{CRIT}} = \frac{3EI}{(r')^3}$$

Torsion of shafts







Elastic deflection

$$\theta = \frac{\ell T}{KG}$$

Failure

$$T_f = \frac{K\sigma_y}{d_0} \text{ (Onset of yield)}$$

$$T_f = \frac{2K\sigma_f}{d_0} \text{ (Brittle fracture)}$$

T = torque (Nm)

θ = angle of twist

G = shear modulus (N/m^2)

ℓ = length (m)

d = diameter (m)

K = see Table 1 (m^4)

σ_y = yield strength (N/m^2)

σ_f = modulus of rupture (N/m^2)

Spring deflection and failure

$$u = \frac{64FR^3n}{Gd^4}$$

$$F_f = \frac{\pi}{32}\frac{d^3\sigma_y}{R}$$

F = force (N)

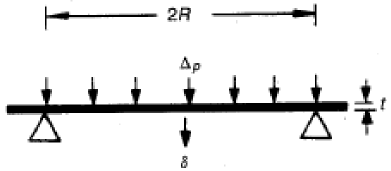
u = deflection (m)

R = coil radius (m)

n = number of turns

Tables

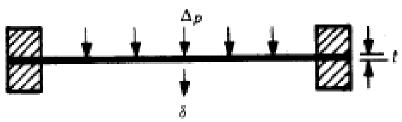
Static and spinning discs



Simple

$$\delta = \frac{3}{4}(1-\nu^2)\frac{\Delta p R^4}{E t^3}$$

$$\sigma_{\max} = \frac{3}{8}(3+\nu)\frac{\Delta p R^2}{t^2}$$



Clamped

$$\delta = \frac{3}{16}(1-\nu^2)\frac{\Delta p R^4}{E t^3}$$

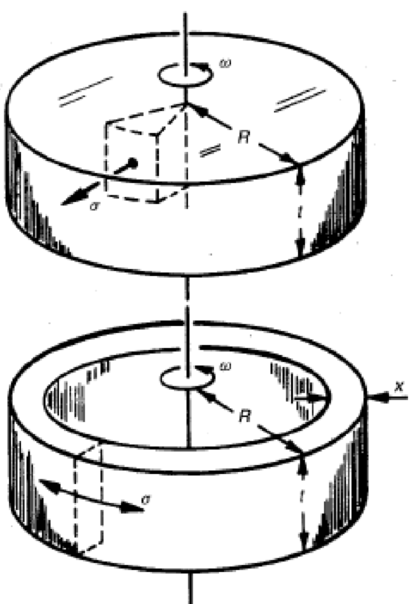
$$\sigma_{\max} = \frac{3}{8}(1+\nu)\frac{\Delta p R^2}{t^2}$$

δ = deflection (m)

E = Young's modulus (N/m)

Δp = pressure diff. (N/m)

ν = Poisson's ratio



Disc

$$u = \frac{\pi}{4}\rho t \omega^2 R^4$$

$$\sigma_{\max} = \frac{1}{8}(3+\nu)\rho\omega^2 R^2$$

Ring

$$u = \pi\rho t\omega^2 R^3 x$$

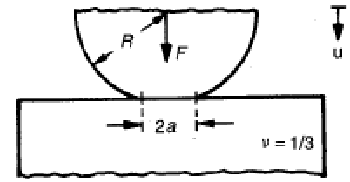
$$\sigma_{\max} = \rho R^2 \omega^2$$

u = energy (J)

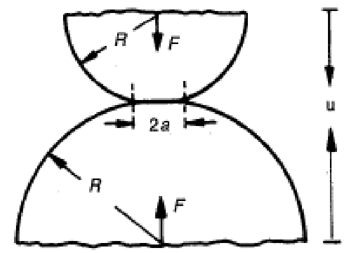
ω = angular vel. (rad/s)

ρ = density kg/m^3

Contact stresses

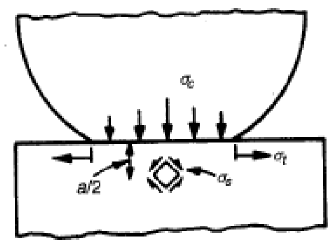


$$\left.\begin{aligned} a &= 0.7\left(\frac{FR}{E}\right)^{1/3} \\ u &= 1.0\left(\frac{F^2}{E^2 R}\right)^{1/3}\end{aligned}\right\} \nu = \frac{1}{3}$$



$$a = \left(\frac{3}{4}\frac{F}{E^*}\frac{R_1 R_2}{(R_1+R_2)}\right)^{1/3}$$

$$u = \left(\frac{9}{16}\frac{F^2}{(E^*)^2}\frac{(R_1+R_2)}{R_1 R_2}\right)^{1/3}$$

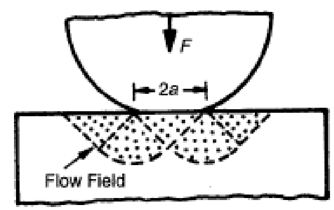


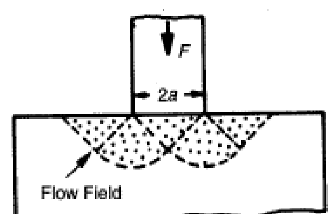
$$(\sigma_c)_{\max} = \frac{3F}{2\pi a^2}$$

$$(\sigma_s)_{\max} = \frac{F}{2\pi a^2}$$

$$(\sigma_t)_{\max} = \frac{F}{6\pi a^2}$$

$R_1 R_2$	radii of spheres (m)
$E_1 E_2$	modulii of spheres (N/m^2)
$\nu_1 \nu_2$	Poisson's ratios
F	load (N)
a	radius of contact (m)
u	displacement (m)
σ	stresses (N/m^2)
σ_y	yield stress (N/m^2)
E^*	$\left(\frac{1-\nu_1^2}{E_1}+\frac{1-\nu_2^2}{E_2}\right)^{-1}$

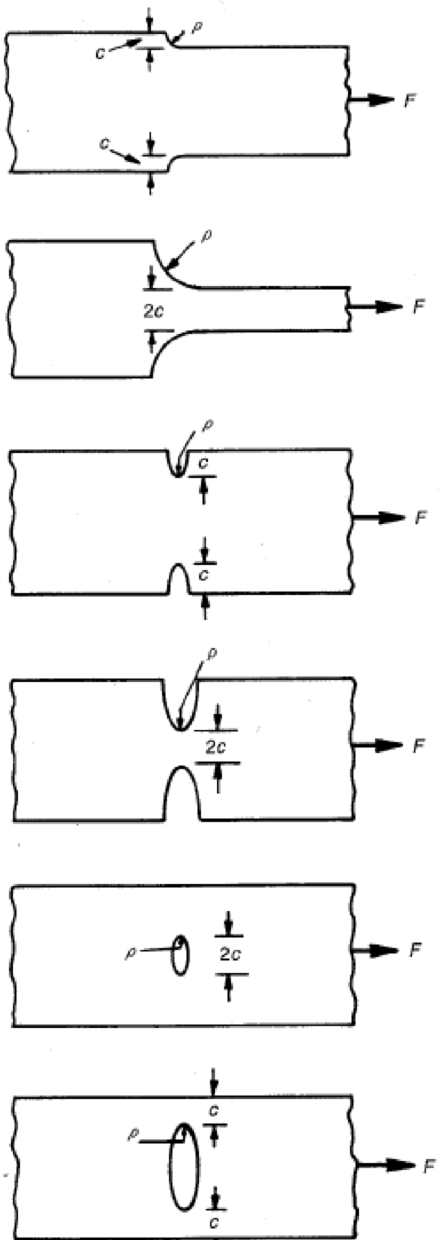




$$\frac{F}{\pi a^2} = 3\sigma_y$$

Tables

Estimates for stress concentrations



$$\frac{\sigma_{max}}{\sigma_{nom}} = 1 + \alpha\left(\frac{c}{\rho}\right)^{1/2}$$

F = force (N)

A_{min} = minimum section (m^2)

$\sigma_{nom} = F/A_{min}$ (N/m^2)

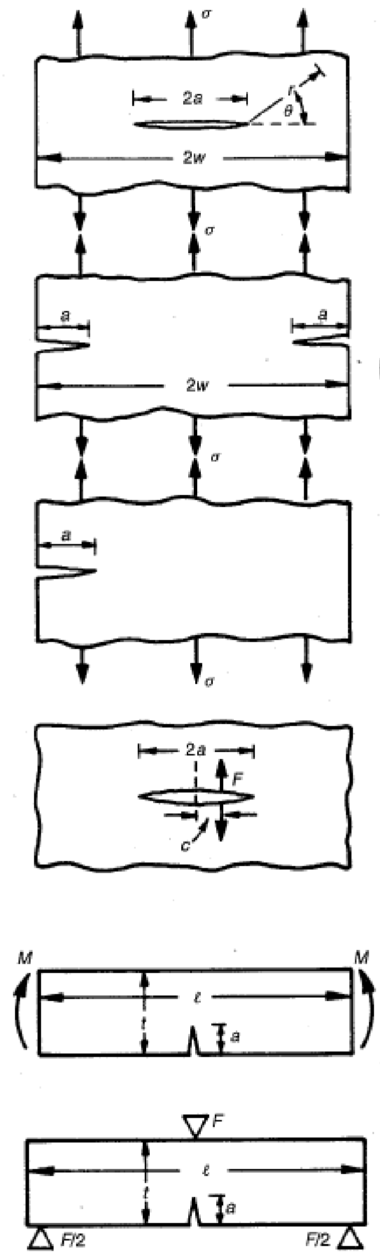
ρ = radius of curvature (m)

c = characteristic length (m)

$\alpha \approx 0.5$ (tension)

$\alpha \approx 2.0$ (torsion)

Sharp cracks



C

1.0 $(a \ll w)$

$\left(\frac{2w}{\pi a}\tan\frac{\pi a}{2w}\right)^{1/2}$

1.1 $(a \ll w)$

$\left(\frac{\frac{2w}{\pi a}\tan\frac{\pi a}{2w}}{1-a/w}\right)^{1/2}$

1.1 $(a \ll w)$

$\frac{1.1\,(1-0.2\,a/w)}{(1-a/w)^{3/2}}$

$\frac{2}{\pi}\left(\frac{a+c}{a-c}\right)^{1/2}$

$\frac{1.1\left(1-\frac{3}{2}\frac{a}{t}\right)}{(1-a/t)^{3/2}}$

$\frac{1.1\left(1-\frac{3}{2}\frac{a}{t}\right)}{(1-a/t)^{3/2}}$

$$K_1 = C\sigma\sqrt{\pi a}$$

failure when

$$K_1 \geq K_{IC}$$

K_1 = stress intensity ($N/m^{3/2}$)

σ = remote stress (N/m^2)

F = load (N)

M = moment (Nm)

a = crack half-length

= surface crack length (m)

w = half-width (centre) (m)

= width (edge crack) (m)

b = sample depth (m)

t = beam thickness (m)

point load on crack face:

$$\sigma = \frac{F}{2ab}$$

moment on beam:

$$\sigma = \frac{6M}{bt}$$

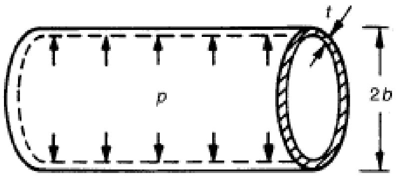
3-point bending:

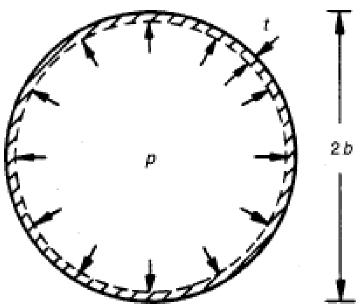
$$\sigma = \frac{3F\ell}{2bt}$$

Tables

Pressure vessels

Thin Walled





Cylinder

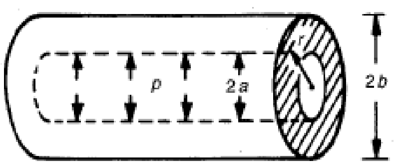
$$\sigma_\theta = \frac{pb}{t}$$
$$\sigma_r = -p/2$$
$$\sigma_z = \frac{pb}{2t} \text{ (closed ends)}$$

Sphere

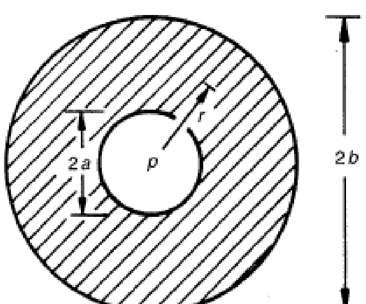
$$\sigma_\theta = \sigma_\phi = \frac{pb}{2t}$$
$$\sigma_r = -p/2$$

p = pressure (N/m^2)
t = wall thickness (m)
a = inner radius (m)
b = outer radius (m)
r = radial coordinate (m)

Thick Walled

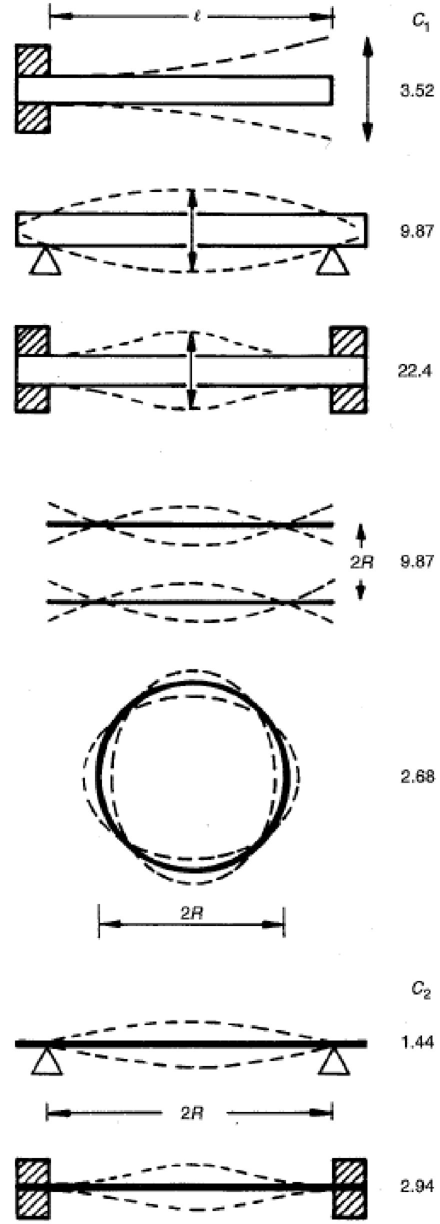


$$\sigma_\theta = \frac{pa^2}{r^2}\left(\frac{b^2 - r^2}{b^2 - a^2}\right)$$
$$\sigma_r = -\frac{pa^2}{r^2}\left(\frac{b^2 + r^2}{b^2 - a^2}\right)$$



$$\sigma_\theta = \sigma_\phi = \frac{pa^3}{2r^3}\left(\frac{b^3 + 2r^3}{b^3 - a^3}\right)$$
$$\sigma_r = -\frac{pa^3}{r^3}\left(\frac{b^3 - r^3}{b^3 - a^3}\right)$$

Vibrating beams, tubes and discs



Beams, tubes

$$F_1 = \frac{C_1}{2\pi}\sqrt{\frac{EI}{m_0 \ell^4}}$$

f = natural frequency (s^{-1})
$m_0 = \rho A$ = mass/length (kg/m)
ρ = density (kg/m^3)
A = section area (m^2)
I = see Table A1

$\left\{ \text{with } A = 2\pi R, \quad I = \pi R^3 t \right.$

$\left\{ \text{with } A = \frac{\ell t^3}{12} \right.$

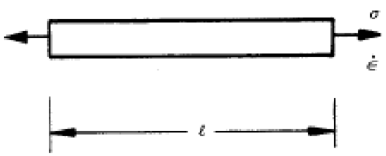
Discs

$$f_1 = \frac{C_2}{2\pi}\sqrt{\frac{Et^3}{m_1 R^4 (1 - \nu^2)}}$$

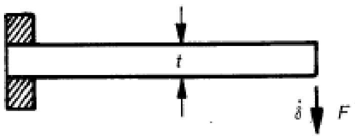
$m_1 = \rho t$ = mass/area (kg/m^2)
t = thickness (m)
R = radius (m)
ν = Poisson's ratio

Tables

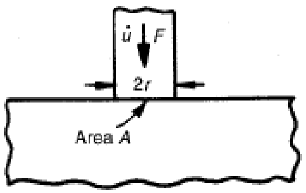
Creep and creep fracture



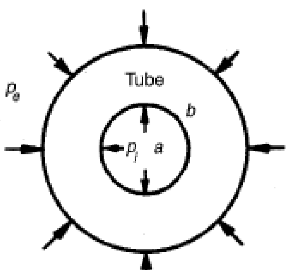
$$\dot{\varepsilon} = \dot{\varepsilon}_0 \left(\frac{\sigma}{\sigma_0}\right)^n$$



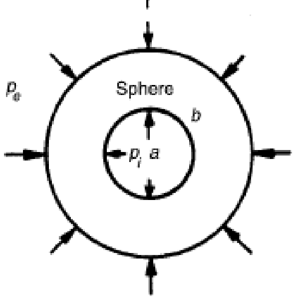
$$\dot{\delta} = \frac{2\dot{\varepsilon}_0}{n+2}\left(\frac{(2n+1)}{n\sigma_0}\frac{E}{bt}\right)^n \left(\frac{\ell}{t}\right)^{n+1} \ell$$



$$\dot{u} = C_1 \dot{\varepsilon}_0 \sqrt{A}\left(\frac{C_2 F}{\sigma_0 A}\right)^n$$



$$\dot{\rho} = 2\dot{\varepsilon}_0 \frac{\rho(1-\rho)}{\left(1-(1-\rho)^{1/n}\right)^n} \times \left(\frac{2}{n}\frac{(p_e - p_i)}{\sigma_0}\right)^n$$



$$\dot{\rho} = \frac{3}{2}\dot{\varepsilon}_0 \frac{\rho(1-\rho)}{\left(1-(1-\rho)^{1/n}\right)^n} \times \left(\frac{3}{2n}\frac{(\rho_e - \rho_i)}{\sigma_0}\right)^n$$

σ = stress N/m^2

F = force (N)

$\dot{\delta}, \dot{u}$ = displacement rates (m/s)

$n, \dot{\varepsilon}_0, \sigma_0$ = creep constants

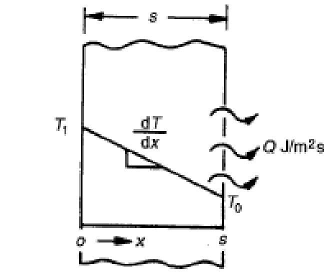
ℓ, b, t = beam dimensions (m)

a, b = radii of pressure vessels (m)

ρ = relative density, $\dfrac{b^3 - a^3}{b^3}$

C_1, C_2 = constants

Flow of heat and matter



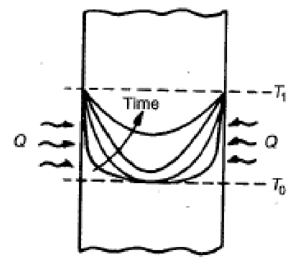
$$Q = -\lambda \nabla T = -\lambda \frac{dT}{dx}$$

Q = heat flux (J/m^{2}s)

T = temperature (K)

x = distance (m)

λ = thermal conductivity (W/mK)



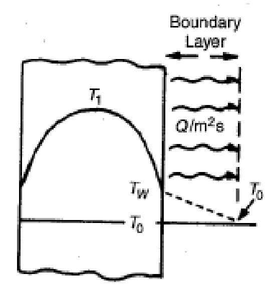
$$\frac{\partial T}{\partial t} = a\nabla^2 T = a\frac{\partial^2 T}{\partial x^2}$$

t = time (s)

ρ = density (kg/m^3)

C = specific heat (J/m^{3}K)

a = thermal diffusivity, $\dfrac{\lambda}{\rho c}$ (m^2/s)



$$Q = h(T_w - T_0)$$

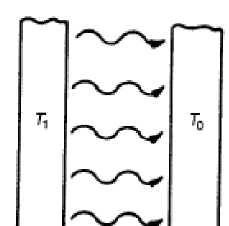
T_w = surface temperature (K)

T = fluid temperature (K)

h = heat transfer coeff. (W/m^{2}K)

= 5–50 W/m^{2}K in air

= 1000–5000 W/m^{2}K in water



$$Q = \varepsilon\sigma(T_1^4 - T_0^4)$$

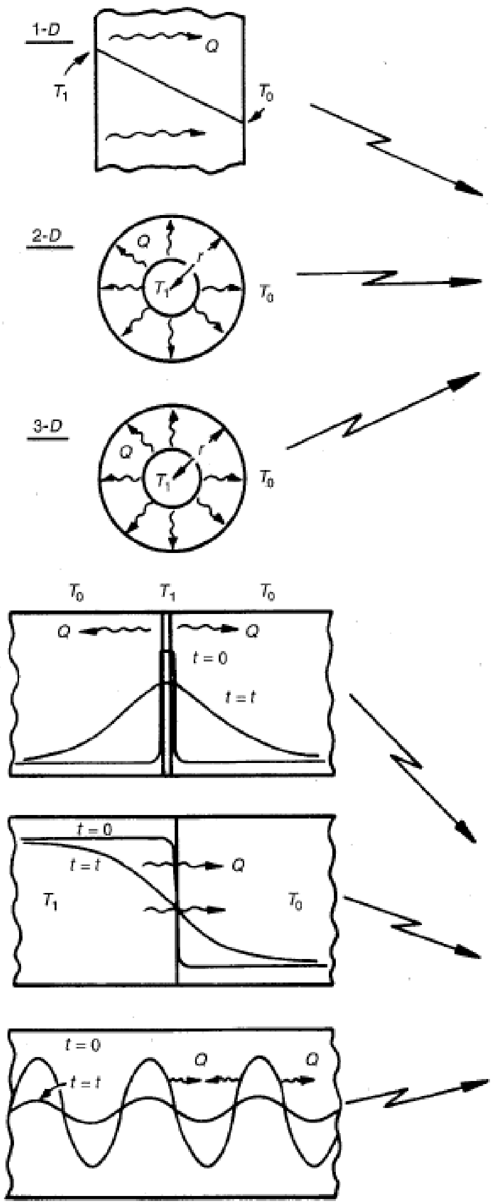
ε = emissivity (1 for black body)

σ = Stefan constant

= 5.67×10^{-8} W/m^{2}K^4

Tables

Solutions for diffusion equations



Steady state

$$\lambda \nabla^2 T = 0$$

$$D \nabla^2 C = 0$$

$$\lambda T(x) = Ax + B (1 - D)$$

$$\lambda T(r) = A \ell n r + B (2 - D)$$

$$\lambda T(r) = A/r + B (3 - D) \text{ etc.}$$

A, B = constants of integration

x = linear distance (m)

r = radial distance (m)

Transient

$$\frac{\partial T}{\partial t} = a \nabla^2 T$$

$$\frac{\partial C}{\partial t} = D \nabla^2 C$$

$$T(x,t) = \frac{A}{\sqrt{at}} \exp - \left(\frac{x^2}{4at} \right) + B$$

$$T(x,t) = A \left(1 + erf \left(\frac{x}{2\sqrt{at}} \right) \right) + B$$

$$T(x,t) = A \sin \lambda x, \exp - \lambda^2 at \text{ etc.}$$

Table B1 Stiffness-limited design at minimum mass (cost, energy, environmental impact*)

*Function and constraints**	*Maximize*†
Tie (tensile strut)	
stiffness, length specified; section area free	E/ρ
Shaft (loaded in torsion)	
stiffness, length, shape specified, section area free	$G^{1/2}/\rho$
stiffness, length, outer radius specified; wall thickness free	G/ρ
stiffness, length, wall-thickness specified; outer radius free	$G^{1/3}/\rho$
Beam (loaded in bending)	
stiffness, length, shape specified; section area free	$E^{1/2}/\rho$
stiffness, length, height specified; width free	E/ρ
stiffness, length, width specified; height free	$E^{1/3}/\rho$
Column (compression strut, failure by elastic buckling)	
buckling load, length, shape specified; section area free	$E^{1/2}/\rho$
Panel (flat plate, loaded in bending)	
stiffness, length, width specified, thickness free	$E^{1/3}/\rho$
Plate (flat plate, compressed in-plane, buckling failure)	
collapse load, length and width specified, thickness free	$E^{1/3}/\rho$
Cylinder with internal pressure	
elastic distortion, pressure and radius specified; wall thickness free	E/ρ
Spherical shell with internal pressure	
elastic distortion, pressure and radius specified, wall thickness free	$E/(1-\nu)\rho$

*To minimize cost, use the above criteria for minimum weight, replacing density ρ by $C_m\rho$, where C_m is the material cost per kg. To minimize energy content, use the above criteria for minimum weight replacing density ρ by $q\rho$ where q is the energy content per kg. To minimize environmental impact, replace density ρ by $I_e\rho$ instead, where I_e is the eco-indicator value for the material (references [1] and [4]).

†E = Young's modulus for tension, the flexural modulus for bending or buckling; G = shear modulus; ρ = density, q = energy content/kg; I_e = eco-indicator value/kg.

Table B7 Electro-mechanical design

Function and constraints	*Maximize**
Bus bars	
minimum life-cost; high current conductor	$1/\rho_e\rho C_m$
Electro-magnet windings	
maximum short-pulse field; no mechanical failure	σ_y
maximize field and pulse-length, limit on temperature rise	$C_p\rho/\rho_e$
Windings, high-speed electric motors	
maximum rotational speed; no fatigue failure	σ_e/ρ_e
minimum ohmic losses; no fatigue failure	$1/\rho_e$
Relay arms	
minimum response time; no fatigue failure	$\sigma_e/E\rho_e$
minimum ohmic losses; no fatigue failure	$\sigma_e^2/E\rho_e$

*C_m = material cost/kg; E = Young's modulus; ρ = density; ρ_e = electrical resistivity; σ_y = yield strength; σ_e = endurance limit.

Table B2 Strength-limited design at minimum mass (cost, energy, environmental impact*)

*Function and constraints**‡	*Maximize*†
Tie (tensile strut)	
stiffness, length specified; section area free	σ_f/ρ
Shaft (loaded in torsion)	
load, length, shape specified, section area free	$\sigma_f^{2/3}/\rho$
load, length, outer radius specified; wall thickness free	σ_f/ρ
load, length, wall-thickness specified; outer radius free	$\sigma_f^{1/2}/\rho$
Beam (loaded in bending)	
load, length, shape specified; section area free	$\sigma_f^{2/3}/\rho$
load length, height specified; width free	σ_f/ρ
load, length, width specified; height free	$\sigma_f^{1/2}/\rho$
Column (compression strut)	
load, length, shape specified; section area free	σ_f/ρ
Panel (flat plate, loaded in bending)	
stiffness, length, width specified, thickness free	$\sigma_f^{1/2}/\rho$
Plate (flat plate, compressed in-plane, buckling failure)	
collapse load, length and width specified, thickness free	$\sigma_f^{1/2}/\rho$
Cylinder with internal pressure	
elastic distortion, pressure and radius specified; wall thickness free	σ_f/ρ
Spherical shell with internal pressure	
elastic distortion, pressure and radius specified, wall thickness free	σ_f/ρ
Flywheels, rotating discs	
maximum energy storage per unit volume; given velocity	ρ
maximum energy storage per unit mass; no failure	σ_f/ρ

*To minimize cost, use the above criteria for minimum weight, replacing density ρ by $C_m\rho$, where C_m is the material cost per kg. To minimize energy content, use the above criteria for minimum weight replacing density ρ by $q\rho$ where q is the energy content per kg. To minimize environmental impact, replace density ρ by $I_e\rho$ instead, where I_e is the eco-indicator value for the material (references [1] and [4]).

†σ_f = failure strength (the yield strength for metals and ductile polymers, the tensile strength for ceramics, glasses and brittle polymers loaded in tension; the flexural strength or modulus of rupture for materials loaded in bending); ρ = density.

‡For design for infinite fatigue life, replace σ_f by the endurance limit σ_e.

Tables

Table B3 Strength-limited design: springs, hinges etc. for maximum performance*

*Function and constraints**‡	*Maximize*†
Springs	
maximum stored elastic energy per unit volume; no failure	σ_f^2/E
maximum stored elastic energy per unit mass; no failure	$\sigma_f^2/E\rho$
Elastic hinges	
radius of bend to be minimized (max flexibility without failure)	σ_f/E
Knife edges, pivots	σ_f^3/E^2 and
minimum contact area, maximum bearing load	H
Compression seals and gaskets	$\sigma_f^{3/2}/E$ and
maximum conformability; limit on contact pressure	$1/E$
Diaphragms	
maximum deflection under specified pressure or force	$\sigma_f^{3/2}/E$
Rotating drums and centrifuges	
maximum angular velocity; radius fixed; wall thickness free	σ_f/ρ

*To minimize cost, use the above criteria for minimum weight, replacing density ρ by $C_m\rho$, where C_m is the material cost per kg. To minimize energy content, use the above criteria for minimum weight replacing density ρ by $q\rho$ where q is the energy content per kg. To minimize environmental impact, replace density ρ by $I_e\rho$ instead, where I_e is the eco-indicator value for the material (references [1] and [4]).

†σ_f = failure strength (the yield strength for metals and ductile polymers, the tensile strength for ceramics, glasses and brittle polymers loaded in tension; the flexural strength or modulus of rupture for materials loaded in bending); ρ = density; H = hardness.

‡For design for infinite fatigue life, replace σ_f by the endurance limit σ_e.

Table B4 Vibration-limited design

Function and constraints	*Maximize**
Ties, columns	
maximum longitudinal vibration frequencies	E/ρ
Beams, all dimensions prescribed	
maximum flexural vibration frequencies	E/ρ
Beams, length and stiffness prescribed	
maximum flexural vibration frequencies	$E^{1/2}/\rho$
Panels, all dimensions prescribed	
maximum flexural vibration frequencies	E/ρ
Panels, length, width and stiffness prescribed	
maximum flexural vibration frequencies	$E^{1/3}/\rho$
Ties, columns, beams, panels, stiffness prescribed	
minimum longitudinal excitation from external drivers, ties	$\eta E/\rho$
minimum flexural excitation from external drivers, beams	$\eta E^{1/2}/\rho$
minimum flexural excitation from external drivers, panels	$\eta E^{1/3}/\rho$

*E = Young's modulus for tension, the flexural modulus for bending; G = shear modulus; ρ = density; η = damping coefficient (loss coefficient).

Table B5 Damage-tolerant design

Function and constraints	*Maximize**
Ties (tensile member)	
Maximize flaw tolerance and strength, load-controlled design	K_{Ic} and σ_f
Maximize flaw tolerance and strength, displacement-control	K_{Ic}/E and σ_f
Maximize flaw tolerance and strength, energy-control	K_{Ic}^2/E and σ_f
Shafts (loaded in torsion)	
Maximize flaw tolerance and strength, load-controlled design	K_{Ic} and σ_f
Maximize flaw tolerance and strength, displacement-control	K_{Ic}/E and σ_f
Maximize flaw tolerance and strength, energy-control	K_{Ic}^2/E and σ_f
Beams (loaded in bending)	
Maximize flaw tolerance and strength, load-controlled design	K_{Ic} and σ_f
Maximize flaw tolerance and strength, displacement-control	K_{Ic}/E and σ_f
Maximize flaw tolerance and strength, energy-control	K_{Ic}^2/E and σ_f
Pressure vessel	
Yield-before-break	K_{Ic}/σ_f
Leak-before-break	K_{Ic}^2/σ_f

*K_{Ic} = fracture toughness; E = Young's modulus; σ_f = failure strength (the yield strength for metals and ductile polymers, the tensile strength for ceramics, glasses and brittle polymers loaded in tension; the flexural strength or modulus of rupture for materials loaded in bending).

Table B6 Thermal and thermo-mechanical design

Function and constraints	*Maximize**
Thermal insulation materials	
minimum heat flux at steady state; thickness specified	$1/\lambda$
minimum temp rise in specified time; thickness specified	$1/a = \rho C_p/\lambda$
minimize total energy consumed in thermal cycle (kilns, etc)	$\sqrt{a}/\lambda = \sqrt{1/\lambda\rho C_p}$
Thermal storage materials	
maximum energy stored/unit material cost (storage heaters)	C_p/C_m
maximize energy stored for given temperature rise and time	$\lambda/\sqrt{a} = \sqrt{\lambda\rho C_p}$
Precision devices	
minimize thermal distortion for given heat flux	λ/α
Thermal shock resistance	$\sigma_f/E\alpha$
maximum change in surface temperature; no failure	
Heat sinks	
maximum heat flux per unit volume; expansion limited	$\lambda/\Delta\alpha$
maximum heat flux per unit mass; expansion limited	$\lambda/\rho\Delta\alpha$
Heat exchangers (pressure-limited)	
maximum heat flux per unit area; no failure under Δp	$\lambda\sigma_f$
maximum heat flux per unit mass; no failure under Δp	$\lambda\sigma_f/\rho$

*λ = thermal conductivity; a = thermal diffusivity; C_p = specific heat capacity; C_m = material cost/kg; T_{max} = maximum service temperature; α = thermal expansion coeff.; E = Young's modulus; ρ = density; σ_f = failure strength (the yield strength for metals and ductile polymers, the tensile strength for ceramics, glasses and brittle polymers).